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Research Article

On the Period-Two Cycles of $x_{n+1} = (\alpha + \beta x_n + \gamma x_{n-k})/(A + Bx_n + Cx_{n-k})$

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We consider the higher order nonlinear rational difference equation $x_{n+1} = (\alpha+\beta x_n+\gamma x_{n-k})/(A+Bx_n+Cx_{n-k})$, $n = 0, 1, 2, \ldots$, where the parameters $\alpha, \beta, \gamma, A, B, C$ are positive real numbers and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are nonnegative real numbers, $k \in \{1, 2, \ldots\}$. We give a necessary and sufficient condition for the equation to have a prime period-two solution. We show that the period-two solution of the equation is locally asymptotically stable.

1. Introduction

Recently, dynamics of nonnegative solutions of higher order rational difference equation has been an area of intense interest. Related to this subject, researches are done by Dehghan et al. [1–4], Zayed [5–7], Huang and Knopf [8, 9], Karatas [10, 11], and others. For the general theory of difference equations, one can refer to the monographes of Kocić and Ladas [12], Elaydi [13], Agarwal [14], Kulenović and Ladas [15], and Camouzis and Ladas [16]. Other related results can be found in [17–24].

Our aim in this paper is to study the higher order nonlinear rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{1}$$

where the parameters $\alpha, \beta, \gamma, A, B, C$ are positive real numbers and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are nonnegative real numbers, $k \in \{1, 2, \ldots\}$. Our concentration is on the periodic character of all positive solutions of (1).

The periodic character of positive solutions of (1) for $k = 1$ has been investigated by the authors in [25]. They showed that the period-two solution of (1) for $k = 1$ is locally asymptotically stable if it exists.

Motivated by the above results, our interest is now to study and generalize the previous results to the general case depicted in (1).

The change of variable

$$x_n = \frac{\gamma}{C} y_n \tag{2}$$

reduces (1) to

$$y_{n+1} = \frac{r + py_n + y_{n-k}}{z + qy_n + y_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{3}$$

where

$$r = \frac{\alpha C}{\gamma^2}, \qquad p = \frac{\beta}{\gamma}, \qquad z = \frac{A}{\gamma}, \qquad q = \frac{B}{C} \tag{4}$$

are positive real numbers and the initial conditions $y_{-k}, \ldots, y_{-1}, y_0$ are nonnegative real numbers.

This paper is organized besides this introduction in three sections. In Section 2, we present some preliminaries and some results which can be mainly deduced from the general situation studied in [12–16, 26]. Our main results are presented in Section 3; we give a necessary and sufficient condition for the equation to have a prime period-two solution, in addition to providing a necessary and sufficient conditions for the prime period-two solution of the equation to be locally asymptotically stable. In order to illustrate the results of the previous section and to support our theoretical discussion, we consider several numerical examples in Section 4; we use MATLAB to see how the behaviors of (1) look like. Finally, we conclude in Section 5 with suggestions for future research.

2. Preliminaries

For the sake of self-containment and convenience, we recall the following definitions and results from [16].

Let I be some interval of real numbers and let

$$f : I^{k+1} \longrightarrow I \tag{5}$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, \ldots, x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f\left(x_n, x_{n-1}, \ldots, x_{n-k}\right), \quad n = 0, 1, 2, \ldots, \tag{6}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A solution of (6) that is constant for all $n \geq -k$ is called an equilibrium solution of (6). If

$$x_n = \overline{x} \quad \text{for } n \geq -k \tag{7}$$

is an equilibrium solution of (6), then $\overline{x}$ is called an equilibrium point or simply an equilibrium of (6).

Definition 1. (i) The equilibrium point $\overline{x}$ of (6) is called locally stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $x_{-k}, \ldots, x_{-1}, x_0 \in I$, and $|x_{-k} - \overline{x}| + \cdots, |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \delta$, we have

$$\left|x_n - \overline{x}\right| < \epsilon, \quad \forall n \geq -k. \tag{8}$$

(ii) The equilibrium point $\overline{x}$ of (6) is called locally asymptotically stable if it is locally stable, and if there exists $\gamma > 0$, if $x_{-k}, \ldots, x_{-1}, x_0 \in I$, and $|x_{-k} - \overline{x}| + \cdots, |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \delta$, we have

$$\lim_{n \to \infty} x_n = \overline{x}. \tag{9}$$

(iii) The equilibrium point $\overline{x}$ of (6) is called a global attractor if for every $x_{-k}, \ldots, x_{-1}, x_0 \in I$, we have

$$\lim_{n \to \infty} x_n = \overline{x}. \tag{10}$$

(iv) The equilibrium point $\overline{x}$ of (6) is called globally asymptotically stable if it is locally stable and a global attractor.

(v) The equilibrium point $\overline{x}$ of (6) is called unstable if it is not stable.

Definition 2. (i) A solution $\{x_n\}$ of (6) is said to be periodic with period P if

$$x_{n+P} = x_n, \quad \forall n \geq -k. \tag{11}$$

(ii) A solution $\{x_n\}$ of (6) is said to be periodic with prime period P if P is the least positive integer for which (11) holds.

