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Research Article

Fixed Point Theorems of Quasicontractions on Cone Metric Spaces with Banach Algebras

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We introduce the concept of quasicontractions on cone metric spaces with Banach algebras, and by a new method of proof, we will prove the existence and uniqueness of fixed points of such mappings. The main result generalizes the well-known theorem of Ćirić (Ćirić 1974).

1. Introduction

Let (X, d) be a complete metric space. Recall that a mapping $T : X \to X$ is called a quasicontraction if, for some $k \in (0, 1)$ and for all $x, y \in X$, one has

$$\begin{aligned} d(Tx, Ty) & \\ &\leqslant k \max \{d(x, y),\ d(x, Tx),\ d(y, Ty), \\ &\qquad d(x, Ty),\ d(y, Tx)\}. \end{aligned} \tag{1}$$

Ćirić [1] introduced and studied quasicontractions as one of the most general classes of contractive-type mappings. He proved the well-known theorem that any quasicontraction T has a unique fixed point. Recently, scholars obtained various similar results on cone metric spaces. See, for instance, [2–5].

In this paper, we study the quasicontractions on metric spaces with Banach algebras, which are introduced in [6] and turn out to be an interesting generalization of classic metric spaces. By a new method of proof, we generalize Ćirić theorem.

Let A always be a real Banach algebra with a multiplication unit e; that is, $ex = xe = x$ for all $x \in A$. An element $x \in A$ is said to be invertible if there is an inverse element $y \in A$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [7].

The following proposition is well known (see [7]).

Proposition 1 (see [7]). *Let A be a Banach algebra with a unit e, and let $x \in A$. If the spectral radius $\rho(x)$ of x is less than 1, that is,*

$$\rho(x) = \lim_{n \to \infty} \left\| x^n \right\|^{1/n} = \inf_{n \geqslant 1} \left\| x^n \right\|^{1/n} < 1, \tag{2}$$

then $e - x$ is invertible. Actually,

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i. \tag{3}$$

A subset P of A is called a cone if

(1) P is nonempty closed and $\{0, e\} \subset P$;

(2) $\alpha P + \beta P \subset P$ for all nonnegative real numbers α, β;

(3) $P^2 = PP \subset P$;

(4) $P \cap (-P) = \{0\}$.

For a given cone $P \subset A$, we can define a partial ordering $\leqslant$ with respect to P by $x \leqslant y$ if and only if $y - x \in P$. And $x \lneqq y$ will stand for $x \leqslant y$ and $x \neq y$, while $x < y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

Remark 2. In the literature on cone metric spaces, authors use $x < y$ to mean $x \leqslant y$ and $x \neq y$ and $x \ll y$ to mean $y - x \in \operatorname{int} P$. To our knowledge, and from a topological point of view, the order relation $y - x \in \operatorname{int} P$ plays a very similar role in cone metric spaces as $x < y$ does in $\mathbb{R}$.

The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in A$,

$$0 \leqslant x \leqslant y \Longrightarrow \|x\| \leqslant M \|y\|. \tag{4}$$

The least positive number satisfying above is called the normal constant of P (see [8]).

In the following, we always assume that P is a cone in A with $\operatorname{int} P \neq \emptyset$ and $\leqslant$ is partial ordering with respect to P.

Definition 3 (see [8]). Let X be a nonempty set. Suppose the mapping $d : X \times X \to A$ satisfies

(1) $0 \leqslant d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \leqslant d(x, z) + d(z, x)$ for all $x, y, z \in X$.

Then, d is called a cone metric on X, and (X, d) is called a cone metric space (with Banach algebra A).

For more details about cone metric spaces with Banach algebras, we refer the readers to [6].

Definition 4 (see [8]). Let (X, d) be a cone metric space, and let $x \in X$ and $\{x_n\}$ be a sequence in X. Then,

(1) $\{x_n\}$ converges to x whenever for each $c \in A$ with $0 < c$ there is a natural number N such that $d(x_n, x) < c$ for all $n \geqslant N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$;

(2) $\{x_n\}$ is a Cauchy sequence whenever for each $c \in A$ with $0 < c$ there is a natural number N such that $d(x_n, x_m) < c$ for all $n, m \geqslant N$;

(3) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

The following facts are often used.

