Research Article

Midpoint Derivative-Based Closed Newton-Cotes Quadrature

Weijing Zhao[1,2] and Hongxing Li[1]

[1] *Faculty of Electronic Information and Electrical Engineering, Dalian University of Technology, Dalian 116024, China*
[2] *College of Air Traffic Management, Civil Aviation University of China, Tianjin 300300, China*

Correspondence should be addressed to Weijing Zhao; 693509394@qq.com

A novel family of numerical integration of closed Newton-Cotes quadrature rules is presented which uses the derivative value at the midpoint. It is proved that these kinds of quadrature rules obtain an increase of two orders of precision over the classical closed Newton-Cotes formula, and the error terms are given. The computational cost for these methods is analyzed from the numerical point of view, and it has shown that the proposed formulas are superior computationally to the same order closed Newton-Cotes formula when they reduce the error below the same level. Finally, some numerical examples show the numerical superiority of the proposed approach with respect to closed Newton-Cotes formulas.

1. Introduction

Definite integration is one of the most important and basic concepts in mathematics. It has numerous applications in fields such as physics and engineering. In several practical problems, we need to calculate integrals. As is known to all, as for $I = \int_a^b f(x)dx$, once the primitive function $F(x)$ of integrand $f(x)$ is known, the definite integral of $f(x)$ over the interval $[a, b]$ is given by Newton-Leibniz formula, that is,

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{1}$$

However, the explicit primitive function $F(x)$ is not available or its primitive function is not easy to obtain, such as $e^{\pm x^2}$, $\sin x^2$, and $\sin x/x$. Moreover, some of the integrand $f(x)$ is only available at certain points x_i, $i = 0, 1, \ldots, n$. It is often the case that the values of $f(x_i)$ come from experimental data, such as sampling [1]. But the need often arises for calculating the definite integral. And how to get high-precision numerical integration formulas becomes one of the challenges in fields of mathematics [2].

The methods of quadrature are usually based on the interpolation polynomials and can be written in the following form:

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^n w_i f(x_i), \tag{2}$$

where there are $n + 1$ distinct integration points at $x_0, x_1, \ldots, x_n$ within the interval $[a, b]$ and $n + 1$ weights w_i, $i = 0, 1, \ldots, n$. If the integration points are uniformly distributed over the interval, so $x_i = x_0 + ih$ in which $h = (b - a)/n$.

These w_i can be derived in several different ways [3–5]. One method is to interpolate $f(x)$ at the $n + 1$ points $x_0, x_1, \ldots, x_n$, using the Lagrange polynomials and then integrating the foresaid polynomials to obtain (2).

The other method is based on the precision of a quadrature formula. Select the w_i, $i = 0, 1, \ldots, n$, so that the error

$$R_n(f) = \int_a^b f(x)\,dx - \sum_{i=0}^n w_i f(x_i), \tag{3}$$

is exactly zero for $f(x) = x^j$, $j = 0, 1, \ldots, n$. Using the method of undetermined coefficients, this approach generates a system of $n + 1$ linear equations for weights w_i.

Since the monomials $1, x, \ldots, x^n$ are linearly independent, the linear system of equations has a unique solution.

The Newton-Cotes formulas are the most well-known numerical integration rules of this type. There are several subclasses of Newton-Cotes formulas that depend on the integer value of n. We list some of them as follows.

Trapezoidal rule ($n = 1$)

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left(f(a) + f(b)\right) - \frac{(b-a)^3}{12}f''(\xi), \quad (4)$$

where $\xi \in (a, b)$.

Simpson's rule ($n = 2$)

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$
$$- \frac{(b-a)^5}{2880}f^{(4)}(\xi), \quad (5)$$

where $\xi \in (a, b)$.

Simpson's 3/8 rule ($n = 3$)

$$\int_a^b f(x)\,dx$$
$$= \frac{(b-a)}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$
$$- \frac{(b-a)^5}{6480}f^{(4)}(\xi), \quad (6)$$

where $\xi \in (a, b)$.

Bool's rule ($n = 4$)

$$\int_a^b f(x)\,dx = \frac{(b-a)}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right)\right.$$
$$+ 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right)$$
$$\left. + 7f(b)\right] - \frac{(b-a)^7}{1935360}f^{(6)}(\xi), \quad (7)$$

where $\xi \in (a, b)$.

Note that when n is an even integer, the degree of precision is $n + 1$. When n is odd, the degree of precision is only n [1, 2].