Definition 3. Let

$$q_i = \frac{\partial f}{\partial u_i}\left(\overline{x}, \overline{x}, \ldots, \overline{x}\right) \tag{12}$$

denote the partial derivatives of $f(u_0, u_1, \ldots, u_k)$ evaluated at the equilibrium $\overline{x}$ of (6). Then the equation

$$y_{n+1} = q_0 y_n + q_1 y_{n-1} + \cdots + q_k y_{n-k}, \quad n = 0, 1, 2, \ldots, \tag{13}$$

is called the linearized equation associated with (6) about the equilibrium point $\overline{x}$. Its characteristic equation is

$$\lambda^{k+1} - q_0 \lambda^k - \cdots - q_{k-1}\lambda - q_k = 0. \tag{14}$$

Theorem 4 (linearized stability). *(a) If all roots of* (14) *lie in the open unit disk* $|\lambda| < 1$, *then the equilibrium* $\overline{x}$ *of* (6) *is locally asymptotically stable.*

(b) If at least one of the roots of (14) *has absolute value greater than one, then* $\overline{x}$ *is unstable.*

The following result from [26] will become handy in the sequel.

Lemma 5. *If*

$$F_n = \begin{vmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{vmatrix}, \tag{15}$$

then F_n *satisfies the following recursive formula:*

$$F_n = a_n F_{n-1} - b_{n-1} c_{n-1} F_{n-2}. \tag{16}$$

The aforementioned lemma leads to the following conclusion.

Corollary 6. *If*

$$D_n = \begin{vmatrix} 0 & 1 & & & \\ -\lambda & 0 & 1 & & \\ & -\lambda & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & -\lambda & 0 \end{vmatrix}, \tag{17}$$

then

$$D_n = \lambda D_{n-2} = \begin{cases} 0 & \text{if } n \text{ is odd}, \\ \lambda^{n/2} & \text{if } n \text{ is even}. \end{cases} \tag{18}$$

3. Main Result

In this section, we give a necessary and sufficient condition for (1) to have a prime period-two solution. We show that the period-two solution of (1) is locally asymptotically stable.

Equation (3) has a unique positive equilibrium given by

$$\overline{y} = \frac{1+p-z+\sqrt{(1+p-z)^2+4r(q+1)}}{2(q+1)}. \tag{19}$$

The linearized equation associated with (3) about the equilibrium is given by

$$z_{n+1} = \frac{p-q\overline{y}}{z+(q+1)\overline{y}} z_n + \frac{1-\overline{y}}{z+(q+1)\overline{y}} z_{n-k}, \tag{20}$$

and its characteristic equation is

$$\lambda^{k+1} - \frac{p-q\overline{y}}{z+(q+1)\overline{y}} \lambda^k - \frac{1-\overline{y}}{z+(q+1)\overline{y}} = 0. \tag{21}$$

Theorem 7. *(a) If*

$$p+z \geq 1, \tag{22}$$

then (3) *has no nonnegative prime period-two solution.*

(b) If

$$p+z < 1, \tag{23}$$

then (3) *has prime period-two solution*

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots \tag{24}$$

if and only if k is odd and

$$r < \frac{(1-p-z)[q(1-p-z)-(1+3p-z)]}{4}, \tag{25}$$

where the values of Φ *and* Ψ *are the positive and distinct solutions of the quadratic equation*

$$t^2 - (1-z-p)t + \frac{p(1-z-p)+r}{q-1} = 0, \quad q > 1. \tag{26}$$

Proof. Assume that there exist distinct nonnegative real numbers Φ and Ψ, such that

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots \tag{27}$$

is a prime period-two solution of (3); there are two cases to be considered.

Case 1 (k is even). In this case Φ and Ψ satisfy

$$\Phi = \frac{r+p\Psi+\Psi}{z+q\Psi+\Psi}, \qquad \Psi = \frac{r+p\Phi+\Phi}{z+q\Phi+\Phi}. \tag{28}$$

Furthermore,

$$z\Phi + (q+1)\Psi\Phi = r + (p+1)\Psi, \tag{29}$$

$$z\Psi + (q+1)\Phi\Psi = r + (p+1)\Phi. \tag{30}$$

Subtracting (30) from (29), we have

$$z(\Phi-\Psi) = (p+1)(\Psi-\Phi), \tag{31}$$

so

$$\begin{gathered}(\Phi-\Psi)(z+p+1) = 0 \\ \Longrightarrow \Phi = \Psi \text{ or } z+p = -1.\end{gathered} \tag{32}$$

This contradicts the hypothesis that Φ and Ψ are distinct nonnegative real numbers. Also, $z+p=-1$ contradicts the hypothesis that z and p are positive real numbers.

