Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2009, Article ID 696971, 12 pages
doi:10.1155/2009/696971

Research Article

Some $\ell(p)$-Type New Sequence Spaces and Their Geometric Properties

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Received 17 March 2009; Accepted 4 August 2009

Recommended by Agacik Zafer

We introduce an $\ell(p)$-type new sequence space and investigate its some topological properties including AK and AD properties. Besides, we examine some geometric properties of this space concerning Banach-Saks type p and Gurarii's modulus of convexity.

1. Introduction

In general, the $\ell(p)$-type spaces have many useful applications because of the properties of the spaces $\ell(p)$ and ℓ_p. In [1], it was shown that the subspaces of Orlicz spaces, which have rich geometric properties, are isomorphic to the space ℓ_p. Also since the space ℓ_p is reflexive and convex, it is natural to consider the geometric structure of these spaces.

Recently there has been a lot of interest in investigating geometric properties of sequence spaces besides topological and some other usual properties. In literature, there are many papers concerning the geometric properties of different sequence spaces. For example; in [2], Mursaleen et al. studied some geometric properties of normed Euler sequence space. Sanhan and Suantai [3] investigated the geometric properties of Cesáro sequence space ces(p) equipped with Luxemburg norm. Further information on geometric properties of sequence spaces can be found in [4–7].

The main purpose of our work is to introduce an ℓ_p-type new sequence space together with matrix domain and its summability methods. Also we investigate some topological properties of this new space as the paranorm, AK and AD properties, and furthermore characterize geometric properties concerning Banach-Saks type p and Gurarii's modulus of convexity.

2. Preliminaries and Notations

Let w be the space of all real-valued sequences. Each linear subspace of w is called a sequence space denoted by λ. We denote by ℓ_1 and ℓ_p absolutely and p-absolutely convergent series, respectively.

A sequence space λ with a linear topology is called a K-space provided that each of the maps $p_i : \lambda \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field, and $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. A K-space λ is called FK-space provided that λ is a complete linear metric space. An FK-space whom topology is normable is called BK-space. An FK-space λ is said to have AK property, if $\phi \subset \lambda$ and $\{e^{(k)}\}$ is a basis for λ, where $e^{(k)}$ is a sequence whose only nonzero term is 1, kth place for each $k \in \mathbb{N}$, and $\phi = \operatorname{span}\{e^{(k)}\}$, the set of all AD-space, thus AK implies AD.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditivity function $g : X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(-x) = g(x)$ and scalar multiplication is continuous. It is well known that the space ℓ_p is AK-space where $1 \leq p < \infty$.

Throughout this work, we suppose that (p_k) is a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Also the summation without limits runs from 0 to ∞. In [8], the linear space $\ell(p)$ was defined by Maddox (see also Simons [9] and Nakano [10]) as follows:

$$\ell(p) = \left\{ x = (x_k) \in w : \sum_n |x_n|^{p_n} < \infty \right\} \tag{2.1}$$

which is a complete space paranormed by

$$g(x) = \left(\sum_n |x_n|^{p_n} \right)^{1/M}. \tag{2.2}$$

Let λ, μ be any two sequence spaces, and let $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we write $Ax = ((Ax)_n)$, the A-transform of x, if $(Ax)_n = \sum_k a_{nk} x_k$ converges for each $n \in \mathbb{N}$.

A matrix $A = (a_{nk})$ is called a triangle if $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ for all $n \in \mathbb{N}$. It is trivial that $A(Bx) = (AB)x$ holds for the triangle matrices A, B and a sequence x. Further, a triangle matrix P uniquely has an invert $P^{-1} = Q$ which is also a triangle matrix. Then if $Px = y$,

$$x = P(Qx) = Q(Px), \quad x = Qy \tag{2.3}$$

hold for all $x \in w$.