In spite of the many accurate and efficient methods for numerical integration being available in [3–5], recently Dehghan et al. [6] improved the precision degree of closed Newton-Cotes quadrature by including the location of boundaries of the interval as two additional variables and rescaling the original integral to fit the optimal boundary locations. In their following work, they have applied this method to Gauss-Legendre quadrature [7], Gauss-Chebyshev quadrature [8], and open Newton-Cotes quadrature [9]. These formulas increase the order of accuracy of standard numerical integration by two orders. They use the method of undermined coefficients to set up nonlinear equations for parameters, which are solved approximately by using a computer algebra system. Burg has proposed derivative-based closed Newton-Cotes numerical quadrature [10], which uses the function values on uniformly spaced intervals and 2 derivative values at the endpoints. The precision of the method in [10] is higher than the standard closed Newton-Cotes quadrature.

The motivation for this research lies in construction of midpoint derivative-based closed Newton-Cotes numerical quadrature rule for Newton-Cotes quadrature which uses the derivative value at the midpoint only. These new schemes are given in Section 2. In Section 3, the error terms are presented. In Section 4, compared with the Newton-Cotes quadrature, computational costs of these methods and run time on a given processor are presented, where the minimum number of subinterval to achieve the same level is calculated along with the number of function and derivative evaluations. The numerical experiments results are shown in Section 5. Finally, conclusions are drawn in Section 6.

2. Midpoint Derivative-Based Closed Newton-Cotes Quadrature

In this section, by adding the high derivative at the midpoint, schemes with higher precision than the Newton-Cotes quadrature rules are presented.

Theorem 1. *Midpoint derivative-based closed Trapezoidal rule ($n = 1$) is*

$$\int_a^b f(x)\,dx \approx T = \frac{b-a}{2}\left(f(a) + f(b)\right)$$
$$- \frac{(b-a)^3}{12}f''\left(\frac{a+b}{2}\right). \quad (8)$$

The precision of this method is 3.

Proof. Since the Trapezoidal rule has degree of precision 1, the formula (8) at least has 1 precision degree. Now, we just need to verify that the quadrature formula (8) is exact for $f(x) = x^2$, x^3.

When $f(x) = x^2$, $\int_a^b x^2\,dx = (1/3)(b^3 - a^3)$; $T = ((b-a)/2)(a^2 + b^2) - 2(b-a)^3/12 = (1/3)(b^3 - a^3)$.

When $f(x) = x^3$, $\int_a^b x^3 dx = (1/4)(b^4 - a^4)$;

$$T = \frac{b-a}{2}\left(a^3 + b^3\right) - \frac{6(b-a)^3}{12}\frac{a+b}{2}$$

$$= \frac{1}{4}\left(b^2 - a^2\right)\left(2b^2 + 2a^2 - 2ab - b^2 - a^2 + 2ab\right) \quad (9)$$

$$= \frac{1}{4}\left(b^2 - a^2\right)\left(b^2 + a^2\right) = \frac{1}{4}\left(b^4 - a^4\right).$$

So the precision of midpoint derivative-based closed Trapezoidal rule is 3. □

Theorem 2. *Midpoint derivative-based closed Simpson's rule* $(n = 2)$ *is*

$$\int_a^b f(x)\,dx \approx S = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$- \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{a+b}{2}\right). \quad (10)$$

The precision of this method is 5.

Proof. Since the Simpson's rule has degree of precision 3, the formula (10) at least has 3 precision degree. Now, we just need to verify that the quadrature formula (10) is exact for $f(x) = x^4$, x^5.

When $f(x) = x^4$, $\int_a^b x^4 dx = (1/5)(b^5 - a^5)$;

$$S = \frac{b-a}{6}\left[a^4 + 4\left(\frac{a+b}{2}\right)^4 + b^4\right] - \frac{24(b-a)^5}{2880}$$

$$= \frac{b-a}{120}\left[20a^4 + 5(a+b)^4 + 20b^4 - (b-a)^4\right] \quad (11)$$

$$= \frac{1}{5}(b-a)\left(b^4 + b^3 a + b^2 a^2 + ba^3 + a^4\right)$$

$$= \frac{1}{5}\left(b^5 - a^5\right).$$

When $f(x) = x^5$, $\int_a^b x^5 dx = (1/6)(b^6 - a^6)$;

$$S = \frac{b-a}{6}\left[a^5 + 4\left(\frac{a+b}{2}\right)^5 + b^5\right]$$

$$- \frac{60(b-a)^5(a+b)}{2880}$$

$$= \frac{b-a}{48}\left[8a^5 + (a+b)^5 + 8b^5 - (a+b)(b-a)^4\right]$$

$$= \frac{1}{6}(b-a)\left(b^5 + b^4 a + b^3 a^2 + b^2 a^3 + ba^4 + a^5\right)$$

$$= \frac{1}{6}\left(b^6 - a^6\right). \quad (12)$$

So the precision of midpoint derivative-based closed Simpson's rule is 5. □

Similarly, we obtain the midpoint derivative-based closed Simpson's 3/8 rule and Bool's rule.