Case 2 (k is odd). (a) If

$$p+z \geq 1, \tag{33}$$

then in this case Φ and Ψ satisfy

$$\Phi = \frac{r+p\Psi+\Phi}{z+q\Psi+\Phi}, \qquad \Psi = \frac{r+p\Phi+\Psi}{z+q\Phi+\Psi}. \tag{34}$$

Furthermore,

$$\Phi(z+q\Psi+\Phi) = r+p\Psi+\Phi, \tag{35}$$

$$\Psi(z+q\Phi+\Psi) = r+p\Phi+\Psi. \tag{36}$$

Subtracting (35) from (36), we have

$$(\Phi+\Psi) = (1-z-p). \tag{37}$$

But, $p+z \geq 1$, this implies that $\Phi+\Psi \leq 0$ which contradicts the hypothesis that Φ, Ψ are distinct positive real numbers.

(b) If

$$p+z < 1, \tag{38}$$

then in this case Φ and Ψ satisfy

$$\Phi = \frac{r+p\Psi+\Phi}{z+q\Psi+\Phi}, \tag{39}$$

$$\Psi = \frac{r+p\Phi+\Psi}{z+q\Phi+\Psi}. \tag{40}$$

Moreover,

$$\Phi(z+q\Psi+\Phi) = r+p\Psi+\Phi, \tag{41}$$

$$\Psi(z+q\Phi+\Psi) = r+p\Phi+\Psi. \tag{42}$$

Subtracting (41) from (42), we have

$$(\Phi+\Psi) = (1-z-p). \tag{43}$$

Furthermore, one adds (41) to (42), makes use of (43), and then does some elementary algebraic manipulation; we have

$$\Phi\Psi = \frac{p(1-z-p)+r}{q-1}. \tag{44}$$

Equation (44) leads to the following conclusion:

$$q > 1, \tag{45}$$

that follows from the facts that

$$\Phi\Psi > 0, \qquad 1-p-z > 0. \tag{46}$$

Notice that when $q = 1$ then adding (41) to (42) gives $2r + 2p(1-p-z) = 0$, which is impossible.

Construct the quadratic equation

$$t^2 - (1-z-p)\,t + \frac{p(1-z-p)+r}{q-1} = 0, \quad q > 1. \tag{47}$$

So Φ and Ψ are the positive and distinct solutions of the above quadratic equation, that is,

$$t = \frac{(1-z-p) \pm \sqrt{(1-z-p)^2 - 4\left((p(1-z-p)+r)/(q-1)\right)}}{2}, \quad q > 1. \tag{48}$$

□

Theorem 8. *Suppose (3) has a prime period-two solution. Then, the period-two solution is locally asymptotically stable.*

Proof. To investigate the local stability of the two cycles

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots \tag{49}$$

we first vectorize (3) by introducing the following change of variables:

$$\begin{aligned} z_1(n) &= y_{n-k}, \\ z_2(n) &= y_{n-k+1}, \\ z_3(n) &= y_{n-k+2}, \\ &\vdots \\ z_k(n) &= y_{n-1}, \\ z_{k+1}(n) &= y_n, \end{aligned} \tag{50}$$

and write (3) in the equivalent form:

$$\begin{pmatrix} z_1(n+1) \\ z_2(n+1) \\ \vdots \\ z_k(n+1) \\ z_{k+1}(n+1) \end{pmatrix} = T \begin{pmatrix} z_1(n) \\ z_2(n) \\ \vdots \\ z_k(n) \\ z_{k+1}(n) \end{pmatrix}, \quad n = 0, 1, 2, \ldots, \tag{51}$$

where

$$T \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \\ z_{k+1} \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \vdots \\ z_{k+1} \\ \dfrac{r + pz_{k+1} + z_1}{z + qz_{k+1} + z_1} \end{pmatrix}. \tag{52}$$

Now Φ and Ψ generate a period-two solution of (3) only if

$$\begin{pmatrix} \Phi \\ \Psi \\ \vdots \\ \Phi \\ \Psi \end{pmatrix} \tag{53}$$

is a fixed point of T^2, the second iterate of T. Furthermore,

$$T^2 \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \\ z_{k+1} \end{pmatrix} = \begin{pmatrix} f_1(z_1, z_2, \ldots, z_{k+1}) \\ f_2(z_1, z_2, \ldots, z_{k+1}) \\ \vdots \\ f_k(z_1, z_2, \ldots, z_{k+1}) \\ f_{k+1}(z_1, z_2, \ldots, z_{k+1}) \end{pmatrix}, \tag{54}$$

where

$$\begin{aligned} f_1(z_1, z_2, \ldots, z_{k+1}) &= z_3, \\ f_2(z_1, z_2, \ldots, z_{k+1}) &= z_4, \\ &\vdots \\ f_k(z_1, z_2, \ldots, z_{k+1}) &= \frac{r + pz_{k+1} + z_1}{z + qz_{k+1} + z_1} \\ f_{k+1}(z_1, z_2, \ldots, z_{k+1}) &= \frac{r + p\left((r + pz_{k+1} + z_1)/(z + qz_{k+1} + z_1)\right) + z_2}{z + q\left((r + pz_{k+1} + z_1)/(z + qz_{k+1} + z_1)\right) + z_2}. \end{aligned} \tag{55}$$

The prime period-two solution of (3) is asymptotically stable if the eigenvalues of the Jacobian matrix J_{T^2}, evaluated at $\begin{pmatrix} \Phi \\ \Psi \\ \vdots \\ \Phi \\ \Psi \end{pmatrix}$ lie inside the unit disk.