Proposition 5 (see [8]). *Let (X, d) be a cone metric space, let P be a normal cone with normal constant M, and let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ $(n \to \infty)$.*

Proposition 6 (see [8]). *Let (X, d) be a cone metric space, let P be a normal cone with normal constant M, and let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.*

2. Main Results

In this section we will define quasicontractions in the setting of cone metric spaces with Banach algebras and prove the fixed point theorem of such mappings.

Definition 7. Let (X, d) be a cone metric space with Banach algebra A. A mapping $T : X \to X$ is called a quasicontraction if for some $k \in P$ with $\rho(k) < 1$ and for all $x, y \in X$, one has

$$d(Tx, Ty) \leqslant ku, \tag{5}$$

where

$$u \in \{d(x, y),\ d(x, Tx),\ d(y, Ty),\ d(x, Ty),\ d(y, Tx)\}. \tag{6}$$

Remark 8. In Definition 7, we only suppose the spectral radius of k is less than 1, while neither $k < e$ nor $\| k \| < 1$ is assumed. In fact, the condition $\rho(k) < 1$ is weaker than that $\| k \| < 1$. See the example in [6].

Theorem 9. *Let (X, d) be a complete cone metric space with a Banach algebra A, and let P be a normal cone with normal constant M. If the mapping $T : X \to X$ is a quasicontraction, then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.*

In the rest of the paper, we choose $x_0 \in X$ and denote $x_n = T^n x_0$. For the sake of clarity, we divide the proof into several steps.

Lemma 10. *Assume that the hypotheses in Theorem 9 are satisfied. Then, for each $n \geqslant 1$, and for all i, j such that $1 \leqslant i, j \leqslant n$, one has*

$$d(x_i, x_j) \leqslant k(e - k)^{-1} d(x_0, x_1). \tag{7}$$

Proof. We present the proof by induction.

When $n = 1$, which implies $i = j = 1$, the conclusion is trivial.

Assume that the statement is true for $n = m$; that is,

$$d(x_i, x_j) \leqslant k(e - k)^{-1} d(x_0, x_1), \quad \text{for } 1 \leqslant i, j \leqslant m. \tag{8}$$

Now, we will prove that the statement is true for $n = m + 1$. Note that in this case, if $1 \leqslant i, j \leqslant m$, then the statement is just (8). Thus, without loss of generality, we suppose that $j = m + 1$ and $1 \leqslant i \leqslant m$ and denote $i = i_0$.

By the definition of quasicontraction, we have

$$d(x_{i_0}, x_{m+1}) \leqslant ku, \tag{9}$$

where

$$\begin{aligned} u \in \{&d(x_{i_0-1}, x_m),\ d(x_{i_0-1}, x_{i_0}),\ d(x_m, x_{m+1}), \\ &d(x_{i_0-1}, x_{m+1}),\ d(x_{i_0}, x_m)\}. \end{aligned} \tag{10}$$

Firstly, we consider the case that $i_0 = 1$; that is,

$$\begin{aligned} u \in \{&d(x_0, x_m),\ d(x_0, x_1),\ d(x_m, x_{m+1}), \\ &d(x_0, x_{m+1}),\ d(x_1, x_m)\}. \end{aligned} \tag{11}$$

If $u = d(x_0, x_m)$, then

$$\begin{aligned} d\left(x_{i_0}, x_{m+1}\right) &\leqslant kd\left(x_0, x_m\right) \\ &\leqslant k\left(d\left(x_0, x_1\right) + d\left(x_1, x_m\right)\right) \\ &\leqslant k\left(d\left(x_0, x_1\right) + k(e-k)^{-1} d\left(x_0, x_1\right)\right) \\ &= k\left(e + k(e-k)^{-1}\right) d\left(x_0, x_1\right) \\ &= k\left(e + \sum_{t=1}^{\infty} k^t\right) d\left(x_0, x_1\right) \\ &= k(e-k)^{-1} d\left(x_0, x_1\right), \end{aligned} \tag{12}$$

and the statement follows.

If $u = d(x_0, x_1)$, then

$$\begin{aligned} d\left(x_{i_0}, x_{m+1}\right) &\leqslant kd\left(x_0, x_1\right) \\ &\leqslant \left(\sum_{t=1}^{\infty} k^t\right) d\left(x_0, x_1\right) \\ &= k(e-k)^{-1} d\left(x_0, x_1\right), \end{aligned} \tag{13}$$

and the statement also follows.