Theorem 3. *Midpoint derivative-based closed Simpson's 3/8 rule* $(n = 3)$ *is*

$$\int_a^b f(x)\,dx \approx S_{3/8}$$

$$= \frac{(b-a)}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$

$$- \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{a+b}{2}\right). \quad (13)$$

The precision of this method is 5.

Proof. Since the Simpson's 3/8 rule has degree of precision 3, the formula (13) at least has 3 precision degree. Similarly, we just need to verify that the quadrature formula (13) is exact for $f(x) = x^4$, x^5.

When $f(x) = x^4$, $\int_a^b x^4 dx = (1/5)(b^5 - a^5)$;

$$S = \frac{b-a}{8}\left[a^4 + 3\left(\frac{2a+b}{3}\right)^4 + 3\left(\frac{a+2b}{3}\right)^4 + b^4\right]$$

$$- \frac{24(b-a)^5}{6480}$$

$$= \frac{b-a}{216}\left[27a^4 + (2a+b)^4 + (a+2b)^4 + 27b^4\right]$$

$$- \frac{(b-a)^5}{270} \quad (14)$$

$$= \frac{b-a}{1080}\left[135a^4 + 5(2a+b)^4 + 5(a+2b)^4\right.$$

$$\left. + 135b^4 - 4(b-a)^4\right]$$

$$= \frac{216}{1080}(b-a)\left(b^4 + b^3 a + b^2 a^2 + ba^3 + a^4\right)$$

$$= \frac{1}{5}\left(b^5 - a^5\right).$$

When $f(x) = x^5$, $\int_a^b x^5 dx = (1/6)(b^6 - a^6)$;

$$S = \frac{b-a}{8}\left[a^4 + 3\left(\frac{2a+b}{3}\right)^4 + 3\left(\frac{a+2b}{3}\right)^4 + b^4\right]$$
$$- \frac{60(b-a)^5(a+b)}{6480}$$
$$= \frac{b-a}{216}\left[27a^5 + (2a+b)^4 + (a+2b)^4 + 27b^5\right]$$
$$- \frac{(b-a)^5(a+b)}{108}$$
$$= \frac{b-a}{216}\left[27a^5 + (2a+b)^4 + (a+2b)^4 + 27b^5\right.$$
$$\left. - 2(a+b)(b-a)^4\right]$$
$$= \frac{36(b-a)}{216}\left(b^5 + b^4a + b^3a^2 + b^2a^3 + ba^4 + a^5\right)$$
$$= \frac{1}{6}\left(b^6 - a^6\right).$$

$\square$

Theorem 4. *Midpoint derivative-based closed Bool's rule ($n = 4$) is*

$$\int_a^b f(x)\,dx$$
$$\approx B = \frac{(b-a)}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right)\right.$$
$$\left. + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right]$$
$$- \frac{(b-a)^7}{1935360}f^{(6)}\left(\frac{a+b}{2}\right).$$

(16)

The precision of this method is 7.

Proof. Since the Bool's rule has degree of precision 5, the formula (16) at least has 5 precision degree. We just need to verify that the quadrature formula (16) is exact for $f(x) = x^6$, x^7 in like manner.

When $f(x) = x^6$, $\int_a^b x^6 dx = (1/7)(b^7 - a^7)$;

$$S = \frac{b-a}{90}\left[7a^6 + 32\left(\frac{3a+b}{4}\right)^6 + 12\left(\frac{a+b}{2}\right)^6\right.$$
$$\left. + 32\left(\frac{a+3b}{4}\right)^6 + 7b^6\right] - \frac{720(b-a)^7}{1935360}$$

$$= \frac{b-a}{11520}\left[896a^6 + (3a+b)^6 + 24(a+b)^6\right.$$
$$\left. + (a+3b)^6 + 896b^6\right] - \frac{(b-a)^7}{2688}$$
$$= \frac{b-a}{80640}\left[6272a^6 + 7(3a+b)^6 + 168(a+b)^6\right.$$
$$\left. + 7(a+3b)^6 + 6272b^6 - 30(b-a)^6\right]$$
$$= \frac{11520}{80640}(b-a)$$
$$\times\left(b^6 + b^5a + b^4a^2 + b^3a^3 + b^2a^4 + ba^5 + a^6\right)$$
$$= \frac{1}{7}\left(b^7 - a^7\right).$$