We have

$$J_{T^2}\begin{pmatrix}\Phi\\ \Psi\\ \vdots\\ \Phi\\ \Psi\end{pmatrix}=\begin{pmatrix}0 & 0 & 1\ 0\ .\ .\ . & 0\\ 0 & 0 & 0\ 1\ .\ .\ . & 0\\ 0 & 0 & 0\ 0\ 1\ .\ . & 0\\ . & . & .\ .\ .\ .\ . & .\\ . & . & .\ .\ .\ .\ . & .\\ 0 & 0 & 0\ 0\ .\ .\ . & 1\\ \dfrac{1-\Phi}{z+q\Psi+\Phi} & 0 & 0\ 0\ .\ .\ . & \dfrac{p-q\Phi}{z+q\Psi+\Phi}\\ \dfrac{(p-q\Psi)(1-\Phi)}{(z+q\Psi+\Phi)(z+q\Phi+\Psi)} & \dfrac{1-\Psi}{z+q\Phi+\Psi} & 0\ 0\ .\ .\ . & \dfrac{(p-q\Phi)(p-q\Psi)}{(z+q\Psi+\Phi)(z+q\Phi+\Psi)}\end{pmatrix}. \tag{56}$$

Now let $P(\lambda)=\det(J_{T^2}-\lambda I)$ be the characteristic polynomial of J_{T^2}. Then, by the Laplace expansion in the $(k+1)$ row,

$$\begin{aligned}
P(\lambda) &= \begin{vmatrix}-\lambda & 0 & 1 & 0 & \cdots & \cdots & 0\\ 0 & -\lambda & 0 & 1 & \cdots & \cdots & 0\\ 0 & 0 & -\lambda & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & 0 & -\lambda & \cdots & 1\\ \dfrac{1-\Phi}{z+q\Psi+\Phi} & 0 & 0 & 0 & \cdots & -\lambda & \dfrac{p-q\Phi}{z+q\Psi+\Phi}\\ \dfrac{(p-q\Psi)(1-\Phi)}{(z+q\Psi+\Phi)(z+q\Phi+\Psi)} & \dfrac{1-\Psi}{z+q\Phi+\Psi} & 0 & 0 & 0 & 0 & \dfrac{(p-q\Psi)(p-q\Phi)}{(z+q\Psi+\Phi)(z+q\Phi+\Psi)}-\lambda\end{vmatrix}\\
&= -\frac{(p-q\Psi)(1-\Phi)}{(z+q\Psi+\Phi)(z+q\Phi+\Psi)}\underbrace{\begin{vmatrix}0 & 1 & 0 & \cdots & \cdots & 0\\ -\lambda & 0 & 1 & \cdots & \cdots & 0\\ 0 & -\lambda & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & -\lambda & \cdots & 1\\ 0 & 0 & 0 & \cdots & -\lambda & \dfrac{p-q\Phi}{z+q\Psi+\Phi}\end{vmatrix}}_{A_k}\\
&\quad+\frac{1-\Psi}{z+q\Phi+\Psi}\underbrace{\begin{vmatrix}-\lambda & 1 & 0 & \cdots & \cdots & 0\\ 0 & 0 & 1 & \cdots & \cdots & 0\\ 0 & -\lambda & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & -\lambda & \cdots & 1\\ \dfrac{1-\Phi}{z+q\Psi+\Phi} & 0 & 0 & \cdots & -\lambda & \dfrac{p-q\Phi}{z+q\Psi+\Phi}\end{vmatrix}}_{B_k}\\
&\quad+\left(\frac{(p-q\Psi)(p-q\Phi)}{(z+q\Psi+\Phi)(z+q\Phi+\Psi)}-\lambda\right)\underbrace{\begin{vmatrix}-\lambda & 0 & 1 & 0 & \cdots & \cdots & 0\\ 0 & -\lambda & 0 & 1 & \cdots & \cdots & 0\\ 0 & 0 & -\lambda & 0 & 1 & \cdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 1\\ 0 & 0 & 0 & 0 & -\lambda & \cdots & 0\\ \dfrac{1-\Phi}{z+q\Psi+\Phi} & 0 & 0 & 0 & \cdots & \cdots & -\lambda\end{vmatrix}}_{C_k}.
\end{aligned} \tag{57}$$

However; by Lemma 5, Corollary 6, and the fact that k is odd,

$$\begin{aligned}A_k &= \frac{p-q\Phi}{z+q\Psi+\Phi}D_{k-1}+\lambda D_{k-2}=\left(\frac{p-q\Phi}{z+q\Psi+\Phi}\right)\lambda^{(k-1)/2},\\
B_k &= -\lambda A_{k-1}+\left(\frac{1-\Phi}{z+q\Psi+\Phi}\right)\\
&= -\lambda\left[\frac{p-q\Phi}{z+q\Psi+\Phi}D_{k-2}+\lambda D_{k-3}\right]+\left(\frac{1-\Phi}{z+q\Psi+\Phi}\right)\\
&= -\lambda^{(k+1)/2}+\left(\frac{1-\Phi}{z+q\Psi+\Phi}\right),\\
C_k &= (-\lambda)^k+\left(\frac{1-\Phi}{z+q\Psi+\Phi}\right)D_{k-1}\\
&= -\lambda^k+\left(\frac{1-\Phi}{z+q\Psi+\Phi}\right)\lambda^{(k-1)/2}.\end{aligned} \tag{58}$$