If $u = d(x_m, x_{m+1})$, then we set $i_1 = m$ and we have

$$d\left(x_{i_0}, x_{m+1}\right) \leqslant kd\left(x_{i_1}, x_{m+1}\right). \tag{14}$$

If $u = d(x_0, x_{m+1})$, then

$$\begin{aligned} d\left(x_{i_0}, x_{m+1}\right) &\leqslant kd\left(x_0, x_{m+1}\right) \\ &\leqslant k\left(d\left(x_0, x_1\right) + d\left(x_1, x_{m+1}\right)\right) \\ &= k\left(d\left(x_0, x_1\right) + d\left(x_{i_0}, x_{m+1}\right)\right), \end{aligned} \tag{15}$$

which implies

$$(e-k)\, d\left(x_{i_0}, x_{m+1}\right) \leqslant kd\left(x_0, x_1\right). \tag{16}$$

Note that $(e-k)^{-1} = \sum_{t=0}^{\infty} k^t \geqslant 0$ and that k and $(e-k)^{-1}$ commute. Multiplying both sides by $(e-k)^{-1}$, we have

$$d\left(x_{i_0}, x_{m+1}\right) \leqslant k(e-k)^{-1} d\left(x_0, x_1\right), \tag{17}$$

and the statement also follows.

If $u = d(x_{i_0}, x_m)$, then

$$\begin{aligned} d\left(x_{i_0}, x_{m+1}\right) &\leqslant kd\left(x_{i_0}, x_m\right) \\ &\leqslant k^2(e-k)^{-1} d\left(x_0, x_1\right) \\ &= \left(\sum_{t=2}^{\infty} k^t\right) d\left(x_0, x_1\right) \\ &\leqslant \left(\sum_{t=1}^{\infty} k^t\right) d\left(x_0, x_1\right) \\ &= k(e-k)^{-1} d\left(x_0, x_1\right), \end{aligned} \tag{18}$$

and the statement also follows.

Secondly, we consider the case that $2 \leqslant i_0 \leqslant m$.

If $u = d(x_{i_0-1}, x_m)$ or $u = d(x_{i_0-1}, x_{i_0})$ or $u = d(x_{i_0}, x_m)$, then, by (8), we have

$$\begin{aligned} d\left(x_{i_0}, x_{m+1}\right) &\leqslant ku \\ &\leqslant k^2(e-k)^{-1} d\left(x_0, x_1\right) \\ &= \left(\sum_{t=2}^{\infty} k^t\right) d\left(x_0, x_1\right) \\ &\leqslant k(e-k)^{-1} d\left(x_0, x_1\right), \end{aligned} \tag{19}$$

and the statement follows.

If $u = d(x_m, x_{m+1})$ or $u = d(x_{i_0-1}, x_{m+1})$, then we set $i_1 = m$ or $i_1 = i_0 - 1 \geqslant 1$, respectively. And we have

$$\begin{aligned} d\left(x_{i_0}, x_{m+1}\right) &\leqslant ku \\ &= kd\left(x_{i_1}, x_{m+1}\right). \end{aligned} \tag{20}$$

In conclusion from discussions of both cases, it results that either the proof is complete, that is,

$$d\left(x_{i_0}, x_{m+1}\right) \leqslant k(e-k)^{-1} d\left(x_0, x_1\right), \tag{21}$$

or there exists an integer i_1 such that

$$d\left(x_{i_0}, x_{m+1}\right) \leqslant kd\left(x_{i_1}, x_{m+1}\right), \quad 1 \leqslant i_1 \leqslant m. \tag{22}$$

As for the latter situation, we continue in a similar way, and come to the result that either

$$d\left(x_{i_i}, x_{m+1}\right) \leqslant k(e-k)^{-1} d\left(x_0, x_1\right), \tag{23}$$

which implies that

$$\begin{aligned} d\left(x_{i_0}, x_{m+1}\right) &\leqslant kd\left(x_{i_1}, x_{m+1}\right) \\ &\leqslant k^2(e-k)^{-1} d\left(x_0, x_1\right) \\ &\leqslant k(e-k)^{-1} d\left(x_0, x_1\right), \end{aligned} \tag{24}$$

and the proof is complete, or there exists an integer i_2 such that

$$d\left(x_{i_1}, x_{m+1}\right) \leqslant kd\left(x_{i_2}, x_{m+1}\right), \quad 1 \leqslant i_2 \leqslant m, \tag{25}$$

which implies that

$$d\left(x_{i_0}, x_{m+1}\right) \leqslant k^2 d\left(x_{i_2}, x_{m+1}\right), \quad 1 \leqslant i_2 \leqslant m. \tag{26}$$