By (λ, μ), we denote the class of all infinite matrices A such that $A : \lambda \to \mu$. The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined by $\lambda_A = \{x = (x_k) \in w : Ax \in \lambda\}$ which is a sequence space. It is well known that the new sequence space λ_A generated by the limitation matrix A from a sequence space λ is the expansion or the contraction of original space λ.

If A is triangle, then one can easily observe that the sequence spaces λ_A and λ are linearly isomorphic, that is, $\lambda_A \cong \lambda$. Let λ be a sequence space. Then the continuous dual λ'_A

of the space λ_A is defined by $\lambda'_A = \{f : f = g \circ A;\ g \in \lambda'\}$. Let X be a seminormed space. A set $Y \subset X$ is called fundamental set if the span of Y is dense in X. An application of Hahn-Banach theorem on fundamental set is as follows: if Y is the subset of a seminormed space X and $f(Y) = 0$ implies $f = 0$ for $f \in X'$, then Y is a fundamental set (see [11]).

By the idea mentioned above, let us give the definitions of some matrices to construct a new sequence space in sequel to this work. We denote $\Delta = (\delta_{nk})$ and $S = (s_{nk})$ by

$$\delta_{nk} = \begin{cases} (-1)^{n-k}, & \text{if } n-1 \le k \le n, \\ 0, & \text{otherwise,} \end{cases} \qquad s_{nk} = \begin{cases} 1, & \text{if } 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Malkowsky and Savas [12], Choudhary and Mishra [13], and Altay and Basar [14] have defined the sequence spaces $Z(u,v;X)$, $\overline{\ell(p)}$, and $\ell(u,v;p)$, respectively. By using the matrix domain, the spaces $Z(u,v;X)$, $\overline{\ell(p)}$, and $\ell(u,v;p)$ may be redefined by $Z(u,v;X) = X_{G(u,v)}$, $\overline{\ell(p)} = (\ell_p)_S$, and $\ell(u,v;p) = (\ell(p))_{G(u,v)}$, respectively.

If $\lambda \subset w$ is a sequence space and $x = (x_k) \in \lambda$, (Sx)-transform with (2.4) corresponds to nth partial sum of the series $\sum_n x_n$ and it is denoted by $s = (s_n)$.

By using (2.4) and any infinite lower triangular matrix A, we can define two infinite lower triangular matrices $\overline{A}$ and $\widehat{A}$ as follows: $\overline{A} = AS$ and $\widehat{A} = \Delta\overline{A}$. Let $x = (x_k)$ be a sequence in λ. By considering the multiplication of infinite lower triangular matrices, we have $A(Sx) = \overline{A}x$, that is,

$$t_n = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \overline{a}_{nv} x_v. \tag{2.5}$$

Also since $\widehat{A} = \Delta\overline{A}$, we have $\widehat{A}x = (\Delta\overline{A})x$, that is,

$$t_n - t_{n-1} = \sum_{v=0}^{n} \widehat{a}_{nv} x_v. \tag{2.6}$$

Now let us write the following equality:

$$z_n = \left(\widehat{A}x\right)_n = \sum_{v=0}^{n} \widehat{a}_{nv} x_v. \tag{2.7}$$

It can be seen that for any sequences x, y and scalar $\alpha \in \mathbb{R}$, $(\widehat{A}(x+y))_n = (\widehat{A}x)_n + (\widehat{A}y)_n$ and $(\widehat{A}(\alpha x))_n = \alpha(\widehat{A}x)_n$. We now define new sequence space as follows:

$$\ell\left(\widehat{A};p\right) = \left\{ x = (x_k) \in w : \sum_n \left| \left(\widehat{A}x\right)_n \right|^{p_n} < \infty \right\}. \tag{2.8}$$

For some special cases of the infinite lower triangular matrix A and the sequence (p_k), we obtain the following spaces.