(17)

When $f(x) = x^7$, $\int_a^b x^7 dx = (1/8)(b^8 - a^8)$;

$$S = \frac{b-a}{90}\left[7a^7 + 32\left(\frac{3a+b}{4}\right)^7 + 12\left(\frac{a+b}{2}\right)^7\right.$$
$$\left. + 32\left(\frac{a+3b}{4}\right)^7 + 7b^7\right] - \frac{5040(b-a)^7(a+b)}{1935360}$$
$$= \frac{b-a}{46080}\left[3584a^7 + (3a+b)^7 + 48(a+b)^7\right.$$
$$\left. + (a+3b)^7 + 3584b^7\right] - \frac{(b-a)^7(a+b)}{768}$$
$$= \frac{b-a}{46080}\left[3584a^7 + (3a+b)^7 + 48(a+b)^7\right.$$
$$\left. + (a+3b)^7 + 3584b^7 - 60(b-a)^6(a+b)\right]$$
$$= \frac{5760(b-a)}{46080}$$
$$\times\left(b^7 + b^6a + b^5a^2 + b^4a^3 + b^3a^4 + b^2a^5 + ba^6 + a^7\right)$$
$$= \frac{1}{8}\left(b^8 - a^8\right).$$

(18)

$\square$

3. The Error Terms of Midpoint Derivative-Based Closed Newton-Cotes Quadrature

In this section, the error terms of midpoint derivative-based closed Newton-Cotes quadrature are given. The error term can be given in mainly 3 different ways [5, 10]. Here, we use the concept of precision to calculate the error term, where the error term is related to the difference between the quadrature formula for the monomial $x^{p+1}/(p+1)!$ and the exact value $(1/(p+1)!)\int_a^b x^{p+1}dx = (b^{p+2} - a^{p+2})/(p+2)!$, where p is the precision of the quadrature formula.

Theorem 5. *Midpoint derivative-based closed Trapezoidal rule ($n = 1$) with the error term is*

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left(f(a) + f(b)\right) - \frac{(b-a)^3}{12} f''\left(\frac{a+b}{2}\right)$$
$$- \frac{(b-a)^5}{480} f^{(4)}(\xi),$$

$$(19)$$

where $\xi \in (a,b)$. Thus, this scheme is fifth order accurate with the error term $R_1[f] = -((b-a)^5/480)f^{(4)}(\xi)$, and the associate composite method is fourth order.

Proof. Let $f(x) = x^4/4!$. So $(1/4!)\int_a^b x^4 dx = (1/120)(b^5 - a^5)$,

$$\frac{b-a}{2}\left(f(a) + f(b)\right) - \frac{(b-a)^3}{12} f''\left(\frac{a+b}{2}\right)$$
$$= \frac{b-a}{2}\left(\frac{a^4}{4!} + \frac{b^4}{4!}\right) - \frac{(b-a)^3}{24}\left(\frac{a+b}{2}\right)^2 \quad (20)$$
$$= \frac{b-a}{96}\left(b^2 + a^2\right)^2.$$

Therefore,

$$\frac{1}{120}\left(b^5 - a^5\right) - \frac{b-a}{96}\left(b^2 + a^2\right)^2$$
$$= \frac{(b-a)}{480}\left(-b^4 + 4ab^3 - 6a^2b^2 + 4a^3b - a^4\right) \quad (21)$$
$$= -\frac{(b-a)}{480}(b-a)^4 = -\frac{(b-a)^5}{480}.$$

This implies that

$$R_1[f] = -\frac{(b-a)^5}{480} f^{(4)}(\xi). \quad (22)$$

$\square$

Theorem 6. *Midpoint derivative-based closed Simpson's rule ($n = 2$) with the error term is*

$$\int_a^b f(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$
$$- \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{a+b}{2}\right) \quad (23)$$
$$- \frac{(b-a)^7}{241920} f^{(6)}(\xi),$$

where $\xi \in (a,b)$. Thus, this scheme is seventh order accurate with the error term $R_2[f] = -((b-a)^7/241920)f^{(6)}(\xi)$, and the associate composite method is sixth order.