Therefore,

$$\begin{aligned}P(\lambda) = &-\frac{(p-q\Psi)(1-\Phi)}{(z+q\Psi+\Phi)(z+q\Phi+\Psi)}\\
&\times\left(\frac{p-q\Phi}{z+q\Psi+\Phi}\right)\lambda^{(k-1)/2}\\
&+\left(\frac{1-\Psi}{z+q\Phi+\Psi}\right)\left[-\lambda^{(k+1)/2}+\left(\frac{1-\Phi}{z+q\Psi+\Phi}\right)\right]\\
&+\left(\frac{(p-q\Psi)(p-q\Phi)}{(z+q\Psi+\Phi)(z+q\Phi+\Psi)}-\lambda\right)\\
&\times\left[-\lambda^k+\left(\frac{1-\Phi}{z+q\Psi+\Phi}\right)\lambda^{(k-1)/2}\right]\\
= {}&\lambda^{k+1}-\frac{(p-q\Psi)(p-q\Phi)}{(z+q\Psi+\Phi)(z+q\Phi+\Psi)}\lambda^k\\
&-\left(\frac{1-\Psi}{z+q\Phi+\Psi}+\frac{1-\Phi}{z+q\Psi+\Phi}\right)\lambda^{(k+1)/2}\\
&+\frac{(1-\Psi)(1-\Phi)}{(z+q\Phi+\Psi)(z+q\Psi+\Phi)}.\end{aligned} \tag{59}$$

Hence, the characteristic polynomial is given by

$$f(\lambda)=\lambda^{k+1}-Q\lambda^k-L\lambda^{(k+1)/2}+\mu=0, \tag{60}$$

where

$$\begin{aligned}Q &= \frac{(p-q\Phi)(p-q\Psi)}{(z+q\Phi+\Psi)(z+q\Psi+\Phi)},\\
L &= \underbrace{\frac{1-\Phi}{z+q\Psi+\Phi}}_{a}+\underbrace{\frac{1-\Psi}{z+q\Phi+\Psi}}_{b},\\
\mu &= \frac{(1-\Phi)(1-\Psi)}{(z+q\Psi+\Phi)(z+q\Phi+\Psi)}=ab.\end{aligned} \tag{61}$$

Assume that $0<\Phi<\Psi$. Then, by (39),

$$1=\frac{(r/\Phi)+p(\Psi/\Phi)+1}{z+q\Psi+\Phi}>\frac{1}{z+q\Psi+\Phi}. \tag{62}$$

Hence,

$$z+q\Psi+\Phi>1. \tag{63}$$

Similarly, we observe that

$$z+q\Phi+\Psi>1. \tag{64}$$

Furthermore, since $p+z<1$, (43) implies the sum of Φ, Ψ is less than 1 and, a fortiori, each is less than 1. Indeed, we have

$$0<\Phi<\min\left\{\Psi,\frac{1}{2}\right\}<1. \tag{65}$$

With that in mind, it is clear that

$$0<a,\ b<1. \tag{66}$$

In addition, with understanding that

$$(\Phi+\Psi)=(1-z-p)>0, \qquad \Phi\Psi=\frac{p(1-z-p)+r}{q-1} \tag{67}$$

and the fact that

$$q>1, \tag{68}$$

we have to establish

$$Q>0, \tag{69}$$

$$Q+L<1+\mu. \tag{70}$$

First, we will establish inequality (69). To this end, observe that inequality (69) is equivalent to

$$\frac{(p-q\Phi)(p-q\Psi)}{(z+q\Phi+\Psi)(z+q\Psi+\Phi)}>0, \tag{71}$$

which is true if and only if

$$\frac{p^2-pq(\Phi+\Psi)+q^2\Phi\Psi}{(z+q\Phi+\Psi)(z+q\Psi+\Phi)}>0, \tag{72}$$

which is true if and only if

$$\frac{p^2-pq(1-p-z)+q^2((p(1-p-z)+r)/(q-1))}{(z+q\Phi+\Psi)(z+q\Psi+\Phi)}>0, \tag{73}$$

which is true if and only if

$$\frac{p^2(q-1)-pq(q-1)(1-p-z)+pq^2(1-p-z)+rq^2}{(q-1)(z+q\Phi+\Psi)(z+q\Psi+\Phi)}>0, \tag{74}$$

which is true if and only if

$$\frac{p^2(q-1) - pq(1-p-z)[q-1-q] + rq^2}{(q-1)(z+q\Phi+\Psi)(z+q\Psi+\Phi)} > 0, \quad (75)$$

which is true if and only if

$$\frac{p^2(q-1) + pq(1-p-z) + rq^2}{(q-1)(z+q\Phi+\Psi)(z+q\Psi+\Phi)} > 0, \quad (76)$$

which is clearly satisfied.