(i) If $p_k = p$ for all $k \in \mathbb{N}$, the space $\ell(\widehat{A};p)$ reduces to the normed space $\ell_p(\widehat{A})$ denoted by

$$\ell_p\left(\widehat{A}\right) = \left\{x = (x_k) \in w : \sum_n \left|\sum_{v=0}^{n} \widehat{a}_{nv} x_v\right|^p < \infty\right\}. \tag{2.9}$$

(ii) If $A = (C,1)$, which is Cesáro matrix order 1, then the space $\ell(\widehat{A};p)$ corresponds to the space $\ell(\widehat{C};p)$ denoted by

$$\ell\left(\widehat{C};p\right) = \left\{x = (x_k) \in w : \sum_n \left|\left(\widehat{C}x\right)_n\right|^{p_n} < \infty\right\}, \tag{2.10}$$

where $(\widehat{C}x)_n = (1/n(n+1)) \sum_{k=1}^{n} kx_k$ for $n \geq 1$ and $(\widehat{C}x)_0 = x_0$.

(iii) If $A = (N,p_n)$, which is Nörlund type matrix, then the space $\ell(\widehat{A};p)$ reduces to the space $\ell(\widehat{N};p) = |\overline{N},p_n|(r)$ (see [15, 16]) denoted by

$$\ell\left(\widehat{N};p\right) = \left\{x = (x_k) \in w : \sum_n \left|\left(\widehat{N}x\right)_n\right|^{p_n} < \infty\right\}, \tag{2.11}$$

where $(\widehat{N}x)_n = (p_n/P_nP_{n-1}) \sum_{k=1}^{n} P_{k-1}x_k$ for $n \geq 1$ and $(\widehat{N}x)_0 = x_0$.

Also if $p_k = p$ for all $k \in \mathbb{N}$, then the spaces $\ell(\widehat{C};p)$ and $\ell(\widehat{N};p) = |\overline{N},p_n|(r)$ reduce to the spaces $\ell_p(\widehat{C})$ and $\ell_p(\widehat{N}) = |\overline{N}_p|$ (see [17]), respectively.

Now let us introduce some definitions of geometric properties of sequence spaces.

Let $(X,\|\cdot\|)$ be a normed linear space, and let $S(X)$ and $B(X)$ be the unit sphere and unit ball of X (for the brevity $X = (X,\|\cdot\|)$), respectively. Consider Clarkson's *modulus of convexity* (Clarkson [18] and Day [19]) defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2};\ x,y \in S(X),\ \|x-y\| = \varepsilon\right\}, \tag{2.12}$$

where $0 \leq \varepsilon \leq 2$. The inequality $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$ characterizes the uniformly convex spaces.

In [20], Gurariĭ's modulus of convexity is defined by

$$\beta_X(\varepsilon) = \inf\left\{1 - \inf_{\alpha \in [0,1]} \|\alpha x + (1-\alpha)y\|;\ x,y \in S(X),\ \|x-y\| = \varepsilon\right\}, \tag{2.13}$$

where $0 \leq \varepsilon \leq 2$. It is easily shown that $\delta_X(\varepsilon) \leq \beta_X(\varepsilon) \leq 2\delta_X(\varepsilon)$ for any $0 \leq \varepsilon \leq 2$. Also if $0 < \beta_X(\varepsilon) < 1$, then X is uniformly convex, and if $\beta_X(\varepsilon) < 1$, then X is strictly convex.

A Banach space X is said to have the Banach-Saks property if every bounded sequence (x_n) in X admits a subsequence (z_n) such that the sequence $\{t_k(z)\}$ is convergent in the norm in X (see [21]), where

$$t_k(z) = \frac{1}{k+1}(z_0 + z_1 + z_2 + \cdots + z_k) \quad (k \in \mathbb{N}). \tag{2.14}$$

Let $1 < p < \infty$. A Banach space is said to have the *Banach-Saks type* p or property (BS_p), if every weakly null sequence (x_k) has a subsequence (x_{kl}) such that for some $C > 0$

$$\left\| \sum_{l=0}^{n} x_{kl} \right\| < C(n+1)^{1/p} \tag{2.15}$$

for all $n \in \mathbb{N}$ (see [22]).