Generally, if the procedure ends by the ℓ-th step with $\ell \leqslant m-1$, that is, there exist $\ell + 1$ integers

$$i_0,\ i_1, \ldots, i_\ell \in \{1, \ldots, m\}, \tag{27}$$

such that

$$\begin{aligned} d\left(x_{i_0}, x_{m+1}\right) &\leqslant kd\left(x_{i_1}, x_{m+1}\right) \\ &\leqslant \cdots \leqslant k^\ell d\left(x_{i_\ell}, x_{m+1}\right), \end{aligned} \tag{28}$$

and such that

$$d\left(x_{i_\ell}, x_{m+1}\right) \leqslant k(e-k)^{-1} d\left(x_0, x_1\right), \tag{29}$$

then

$$\begin{aligned} d\left(x_{i_0}, x_{m+1}\right) &\leqslant k^{\ell+1}(e-k)^{-1} d\left(x_0, x_1\right) \\ &= \left(\sum_{t=\ell+1}^{\infty} k^t\right) d\left(x_0, x_1\right) \\ &\leqslant k(e-k)^{-1} d\left(x_0, x_1\right). \end{aligned} \tag{30}$$

Hence, the proof is complete.

Finally, if the procedure continues more than m steps, then there exist $m+1$ integers

$$i_0,\ i_1, \ldots, i_m \in \{1, \ldots, m\}, \tag{31}$$

such that

$$\begin{aligned} d\left(x_{i_0}, x_{m+1}\right) &\leqslant k d\left(x_{i_1}, x_{m+1}\right) \\ &\leqslant \cdots \leqslant k^m d\left(x_{i_m}, x_{m+1}\right). \end{aligned} \tag{32}$$

Thus, there must exist two integers, p and q, say, such that

$$0 \leqslant p < q \leqslant m, \quad i_p = i_q. \tag{33}$$

From (32), one sees that

$$\begin{aligned} d\left(x_{i_p}, x_{i_{m+1}}\right) &\leqslant k^{q-p} d\left(x_{i_q}, x_{m+1}\right) \\ &= k^{q-p} d\left(x_{i_p}, x_{m+1}\right), \end{aligned} \tag{34}$$

and therefore

$$\left(e-k^{q-p}\right) d\left(x_{i_p}, x_{m+1}\right) \leqslant 0. \tag{35}$$

Note that

$$\rho\left(k^{q-p}\right) \leqslant \rho(k)^{q-p} < 1, \tag{36}$$

which implies $e-k^{q-p}$ is invertible. And since that

$$\left(e-k^{q-p}\right)^{-1} = \sum_{t=0}^{\infty} k^{(q-p)t} \geqslant 0, \tag{37}$$

we have

$$d\left(x_{i_p},\ x_{m+1}\right) \leqslant 0. \tag{38}$$

So,

$$d\left(x_{i_p},\ x_{m+1}\right) = 0, \tag{39}$$

$$\begin{aligned} d\left(x_{i_0}, x_{m+1}\right) &\leqslant k^p d\left(x_{i_p}, x_{m+1}\right) \\ &= 0 \\ &\leqslant k(e-k)^{-1} d\left(x_0, x_1\right) \end{aligned} \tag{40}$$

Therefore, by induction, the statement is proved. □

Remark 11. Lemma 10 simply says that

$$d\left(x_i, x_j\right) \leqslant k(e-k)^{-1} d\left(x_0, x_1\right), \quad \forall i,\ j \geqslant 1. \tag{41}$$

Lemma 12. *Assume that the hypotheses in Theorem 9 are satisfied. Then, $\{x_n\}$ is a Cauchy sequence.*

Proof. For $1 < m < n$, denote that

$$C(m, n) = \left\{d\left(x_i, x_j\right) \mid m \leqslant i, j \leqslant n\right\}. \tag{42}$$

By the definition of quasicontraction, it follows that, for each $u \in C(m, n)$, there exists $v \in C(m-1, n)$, such that

$$u \leqslant kv. \tag{43}$$

Consequently,

$$\begin{aligned} d\left(x_m, x_n\right) &\leqslant k u_1 \\ &\leqslant k^2 u_2 \\ &\leqslant \cdots \leqslant k^{m-1} u_{m-1} \\ &\leqslant k^m (e-k)^{-1} d\left(x_0, x_1\right), \end{aligned} \tag{44}$$

where

$$u_1 \in C(m-1, n), \tag{45}$$

$$u_2 \in C(m-2, n), \ldots, u_{m-1} \in C(1, n), \tag{46}$$

and the last inequality comes from Lemma 10.