Next we will establish inequality (70). Observe that inequality (70) is equivalent to

$$\begin{aligned}&\frac{1-\Phi}{z+q\Psi+\Phi} + \frac{1-\Psi}{z+q\Phi+\Psi}\\&\quad + \frac{(p-q\Phi)(p-q\Psi) - (1-\Phi)(1-\Psi)}{(z+q\Phi+\Psi)(z+q\Psi+\Phi)} < 1,\end{aligned} \quad (77)$$

which is true if and only if

$$\begin{aligned}&\frac{1-\Phi}{z+q\Psi+\Phi} + \frac{1-\Psi}{z+q\Phi+\Psi}\\&\quad + \frac{p^2-1+(q^2-1)\Phi\Psi + (1-pq)(\Phi+\Psi)}{(z+q\Phi+\Psi)(z+q\Psi+\Phi)} < 1,\end{aligned} \quad (78)$$

which is true if and only if

$$\begin{aligned}&\frac{1-\Phi}{z+q\Psi+\Phi} + \frac{1-\Psi}{z+q\Phi+\Psi}\\&\quad + \big(p^2 - 1 + (q-1)(q+1)\\&\qquad \times ((p(1-p-z)+r)/(q-1))\\&\qquad + (1-pq)(1-p-z)\big)\\&\quad \times ((z+q\Phi+\Psi)(z+q\Psi+\Phi))^{-1} < 1,\end{aligned} \quad (79)$$

which is true if and only if

$$\begin{aligned}&\frac{1-\Phi}{z+q\Psi+\Phi} + \frac{1-\Psi}{z+q\Phi+\Psi}\\&\quad + \frac{p^2-1+r(q+1)+(1-p-z)(p+1)}{(z+q\Phi+\Psi)(z+q\Psi+\Phi)} < 1,\end{aligned} \quad (80)$$

which is true if and only if

$$\begin{aligned}&\frac{1-\Phi}{z+q\Psi+\Phi} + \frac{1-\Psi}{z+q\Phi+\Psi}\\&\quad + \frac{r(q+1)-z(p+1)}{(z+q\Phi+\Psi)(z+q\Psi+\Phi)} < 1,\end{aligned} \quad (81)$$

which is true if and only if

$$\begin{aligned}&(1-\Phi)(z+q\Phi+\Psi)\\&\quad + (1-\Psi)(z+q\Psi+\Phi)\\&\quad + r(q+1) - z(p+1)\\&\quad < (z+q\Psi+\Phi)(z+q\Phi+\Psi).\end{aligned} \quad (82)$$

Now observe that the righthand side of (82) is

$$\begin{aligned}I &= (z+q\Psi+\Phi)(z+q\Phi+\Psi)\\&= z^2 + z(q+1)(\Psi+\Phi)\\&\quad + q(\Psi^2+\Phi^2) + (1+q^2)\Psi\Phi\\&= z^2 + z(q+1)(\Psi+\Phi)\\&\quad + q((\Psi+\Phi)^2 - 2\Psi\Phi) + (1+q^2)\Psi\Phi\\&= z^2 + z(q+1)(\Psi+\Phi)\\&\quad + q(\Psi+\Phi)^2 + (q-1)^2\Psi\Phi\\&= z^2 + z(q+1)(1-p-z)\\&\quad + q(1-p-z)^2 + (q-1)^2\left(\frac{p(1-p-z)+r}{q-1}\right)\\&= z^2 + r(q-1) + (1-p-z)\\&\quad \times (z(q+1) + q(1-p-z) + p(q-1))\\&= z^2 + r(q-1) + (1-p-z)(q-p+z)\\&= r(q-1) - z(p-1) - (p-q)(1-p-z)\\&= r(q-1) + z(1-q) + q(1-p) + p(1-p)\\&= (z-r)(1-q) + (p-q)(p-1)\\&= z - r - qz + qr + p^2 - p - qp + q.\end{aligned} \quad (83)$$

The lefthand side of (82) is

$$\begin{aligned}II &= (1-\Phi)(z+q\Phi+\Psi)\\&\quad + (1-\Psi)(z+q\Psi+\Phi) + r(q+1) - z(p+1)\\&= 2z + q(\Phi+\Psi) + (\Phi+\Psi) - z(\Phi+\Psi)\\&\quad - q(\Psi^2+\Phi^2) - 2\Phi\Psi + r(q+1) - z(p+1)\\&= 2z + (q+1-z)(\Phi+\Psi) - q(\Psi^2+\Phi^2)\\&\quad - 2\Phi\Psi + r(q+1) - z(p+1)\\&= 2z + (q+1-z)(\Phi+\Psi) - q\left[(\Psi+\Phi)^2 - 2\Phi\Psi\right]\\&\quad - 2\Phi\Psi + r(q+1) - z(p+1)\\&= 2z + (q+1-z)(\Phi+\Psi) - q(\Psi+\Phi)^2\\&\quad + 2(q-1)\Phi\Psi + r(q+1) - z(p+1)\\&= 2z + (q+1-z)(\Phi+\Psi) - q(\Psi+\Phi)^2\\&\quad + 2(q-1)\left(\frac{p(1-p-z)+r}{q-1}\right) + r(q+1) - z(p+1)\end{aligned}$$