3. Some Topological Properties of the Space $\ell(\widehat{A}; p)$

In this section, we investigate some topological properties of the sequence space $\ell(\widehat{A}; p)$ as the paranorm AK property and AD property. Let us begin the following theorem.

Theorem 3.1. (i) *The space $\ell(\widehat{A}; p)$ is complete linear metric space with respect to the paranorm defined by*

$$h(x) = \left(\sum_n |(Ax)_n|^{p_n} \right)^{1/M}. \tag{3.1}$$

(ii) *If the sequence (p_n) is constant sequence and $p \geq 1$, then $\ell_p(\widehat{A})$ is a Banach space normed by*

$$\|z\|_{\ell_p} = \|x\|_{\ell_p(\widehat{A})} = \left(\sum_n |(Ax)_n|^{p} \right)^{1/p}. \tag{3.2}$$

Proof. The proof of (ii) is routine verification by using standard techniques and hence it is omitted.

The proof of (i) is that the linearity of $\ell(\widehat{A}; p)$ with respect to the coordinatewise addition and scalar multiplication follows from the following inequalities which are satisfied for $x, y \in \ell(\widehat{A}; p)$:

$$\left(\sum_n \left| \left(\widehat{A}(x+y) \right)_n \right|^{p_n} \right)^{1/M} \leq \left(\sum_n \left| \left(\widehat{A}x \right)_n \right|^{p_n} \right)^{1/M} + \left(\sum_n \left| \left(\widehat{A}y \right)_n \right|^{p_n} \right)^{1/M} \tag{3.3}$$

and $|\alpha|^{p_n} \leq \max\{1, |\alpha|^M\}$ for any $\alpha \in \mathbb{R}$ (see [23]). After this step, we must show that the space $\ell(\widehat{A}; p)$ holds the paranorm property and the completeness with respect to given paranorm.

It is easy to show that $h(\theta) = 0$, and $h(x) = h(-x)$ for all $x \in \ell(\widehat{A}; p)$. Besides, from (3.3) we obtain $h(x + y) \leq h(x) + h(y)$ for all $x, y \in \ell(\widehat{A}; p)$. To complete the paranorm conditions for the space $\ell(\widehat{A}; p)$, it remains to show the continuity of the scalar multiplication. Let (x^m) be any sequence in $\ell(\widehat{A}; p)$ such that $h(x^m - x) \to 0$, and let (α_m) be also any sequence of scalars such that $|\alpha_m - \alpha| \to 0$ $(m \to \infty)$. From subadditivity of h, we give the inequality $h(x^m) \leq h(x) + h(x^m - x)$. Hence $\{h(x^m)\}$ is bounded and we have

$$h(\alpha_m x^m - \alpha x) = \left(\sum_n \left| (\alpha_m - \alpha) \sum_{v=0}^{n} \widehat{a}_{nv} x_v^m + \alpha \sum_{v=0}^{n} \widehat{a}_{nv} (x_v^m - x_v) \right|^{p_n} \right)^{1/M} \tag{3.4}$$

which tends to zero as $m \to \infty$. Consequently we obtain that h is a paranorm over the space $\ell(\widehat{A}; p)$. To prove the completeness of the space $\ell(\widehat{A}; p)$, let us take any Cauchy sequence (x^i) in the space $\ell(\widehat{A}; p)$. Then for a given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that $h(x^i - x^j) < \varepsilon$ for all $i, j \geq n_0(\varepsilon)$. By using the definitions of the Cauchy sequence and the paranorm, we have, for each fixed n,

$$\left| \left(\widehat{A} x^i\right)_n - \left(\widehat{A} x^j\right)_n \right| \leq \left(\sum_n \left| (\widehat{A}\left(x^i\right)_n - (\widehat{A}\left(x^j\right)_n \right|^{p_n} \right)^{1/M} < \varepsilon \tag{3.5}$$