Proof. Let $f(x) = x^6/6!$. So $(1/6!)\int_a^b x^6 dx = (1/5040)(b^7 - a^7)$,

$$\frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}\left(\frac{a+b}{2}\right)$$
$$= \frac{b-a}{6}\left(\frac{a^6}{6!} + 4\frac{(a+b)^6}{2^6 \cdot 6!} + \frac{b^6}{6!}\right) - \frac{(b-a)^5}{5760}\left(\frac{a+b}{2}\right)^2$$
$$= \frac{(b-a)}{34560}$$
$$\times \left(7b^6 + 6ab^5 + 9a^2b^5 + 4a^3b^3 + 9a^4b^2 + 6a^5b + 7a^6\right).$$

$$(24)$$

Therefore,

$$\frac{1}{5040}\left(b^7 - a^7\right) - \frac{(b-a)}{34560}$$
$$\times \left(7b^6 + 6ab^5 + 9a^2b^5 + 4a^3b^3 + 9a^4b^2 + 6a^5b + 7a^6\right)$$
$$= \frac{48(b-a)}{241920}$$
$$\times \left(b^6 + b^5a + b^4a^2 + b^3a^3 + b^2a^4 + ba^5 + a^6\right)$$
$$- \frac{7(b-a)}{241920}$$
$$\times \left(7b^6 + 6ab^5 + 9a^2b^5 + 4a^3b^3 + 9a^4b^2 + 6a^5b + 7a^6\right)$$
$$= \frac{(b-a)}{241920}$$
$$\times \left(-b^6 + 6b^5a - 15b^4a^2 + 20b^3a^3 - 15b^2a^4 + 6ba^5 - a^6\right)$$
$$= -\frac{(b-a)^7}{241920}.$$

$$(25)$$

This implies that

$$R_2[f] = -\frac{(b-a)^7}{241920} f^{(6)}(\xi). \quad (26)$$

$\square$

Theorem 7. *Midpoint derivative-based closed Simpson's 3/8 rule ($n = 3$) with the error term is*

$$\int_a^b f(x)\,dx$$
$$= \frac{(b-a)}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$
$$- \frac{(b-a)^5}{6480} f^{(4)}\left(\frac{a+b}{2}\right) - \frac{23(b-a)^7}{9797760} f^{(6)}(\xi),$$

$$(27)$$

where $\xi \in (a, b)$. It has a seventh order leading order error term and is sixth order accurate in its composite form. And the error term of this method is $R_3[f] = -(23(b-a)^7/9797760)f^{(6)}(\xi)$.

Proof. Let $f(x) = x^6/6!$. So $(1/6!) \int_a^b x^6 dx = (1/5040)(b^7 - a^7)$,

$$\frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]$$

$$- \frac{(b-a)^5}{6480}f^{(4)}\left(\frac{a+b}{2}\right)$$

$$= \frac{b-a}{8}\left(\frac{a^6}{6!} + 3\frac{(2a+b)^6}{3^6 \cdot 6!} + 3\frac{(a+2b)^6}{3^6 \cdot 6!} + \frac{b^6}{6!}\right)$$

$$- \frac{(b-a)^5}{12960}\left(\frac{a+b}{2}\right)^2 \tag{28}$$

$$= \frac{(b-a)}{9797760}\left(1967b^6 + 1806ab^5 + 2289a^2b^5\right.$$

$$+ 1484a^3b^3 + 2289a^4b^2$$

$$\left. + 1806a^5b + 1967a^6\right).$$

Therefore,

$$\frac{1}{5040}\left(b^7 - a^7\right) - \frac{(b-a)}{9797760}$$

$$\times \left(1967b^6 + 1806ab^5 + 2289a^2b^5 + 1484a^3b^3\right.$$

$$\left. + 2289a^4b^2 + 1806a^5b + 1967a^6\right)$$

$$= \frac{1944(b-a)}{9797760}$$

$$\times \left(b^6 + b^5a + b^4a^2 + b^3a^3 + b^2a^4 + ba^5 + a^6\right)$$

$$- \frac{(b-a)}{9797760}\left(1967b^6 + 1806ab^5 + 2289a^2b^5 + 1484a^3b^3\right.$$