$$\begin{aligned}
&= 2z + 2r + (\Phi + \Psi)\left[1 - z + q - q(\Psi + \Phi) + 2p\right] \\
&\quad + r(q+1) - z(p+1) \\
&= 2z + 2r + (1 - p - z)(1 - z + q - q(1 - p - z) + 2p) \\
&\quad + r(q+1) - z(p+1) \\
&= 2z + 2r + (1 - p - z)(1 - z + qp + qz + 2p) \\
&\quad + r(q+1) - z(p+1) \\
&= 3r + 1 + p - z + rq + pq + zq - 2pz \\
&\quad - 2pzq + z^2 - qz^2 - qp^2 - 2p^2 - 2pz.
\end{aligned} \tag{84}$$

Hence, inequality (70) is true if and only if

$$\begin{aligned}
&3r + 1 + p - z + rq + pq + zq - 2pz \\
&\qquad - 2pzq + z^2 - qz^2 - qp^2 - 2p^2 - 2pz \\
&\qquad < z - r - qz + qr + p^2 - p - qp + q
\end{aligned} \tag{85}$$

or equivalently

$$\begin{aligned}
&4r < q - qp - qz - 1 - 3p + z - pq + qp^2 + qzp + p \\
&\qquad + 3p^2 - zp - zq + qzp + qz^2 + z + 3pz - z^2 \\
&\Longleftrightarrow 4r < (1 - p - z)\left[q - qp - qz - 1 - 3p + z\right] \\
&\Longleftrightarrow 4r < (1 - p - z)\left[q(1 - p - z) - (1 + 3p - z)\right] \\
&\Longleftrightarrow r < \frac{(1 - p - z)\left[q(1 - p - z) - (1 + 3p - z)\right]}{4},
\end{aligned} \tag{86}$$

which is clearly satisfied (condition (25)).

Now, by applying Theorem 4 we shall show that the zeros of f in (60) lie in the open unit disk $|\lambda| < 1$. To do so, suppose to the contrary that f has a zero λ such that $|\lambda| \geq 1$. Then, by the triangle inequality,

$$\begin{aligned}
&\left(|\lambda|^{(k+1)/2} - a\right)\left(|\lambda|^{(k+1)/2} - b\right) \\
&\qquad \leq \left|\left(\lambda^{(k+1)/2} - a\right)\left(\lambda^{(k+1)/2} - b\right)\right| = Q|\lambda|^k.
\end{aligned} \tag{87}$$

Thus,

$$f(|\lambda|) = |\lambda|^{k+1} - Q|\lambda|^k - L|\lambda|^{(k+1)/2} + \mu \leq 0. \tag{88}$$

However, by the Descartes' Rule of Signs f has either two or no positive zeros. Furthermore,

$$\begin{aligned}
f(0) &= \mu > 0, \\
f\left(\sqrt[(k+1)/2]{a}\right) &< 0, \\
f(1) &= 1 + \mu - Q - L > 0,
\end{aligned} \tag{89}$$

and so, by the Intermediate Value Theorem, $f(x)$ has two positive zeros in the open interval $(0, 1)$. Moreover, since $f(1) > 0$, we conclude that $f(x) > 0$ for all $x \geq 1$ which contradicts inequality (88).

The proof is complete. □

Remark 9. The characteristic equation of the linearized equation at the equilibrium solution is given by

$$\lambda^{k+1} - \frac{p - q\overline{y}}{z + (q+1)\overline{y}}\lambda^k - \frac{1 - \overline{y}}{z + (q+1)\overline{y}} = 0. \tag{90}$$

Since the magnitude of the constant term is less than 1, the equation has at least one root inside the unit disk. As such, by the Stable Manifold Theorem, there is a manifold of solutions, of dimension bigger than or equal to 1, that converge to the equilibrium solution. Hence, the period-two solution cannot be globally asymptotically stable.

4. Numerical Examples

In order to illustrate the results of the previous section and to support our theoretical discussion, we consider several numerical examples generated by MATLAB.

Case 1 (k is even). For this case we consider the following example:

$$y_{n+1} = \frac{0.0001 + 0.4y_n + y_{n-2}}{0.5 + 20y_n + y_{n-2}}. \tag{91}$$

The dynamics of (91) is shown in Figure 1, no prime period-two solution.

Case 2 (k is odd). There are two cases to be considered.

Subcase 2.1 $(p + z \geq 1)$. For this case we consider the following example:

$$y_{n+1} = \frac{0.01 + 0.5y_n + y_{n-3}}{0.7 + 2y_n + y_{n-3}}. \tag{92}$$

The dynamics of (92) is shown in Figure 2, no prime period-two solution.