for every $i, j \geq n_0(\varepsilon)$. Hence we obtain that the sequence $\{\widehat{A}(x^0)_n, \widehat{A}(x^1)_n, \widehat{A}(x^2)_n, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $n \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, that is, $(\widehat{A}(x^j)_n \to (\widehat{A}x)_n$ as $j \to \infty$, where $\{(\widehat{A}x)_n\} = \{(\widehat{A}x)_0, (\widehat{A}x)_1, (\widehat{A}x)_2, \ldots\}$. Now let us choose $m \in \mathbb{N}$ such that $\sum_{n=0}^{m} |(\widehat{A}x^i)_n - (\widehat{A}x^j)_n|^{p_n} < \varepsilon^M$ for each $m \in \mathbb{N}$ and $i, j \geq n_0(\varepsilon)$. By taking $j \to \infty$ and for every $i \geq n_0(\varepsilon)$, we get

$$\sum_{n=0}^{m} \left| \left(\widehat{A} x^i\right)_n - \left(\widehat{A} x\right)_n \right|^{p_n} < \varepsilon^M. \tag{3.6}$$

Again taking $m \to \infty$ and for every $i \geq n_0(\varepsilon)$, it is obtained that $h(x^i - x) < \varepsilon$. We write the following equality:

$$\left| \left(\widehat{A} x\right)_n \right| = \left| \left(\widehat{A} x\right)_n + \left(\widehat{A} x^i\right)_n - \left(\widehat{A} x^i\right)_n \right|. \tag{3.7}$$

By using (3.7) and Minkowski's inequality, we get

$$\left(\sum_n \left| \left(\widehat{A} x\right)_n \right|^{p_n} \right)^{1/M} \leq h\left(x^i\right) + h\left(x^i - x\right) \tag{3.8}$$

which implies $x \in \ell(\widehat{A}; p)$. It follows $x^i \to x$ as $i \to \infty$. Consequently, since (x^i) is any Cauchy sequence, we obtain that the space $\ell(\widehat{A}; p)$ is complete. This completes the proof. □

Theorem 3.2. *The space $\ell(\widehat{A}; p)$ is linearly isomorphic to the space $\ell(p)$.*

Proof. Let us define $\widehat{A}$-transform between the spaces $\ell(\widehat{A};p)$ and $\ell(p)$ such that $x \to z = \widehat{A}x$. We have to show that the transformation $\widehat{A}$ is linear, injective and surjective. The linearity of $\widehat{A}$ is obvious. Moreover it is injective because of $x = \theta$ whenever $\widehat{A}x = \theta$. For the surjective property, let $y \in \ell(p)$. From (2.3) and (2.7), there exists a matrix $\widehat{B}$ such that $x_n = (\widehat{B}y)_n$. We have

$$h(x) = \left(\sum_n \left|\left(\widehat{A}\left(\widehat{B}y\right)\right)_n\right|^{p_n}\right)^{1/M} = \left(\sum_n |y_n|^{p_n}\right)^{1/M} = g(y) < \infty. \tag{3.9}$$

Hence we obtain that the transformation $\widehat{A}$ is surjective. Consequently, the spaces $\ell(\widehat{A};p)$ and $\ell(p)$ are linearly isomorphic spaces. □

Theorem 3.3. *The space $\ell_p(\widehat{A})$ has AD property.*

Proof. Let $f \in (\ell_p(\widehat{A}))'$. Then $f(x) = g(\widehat{A}x)$ for some $g \in \ell_p'$. Since ℓ_p has AK property and $\ell_p' \cong \ell_q$ where $1/p + 1/q = 1$,

$$f(x) = \sum_n a_n\left(\widehat{A}x\right)_n \tag{3.10}$$

for some $a = (a_n) \in \ell_q$. Also since $\ell_p(\widehat{A}) \cong \ell_p$ and the inclusion $\phi \subset \ell_p$ holds, we have $\phi \subset \ell_p(\widehat{A})$. For any $f \in (\ell_p(\widehat{A}))'$ and $e^{(k)} \in \phi$, we have