By the normality of P, and noting that $\| k^m \| \to 0$ $(m \to \infty)$, we have

$$\begin{aligned} \left\|d\left(x_m, x_n\right)\right\| &\leqslant M \left\|k^m\right\| \left\|(e-k)^{-1}\right\| \\ &\quad \times \left\|d\left(x_0, x_1\right)\right\| \longrightarrow 0 \quad (n > m \longrightarrow \infty). \end{aligned} \tag{47}$$

The proof is complete. □

Now, we finish the remaining part of the proof of Theorem 9.

Proof. By Lemma 12 and the completeness of (X, d), there is $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$. Then,

$$\begin{aligned} d\left(x^*, Tx^*\right) &\leqslant d\left(x^*, x_n\right) + d\left(x_n, Tx^*\right) \\ &\leqslant d\left(x^*, x_n\right) + ku, \end{aligned} \tag{48}$$

where

$$\begin{aligned} u \in \{&d\left(x_{n-1}, x^*\right), d\left(x_{n-1}, x_n\right), d\left(x^*, Tx^*\right), \\ &d\left(x_{n-1}, Tx^*\right), d\left(x^*, x_n\right)\}. \end{aligned} \tag{49}$$

If $u = d(x_{n-1}, x^*)$ or $u = d(x_{n-1}, x_n)$ or $u = d(x^*, x_n)$, then $\| u \| \to 0$ $(n \to \infty)$. Hence,

$$\begin{aligned} \left\|d\left(x^*, Tx^*\right)\right\| &\leqslant M \left\|d\left(x^*, x_n\right)\right\| + \|k\| \|u\| \longrightarrow 0 \\ &\quad (n \longrightarrow \infty). \end{aligned} \tag{50}$$

If $u = d(x^*, Tx^*)$, then

$$(e-k)\, d(x^*, Tx^*) \leqslant d(x^*, x_n). \tag{51}$$

Hence,

$$\|d(x^*, Tx^*)\| \leqslant M \left\|(e-k)^{-1}\right\| \; \|d(x^*, x_n)\| \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{52}$$

If $u = d(x_{n-1}, Tx^*)$, then

$$\begin{aligned} d(x^*, Tx^*) &\leqslant d(x^*, x_n) + kd(x_{n-1}, Tx^*) \\ &\leqslant d(x^*, x_n) + kd(x_{n-1}, x^*) + kd(x^*, Tx^*). \end{aligned} \tag{53}$$

Hence,

$$\begin{aligned} \|d(x^*, Tx^*)\| &\leqslant M \left\|(e-k)^{-1}\right\| \\ &\times (\|d(x^*, x_n)\| + \|k\| \, \|d(x_{n-1}, x^*)\|) \longrightarrow 0, \end{aligned} \tag{54}$$

as $n \to \infty$.

In each case, we have $\| d(x^*, Tx^*) \| = 0$. Thus, $Tx^* = x^*$.
Now, if y^* is another fixed point, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leqslant ku, \tag{55}$$

where

$$\begin{aligned} u \in \{& d(x^*, y^*),\ d(x^*, Tx^*),\ d(y^*, Ty^*), \\ & d(x^*, Ty^*),\ d(y^*, Tx^*)\}. \end{aligned} \tag{56}$$

If $u = d(x^*, Tx^*) = d(y^*, Ty^*) = 0$, then $d(x^*, y^*) = 0$.
If $u = d(x^*, y^*) = d(x^*, Ty^*) = d(y^*, Tx^*)$, then

$$(e-k)\, d(x^*, y^*) \leqslant 0, \tag{57}$$

which implies

$$d(x^*, y^*) = 0. \tag{58}$$

Thus, the fixed point is unique. And we obtain Theorem 9. □

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