Subcase 2.2 $(p + z < 1$ *and* $r < ((1-p-z)[q(1-p-z)-(1+3p-z)]/4))$. For this case we consider the following example:

$$y_{n+1} = \frac{0.01 + 0.4y_n + y_{n-3}}{0.3 + 10y_n + y_{n-3}}. \tag{93}$$

The dynamics of (93) is shown in Figure 3; it has prime period-two solution.

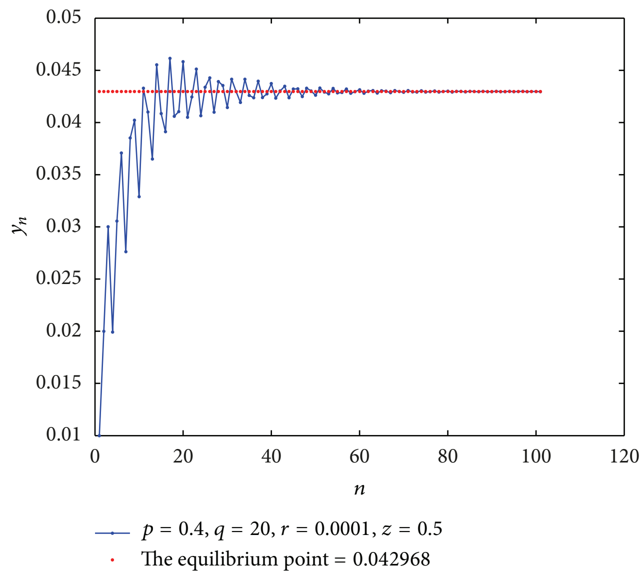

FIGURE 1: Dynamics of $y_{n+1} = (0.0001 + 0.4y_n + y_{n-2})/(0.5 + 20y_n + y_{n-2})$.

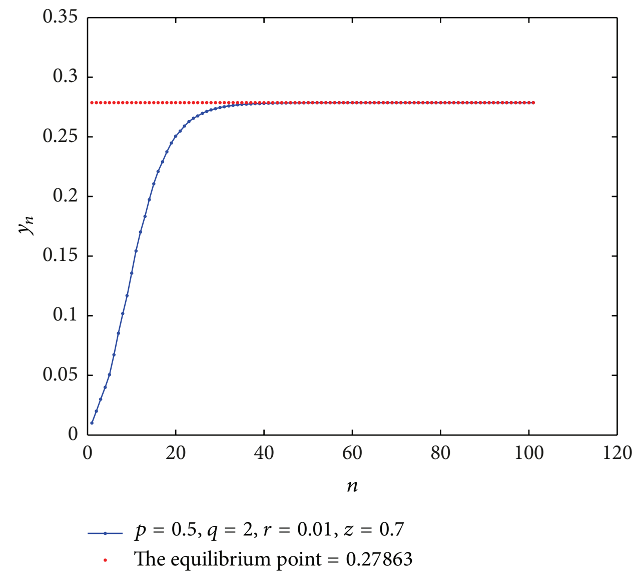

FIGURE 2: Dynamics of $y_{n+1} = (0.01+0.5y_n+y_{n-3})/(0.7+2y_n+y_{n-3})$.

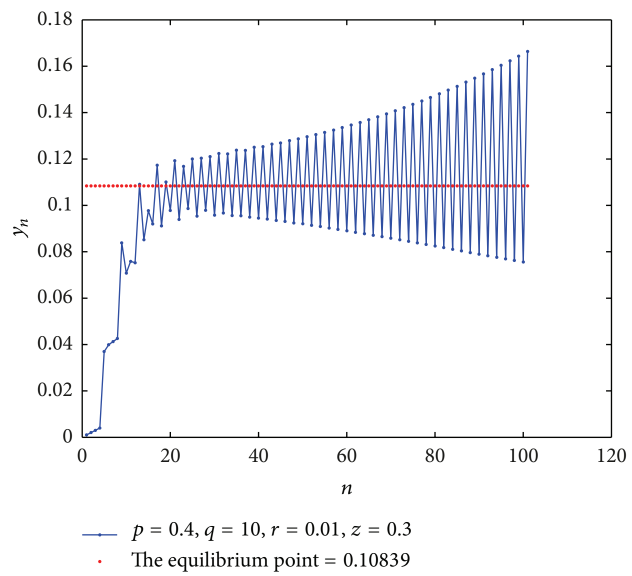

FIGURE 3: Dynamics of $y_{n+1} = (0.01+0.4y_n+y_{n-3})/(0.3+10y_n+y_{n-3})$.

5. Conclusion

In this paper, we showed that the period-two solution of the higher order nonlinear rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{A + Bx_n + Cx_{n-k}}, \quad n = 0, 1, 2, \ldots, \tag{94}$$

where the parameters $\alpha, \beta, \gamma, A, B, C$ are positive real numbers and the initial conditions $x_{-k}, \ldots, x_{-1}, x_0$ are nonnegative real numbers, $k \in \{1, 2, \ldots\}$, is locally asymptotically stable if it exists.

We consider the aforementioned result as a step forward in investigating bigger classes of difference equations which afford the ELAS property; that is, the existence of a periodic solution implies its local asymptotic stability.

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