$$f\left(e^{(k)}\right) = \sum_n a_n\left(\widehat{A}e^{(k)}\right)_n = \left(\widehat{H}a\right)_k, \tag{3.11}$$

where $\widehat{H}$ is transpose of the matrix $\widehat{A}$. Hence from Hahn-Banach theorem, $\phi \subset \ell_p(\widehat{A})$ is dense in $\ell_p(\widehat{A})$ if and only if $\widehat{H}a = \theta$ for $a \in \ell_q$ implies $a = \theta$. Besides, since the null space of the operator $\widehat{H}$ on w is $\{\theta\}$, $\ell_p(\widehat{A})$ has *AD* property. Hence the proof is completed. □

4. Some Geometric Properties of the space $\ell_p(\widehat{A})$

In this section, we give some geometric properties for the space $\ell_p(\widehat{A})$.

Theorem 4.1. *The space $\ell_p(\widehat{A})$ has the Banach-Saks of type p.*

Proof. Let (ε_n) be a sequence of positive numbers for which $\sum_{n=1}^{\infty} \varepsilon_n \leq 1/2$. Let (x_n) be a weakly null sequence in $B(\ell_p(\widehat{A}))$. Set $z_0 = x_0 = 0$ and $z_1 = x_{n_1} = x_1$. Then there exists $s_1 \in \mathbb{N}$

such that

$$\left\| \sum_{i=s_1+1}^{\infty} z_1(i)e^{(i)} \right\|_{\ell_p(\widehat{A})} < \varepsilon_{1.} \tag{4.1}$$

Since (x_n) is a weakly null sequence implies that $x_n \to 0$ with respect to the coordinatwise, there is an $n_2 \in \mathbb{N}$ such that

$$\left\| \sum_{i=0}^{s_1} x_n(i)e^{(i)} \right\|_{\ell_p(\widehat{A})} < \varepsilon_1, \tag{4.2}$$

where $n \geq n_2$. Set $z_2 = x_{n_2}$. Then there exists an $s_2 > s_1$ such that

$$\left\| \sum_{i=s_2+1}^{\infty} z_2(i)e^{(i)} \right\|_{\ell_p(\widehat{A})} < \varepsilon_2. \tag{4.3}$$

By using the fact that $x_n \to 0$ (coordinatwise), there exists an $n_3 > n_2$ such that

$$\left\| \sum_{i=0}^{s_2} x_n(i)e^{(i)} \right\|_{\ell_p(\widehat{A})} < \varepsilon_2, \tag{4.4}$$

where $n \geq n_3$.

If we continue this process, we can find two increasing subsequences (s_i) and (n_i) such that

$$\left\| \sum_{i=0}^{s_j} x_n(i)e^{(i)} \right\|_{\ell_p(\widehat{A})} < \varepsilon_j \tag{4.5}$$

for each $n \geq n_{j+1}$ and

$$\left\| \sum_{i=s_j+1}^{\infty} z_j(i)e^{(i)} \right\|_{\ell_p(\widehat{A})} < \varepsilon_j, \tag{4.6}$$