$$\left. + 2289a^4b^2 + 1806a^5b + 1967a^6\right)$$

$$= \frac{23(b-a)}{9797760}\left(-b^6 + 6b^5a - 15b^4a^2 + 20b^3a^3\right.$$

$$\left. - 15b^2a^4 + 6ba^5 - a^6\right)$$

$$= -\frac{23(b-a)^7}{9797760}. \tag{29}$$

This implies that

$$R_3[f] = -\frac{23(b-a)^7}{9797760}f^{(6)}(\xi). \tag{30}$$

$\square$

Theorem 8. *Midpoint derivative-based closed Bool's rule ($n = 4$) with the error term is*

$$\int_a^b f(x)\,dx$$

$$= \frac{(b-a)}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right)\right.$$

$$\left. + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] \tag{31}$$

$$- \frac{(b-a)^7}{1935360}f^{(6)}\left(\frac{a+b}{2}\right) - \frac{17(b-a)^9}{45 \cdot 2^{11} \cdot 8!}f^{(8)}(\xi),$$

where $\xi \in (a, b)$. It has a ninth order leading order error term and is eighth order accurate in its composite form. And the error term of this method is $R_4[f] = -(17(b-a)^9/45\cdot2^{11}\cdot8!)f^{(8)}(\xi)$.

Proof. Similarly, let $f(x) = x^8/8!$. So $(1/8!) \int_a^b x^8 dx = (1/362880)(b^9 - a^9)$,

$$\frac{(b-a)}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right)\right.$$

$$\left. + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right]$$

$$- \frac{(b-a)^7}{1935360}f^{(6)}\left(\frac{a+b}{2}\right)$$

$$= \frac{b-a}{90}\left(\frac{7a^8}{8!} + 32\frac{(3a+b)^8}{4^8 \cdot 8!} + 12\frac{(a+b)^8}{2^8 \cdot 8!}\right.$$

$$\left. + 32\frac{(a+3b)^8}{4^8 \cdot 8!} + \frac{7b^8}{8!}\right)$$

$$- \frac{(b-a)^7}{3870720}\left(\frac{a+b}{2}\right)^2$$

$$= \frac{(b-a)}{45 \cdot 2^{11} \cdot 8!} \left(10257b^8 + 10104ab^7 + 10716a^2b^6 \right.$$

$$+ 9288a^3b^5 + 11430a^4b^4 + 9288a^5b^3$$

$$\left. + 10716a^6b^2 + 10104a^7b + 10257a^8 \right).$$

(32)

Therefore,

$$\frac{1}{362880} \left(b^9 - a^9 \right) - \frac{(b-a)}{45 \cdot 2^{11} \cdot 8!}$$

$$\times \left(10257b^8 + 10104ab^7 + 10716a^2b^6 + 9288a^3b^5 \right.$$

$$+ 11430a^4b^4 + 9288a^5b^3 + 10716a^6b^2$$

$$\left. + 10104a^7b + 10257a^8 \right)$$

$$= \frac{10240(b-a)}{45 \cdot 2^{11} \cdot 8!} \left(b^8 + b^7a + b^6a^2 + b^5a^3 + b^4a^4 \right.$$

$$\left. + b^3a^5 + b^2a^6 + ba^7 + a^8 \right)$$

$$- \frac{(b-a)}{45 \cdot 2^{11} \cdot 8!} \left(10257b^8 + 10104ab^7 + 10716a^2b^6 \right.$$

$$+ 9288a^3b^5 + 11430a^4b^4 + 9288a^5b^3$$

$$\left. + 10716a^6b^2 + 10104a^7b + 10257a^8 \right)$$

$$= \frac{17(b-a)}{45 \cdot 2^{11} \cdot 8!} \left(-b^8 + 8b^7a - 28b^6a^2 + 56b^5a^3 - 70b^4a^4 \right.$$

$$\left. + 56b^3a^5 - 28b^2a^6 + 8ba^7 - a^6 \right)$$

$$= -\frac{17(b-a)^9}{45 \cdot 2^{11} \cdot 8!}.$$

(33)

This implies that

$$R_4[f] = -\frac{17(b-a)^9}{45 \cdot 2^{11} \cdot 8!} f^{(8)}(\xi).$$

(34)

□

Precision, the orders and the error terms for midpoint derivative-based closed Newton-Cotes quadrature are summarized in Table 1.