where $z_j = x_{n_j}$. Hence,

$$\begin{aligned}
\left\|\sum_{j=0}^{n} z_j\right\|_{\ell_p(\widehat{A})} &= \left\|\sum_{j=0}^{n}\left(\sum_{i=0}^{s_{j-1}} z_j(i)e^{(i)} + \sum_{i=s_{j-1}+1}^{s_j} z_j(i)e^{(i)} + \sum_{i=s_j+1}^{\infty} z_j(i)e^{(i)}\right)\right\|_{\ell_p(\widehat{A})} \\
&\le \left\|\sum_{j=0}^{n}\left(\sum_{i=s_{j-1}+1}^{s_j} z_j(i)e^{(i)}\right)\right\|_{\ell_p(\widehat{A})} + \left\|\sum_{j=0}^{n}\left(\sum_{i=0}^{s_{j-1}} z_j(i)e^{(i)}\right)\right\|_{\ell_p(\widehat{A})} \\
&\quad + \left\|\sum_{j=0}^{n}\left(\sum_{i=s_j+1}^{\infty} z_j(i)e^{(i)}\right)\right\|_{\ell_p(\widehat{A})} \\
&\le \left\|\sum_{j=0}^{n}\left(\sum_{i=s_{j-1}+1}^{s_j} z_j(i)e^{(i)}\right)\right\|_{\ell_p(\widehat{A})} + 2\sum_{j=0}^{n}\varepsilon_j.
\end{aligned} \tag{4.7}$$

On the other hand, since $x_n \in B(\ell_p(\widehat{A}))$ and $\|x\|_{\ell_p(\widehat{A})} = \left(\sum_{i=0}^{\infty} \left| \sum_{v=0}^{i} \widehat{a}_{iv} x_v\right|^p\right)^{1/p}$, it can be seen that $\|x\|_{\ell_p(\widehat{A})} < 1$. Therefore $\|x\|_{\ell_p(\widehat{A})}^p < 1$. We have

$$\begin{aligned}
\left\|\sum_{j=0}^{n}\left(\sum_{i=s_{j-1}+1}^{s_j} z_j(i)e^{(i)}\right)\right\|_{\ell_p(\widehat{A})}^p &= \sum_{j=0}^{n}\sum_{i=s_{j-1}+1}^{s_j}\left|\sum_{v=0}^{i}\widehat{a}_{iv} z_j(v)\right|^p \\
&\le \sum_{j=0}^{n}\sum_{i=0}^{\infty}\left|\sum_{v=0}^{i}\widehat{a}_{iv} z_j(v)\right|^p \\
&\le (n+1).
\end{aligned} \tag{4.8}$$

Hence we obtain

$$\left\|\sum_{j=0}^{n}\left(\sum_{i=s_{j-1}+1}^{s_j} z_j(i)e^{(i)}\right)\right\|_{\ell_p(\widehat{A})} \le (n+1)^{1/p}. \tag{4.9}$$

By using the fact $1 \le (n+1)^{1/p}$ for all $n \in \mathbb{N}$ and $1 \le p < \infty$, we have

$$\left\|\sum_{j=0}^{n} z_j\right\|_{\ell_p(\widehat{A})} \le (n+1)^{1/p} + 1 \le 2(n+1)^{1/p}. \tag{4.10}$$

Hence $\ell_p(\widehat{A})$ has the Banach-Saks type p. This completes the proof. □

Theorem 4.2. *Gurarii's modulus of convexity for the normed space* $\ell_p(\widehat{A})$ *is*

$$\beta_{\ell_p(\widehat{A})}(\varepsilon) \leq 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}, \tag{4.11}$$

where $0 \leq \varepsilon \leq 2$.

Proof. We have $x \in \ell_p(\widehat{A})$. Then we have

$$\|x\|_{\ell_p(\widehat{A})} = \left\|\widehat{A}x\right\|_{l_p} = \left(\sum_n \left|\left(\widehat{A}x\right)_n\right|^p\right)^{1/p}. \tag{4.12}$$

Let $0 \leq \varepsilon \leq 2$ and consider the following sequences:

$$\begin{aligned} x = (x_n) &= \left(\widehat{B}\left(\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}\right), \widehat{B}\left(\frac{\varepsilon}{2}\right), 0, 0, \ldots\right), \\ y = (y_n) &= \left(\widehat{B}\left(\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}\right), \widehat{B}\left(-\frac{\varepsilon}{2}\right), 0, 0, \ldots\right), \end{aligned} \tag{4.13}$$

where $\widehat{B}$ is the inverse of the matrix $\widehat{A}$. Since $z_n = (\widehat{A}x)_n$ and $t_n = (\widehat{A}y)_n$, we have

$$\begin{aligned} z = (z_n) &= \left(\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}, \left(\frac{\varepsilon}{2}\right), 0, 0, \ldots\right), \\ t = (t_n) &= \left(\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}, \left(-\frac{\varepsilon}{2}\right), 0, 0, \ldots\right). \end{aligned} \tag{4.14}$$

By using sequences given above, we obtain the following equalities:

$$\begin{aligned} \|x\|^p_{\ell_p(\widehat{A})} = \left\|\widehat{A}x\right\|^p_{l_p} &= \left|\left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}\right|^p + \left|\frac{\varepsilon}{2}\right|^p \\ &= 1 - \left(\frac{\varepsilon}{2}\right)^p + \left(\frac{\varepsilon}{2}\right)^p \\ &= 1, \end{aligned}$$

$$\begin{aligned}
\|y\|^p_{\ell_p(\widehat{A})} &= \left\|\widehat{A}y\right\|^p_{l_p} = \left|\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}\right|^p + \left|-\frac{\varepsilon}{2}\right|^p \\
&= 1-\left(\frac{\varepsilon}{2}\right)^p + \left(\frac{\varepsilon}{2}\right)^p \\
&= 1, \\
\|x-y\|_{\ell_p(\widehat{A})} &= \left\|\widehat{A}x - \widehat{A}y\right\|_{l_p} \\
&= \left(\left|\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p} - \left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}\right|^p + \left|\frac{\varepsilon}{2} - \left(-\frac{\varepsilon}{2}\right)\right|^p\right)^{1/p} \\
&= \varepsilon.
\end{aligned} \tag{4.15}$$

To complete the conditions of $\beta_{\ell_p(\widehat{A})}(\varepsilon)$ for Gurarii's modulus of convexity, it remains to show the infimum of $\|\alpha x + (1-\alpha)t\|_{\ell_p(\widehat{A})}$ for $0 \le \alpha \le 1$. We have

$$\begin{aligned}
&\inf_{0\le\alpha\le1} \|\alpha x + (1-\alpha)y\|_{\ell_p(\widehat{A})} \\
&\quad= \inf_{0\le\alpha\le1} \left\|\alpha\widehat{A}x + (1-\alpha)\widehat{A}y\right\|_{l_p} \\
&\quad= \inf_{0\le\alpha\le1} \left[\left|\alpha\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p} + (1-\alpha)\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}\right|^p + \left|\alpha\left(\frac{\varepsilon}{2}\right) + (1-\alpha)\left(-\frac{\varepsilon}{2}\right)\right|^p\right]^{1/p} \\
&\quad= \inf_{0\le\alpha\le1} \left[1-\left(\frac{\varepsilon}{2}\right)^p + |2\alpha-1|^p\left(\frac{\varepsilon}{2}\right)^p\right]^{1/p} \\
&\quad= \left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}.
\end{aligned} \tag{4.16}$$

Consequently we get for $p \ge 1$

$$\beta_{\ell_p(\widehat{A})}(\varepsilon) \le 1 - \left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{1/p}. \tag{4.17}$$

This is the desired result. Hence the proof is completed. □

Corollary 4.3. (i) *If* $\varepsilon = 2$, *then* $\beta_{\ell_p(\widehat{A})}(\varepsilon) \le 1$ *and hence* $\ell_p(\widehat{A})$ *is strictly convex.*

(ii) *If* $0 < \varepsilon < 2$, *then* $0 < \beta_{\ell_p(\widehat{A})}(\varepsilon) < 1$ *and hence* $\ell_p(\widehat{A})$ *is uniformly convex.*

Corollary 4.4. *If* $\alpha = 1/2$, *then* $\delta_{\ell_p(\widehat{A})}(\varepsilon) = \beta_{\ell_p(\widehat{A})}(\varepsilon)$.

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