4. Computational Efficiency in Composite Form

In this section, in order to compare the computational efficiency of the closed Newton-Cotes and the midpoint derivative-based quadrature formula, the number of calculations required by each quadrature formula to obtain a certain

TABLE 1: Precision, the orders and the error terms for midpoint derivative-based closed Newton-Cotes quadrature.

n	Precision	Order	Error terms
Trapezoidal rule ($n = 1$)	3	5	$-\dfrac{(b-a)^5}{480} f^{(4)}(\xi)$
Simpson's rule ($n = 2$)	5	7	$-\dfrac{(b-a)^7}{241920} f^{(6)}(\xi)$
Simpson's 3/8 rule ($n = 3$)	5	7	$-\dfrac{23(b-a)^7}{9797760} f^{(6)}(\xi)$
Bool's rule ($n = 4$)	7	9	$-\dfrac{17(b-a)^9}{45 \cdot 2^{11} \cdot 8!} f^{(8)}(\xi)$

level of accuracy of 10^{-10} and 10^{-5} is calculated for the following integrals $\int_0^1 (dx/(1+x))$ and $\int_0^2 e^x dx$, respectively.

In Tables 2 and 3, the number of function and derivative evaluations for the various quadrature formula presented for $\int_0^1 (dx/(1+x))$ and $\int_0^2 e^x dx$ are listed, respectively, using Matlab 6.5.

Take $\int_0^1 (dx/(1+x))$ as an example, for the composite Trapezoidal rule, 25002 function evaluations are required, and the computing time is 0.125 seconds; while for the composite midpoint derivative Trapezoidal rule, 106 function evaluations and 105 second derivative evaluations are required (total = 211), and the computing time is 0.031 seconds on the same processor. So the midpoint derivative Trapezoidal rule is less time-consuming than Trapezoidal rule when they obtain the same level of accuracy.

In order to compare the different methods with the same computational cost, the numerical experiments between Trapezoidal rule and Midpoint derivative Trapezoidal rule are performed. We choose the following two integrals $\int_0^2 e^x dx$ and $\int_0^1 (\ln(1+x)/(1+x^2))dx$ as examples and compare the CPU time for when they reach the same level of accuracy of 10^{-3}, 10^{-6}, and 10^{-9}. The comparative experimental results are shown in Tables 4 and 5.

5. Numerical Results

So far, we have proposed midpoint derivative-based closed Newton-Cotes quadrature in Section 2 and demonstrate the results that the proposed methods use fewer evaluations in Section 4.

In this section, many numerical experiments are carried out to determine whether the novel methods are of high precision. In order to compare the precision of Newton-Cotes quadrature and the midpoint derivative-based closed Newton-Cotes quadrature, we calculate the following integrals: $\int_0^1 (4dx/(1+x^2))$, $\int_0^2 e^x dx$. The comparison results are shown in Tables 6, 7, 8, 9, 10, 11, 12, and 13.

In Tables 6, 7, 8, 9, 10, 11, 12, and 13, the item Int. stands for the number of composite intervals.

Let us define Error = |Exact value − Approximate value|.

TABLE 2: Computational cost to estimate $\int_0^1 (dx/(1+x))$.

Formula	Order	Subintervals	Func. eval	Mid. deriv	Total
Trapezoidal rule	2	25001	25002	0	25002
Simpson's rule	4	67	135	0	135
Simpson's 3/8 rule	4	55	166	0	166
Bool's rule	6	10	41	0	41
Mid. deriv. Trapezoidal rule	4	105	106	105	211
Mid. deriv. Simpson's rule	6	14	29	14	43
Mid. deriv. Simpson's 3/8 rule	6	12	37	12	49
Mid. deriv. Bool's rule	8	5	21	5	26

TABLE 3: Computational cost to estimate $\int_0^2 e^x dx$.

Formula	Order	Subintervals	Func. eval	Mid. deriv	Total
Trapezoidal rule	2	462	463	0	463
Simpson's rule	4	8	17	0	17
Simpson's 3/8 rule	4	7	22	0	22
Bool's rule	6	2	9	0	9
Mid. deriv. Trapezoidal rule	4	13	14	13	27
Mid. deriv. Simpson's rule	6	3	7	3	10
Mid. deriv. Simpson's 3/8 rule	6	3	10	3	13
Mid. deriv. Bool's rule	8	1	5	1	6

TABLE 4: CPU time for $\int_0^2 e^x dx$.

Level of accuracy	Trapezoidal rule	Midpoint derivative Trapezoidal rule
10^{-3}	0.015 s	0 s
10^{-6}	0.016 s	0 s
10^{-9}	0.031 s	0.015 s

TABLE 5: CPU time for $\int_0^1 (\ln(1+x)/(1+x^2))dx$.

Level of accuracy	Trapezoidal rule	Midpoint derivative Trapezoidal rule
10^{-3}	0.016 s	0.015 s
10^{-6}	0.016 s	0.015 s
10^{-9}	0.047 s	0.032 s

It can be seen from Tables 6–13 that midpoint derivative-based closed Newton-Cotes quadrature formulas have a much higher accuracy than classical closed Newton-Cotes quadrature formulas.

6. Conclusion

We briefly summarize our main conclusions in this paper as follows.

(1) A family of numerical integration formulas of closed Newton-Cotes quadrature rules is presented, which uses the derivative value at the midpoint.

(2) It is proved that these kinds of quadrature rules obtain an increase of two orders of precision over the classical closed Newton-Cotes formula, and the error terms are given.

(3) The computational cost for these methods is analyzed for several examples. And it has shown that the proposed formulas are superior computationally to the same order closed Newton-Cotes formulas when they reduce the error below the same level.

(4) Finally, some numerical examples are given to show the efficiency of the proposed approach.

Dehghan's technique may be applied for midpoint derivative-based closed Newton-Cotes quadrature and how to accelerate the convergence of the quadrature formulas by using Richardson extrapolation algorithm will be achieved by further research.

Acknowledgment

TABLE 6: Exact value of $\int_0^1 (4dx/(1+x^2)) = \pi \approx 3.1415926536$.

Int.	Trapezoidal rule		Midpoint derivative Trapezoidal rule	
	Approximate value	Error	Approximate value	Error
1	3	0.1415926536	3.0853333333	0.0562593203
2	3.1	0.0415926536	3.1414302104	0.0001624432
4	3.1311764706	0.0104161830	3.1415916562	0.0000009974

TABLE 7

Int.	Simpson's rule		Midpoint derivative Simpson's rule	
	Approximate value	Error	Approximate value	Error
1	3.1333333333	0.0082593203	3.1463040000	0.0047113464
2	3.1415686275	0.0000240261	3.1416054730	0.0000128194
4	3.1415925024	0.0000001512	3.1415927140	0.0000000604

TABLE 8

Int.	Simpson's 3/8 rule		Midpoint derivative Simpson's 3/8 rule	
	Approximate value	Error	Approximate value	Error
1	3.1384615385	0.0031311151	3.1442262792	0.0026336256
2	3.1415834498	0.0000092038	3.1415998256	0.0000071720
4	3.1415925939	0.0000000596	3.1415926879	0.0000000343

TABLE 9

Int.	Bool's rule		Midpoint derivative Bool's rule	
	Approximate value	Error	Approximate value	Error
1	3.1421176471	0.0005249935	3.1414398566	0.0001527970
2	3.1415940941	0.0000014405	3.1415922411	0.0000004125
4	3.1415926611	0.0000000075	3.1415926536	0.0000000000

TABLE 10: Exact value of $\int_0^2 e^x dx = e^2 - 1 \approx 6.3890560989$.

Int.	Trapezoidal rule		Midpoint derivative Trapezoidal rule	
	Approximate value	Error	Approximate value	Error
1	8.3890560989	2.0000000000	6.5768682133	0.1878121144
2	6.9128098779	0.5237537790	6.4019423495	0.0128862506
4	6.5216101094	0.1325540105	6.3898812442	0.0008251453

TABLE 11

Int.	Simpson's rule		Midpoint derivative Simpson's rule	
	Approximate value	Error	Approximate value	Error
1	6.4207278043	0.0316717054	6.3905246728	0.0014685739
2	6.3912101867	0.0021540878	6.3890815720	0.0000254731
4	6.3891937254	0.0001376265	6.3890565078	0.0000004089

TABLE 12

Int.	Simpson's 3/8 rule		Midpoint derivative Simpson's 3/8 rule	
	Approximate value	Error	Approximate value	Error
1	6.4033154765	0.0142593776	6.3898918626	0.0008357637
2	6.3900166237	0.0009605248	6.3890705727	0.0000144738
4	6.3891173168	0.0000612179	6.3890563312	0.0000002323

TABLE 13

Int.	Bool's rule		Midpoint derivative Bool's rule	
	Approximate value	Error	Approximate value	Error
1	6.3892423455	0.0001862466	6.3890628650	0.0000067661
2	6.3890592947	0.0000031958	6.3890561271	0.0000000282
4	6.3890561500	0.0000000511	6.3890560990	0.0000000001

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