Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 7, 1996, 327–347

NIRENBERG–GAGLIARDO INTERPOLATION INEQUALITY AND REGULARITY OF SOLUTIONS OF NONLINEAR HIGHER ORDER EQUATIONS

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Dedicated to Professor Louis Nirenberg

1. Introduction

Well-known counterexamples in [3, 4] show that quasilinear elliptic equations in divergence form

$$\sum_{|\alpha|\le m}(-1)^{|\alpha|}D^\alpha A_\alpha(x,u,\dots,D^m u)=0 \tag{1.1}$$

with $m>1$ can have unbounded generalized solutions, even when $A_\alpha(x,\xi)$ are analytic functions of their arguments satisfying natural growth conditions for $|\xi|\to\infty$. Here $x=(x_1,\dots,x_n)$, $\alpha=(\alpha_1,\dots,\alpha_n)$ is a vector with nonnegative integer-valued components,

$$|\alpha|=\alpha_1+\dots+\alpha_n,$$
$$D^\alpha=\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\dots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}\quad\text{and}\quad D^k u=\{D^\alpha u:|\alpha|=k\}.$$

Under the ellipticity condition in the form

$$\sum_{|\alpha|=m}A_\alpha(x,\xi)\xi_\alpha\ge C'\sum_{|\alpha|=m}|\xi_\alpha|^p-C''\sum_{|\beta|<m}|\xi_\beta|^{p(\beta)}-f(x) \tag{1.2}$$

1991 *Mathematics Subject Classification*. Primary 35J60.

The second author was partially supported by a grant from Ukrainian Foundation of Fundamental Research and by grant N U97200 from International Science Foundation.

with $p(\beta)$, and $f(x)$ satisfying certain assumptions, J. Freehse, I. V. Skrypnik, K. Widman, V. A. Solonnikov and others proved boundedness, continuity and Hölder continuity of solutions of equation (1.1) if $n - mp$ is zero or sufficiently small [9]. Counterexamples show that the last condition cannot be dropped.

In [8] a class of equations (1.1) was introduced all of whose generalized solutions satisfy Hölder's condition without any assumptions concerning the relation between m, n and p. For this class, condition (1.2) is replaced by

$$\begin{aligned}\sum_{1\le|\alpha|\le m} A_\alpha(x,\xi)\xi_\alpha \ge C' \sum_{|\alpha|=m} |\xi_\alpha|^p + C' \sum_{|\alpha|=1} |\xi_\alpha|^q \\ - C'' \sum_{1<|\alpha|<m} |\xi_\alpha|^{p_\alpha} - f(x)\end{aligned} \tag{1.3}$$

with $q > mp$, positive constants C', C'' and numbers p_α satisfying certain conditions. The study of the regularity of solutions of equation (1.1) in [9] was based on the Nirenberg–Gagliardo interpolation inequality [7].

In [10, 11] the regularity of generalized solutions for quasilinear parabolic higher order equations was established under an analog of condition (1.3).

In this paper we study the regularity problem for equation (1.1) in the degenerate case and we also establish a new analog of the Nirenberg–Gagliardo inequality for the weighted case. We assume that the functions $A_\alpha(x,\xi)$ are Carathéodory functions and satisfy

$$\begin{aligned}&\sum_{1\le|\alpha|\le m} A_\alpha(x,\xi)\xi_\alpha \\ &\qquad\ge C' \sum_{|\alpha|=m} v_\alpha(x)|\xi_\alpha|^q + C' \sum_{|\alpha|=1} v_\alpha(x)|\xi_\alpha|^q \\ &\qquad\quad - C'' \sum_{1<|\alpha|<m} v_\alpha(x)|\xi_\alpha|^{p_\alpha} - C''|\xi_0|^{p_0} v_1(x) - f(x)v_1(x),\end{aligned} \tag{1.4}$$

$$\begin{aligned}&\sum_{1\le|\alpha|\le m} v_\alpha(x)^{-1/(p_\alpha-1)}|A_\alpha(x,\xi)|^{p_\alpha/(p_\alpha-1)} \\ &\qquad + v_1(x)^{-1/(p_0-1)}|A_0(x,\xi)|^{p_0/(p_0-1)} \\ &\qquad\le C''\Big\{\sum_{1\le|\alpha|\le m} v_\alpha(x)|\xi_\alpha|^{p_\alpha} + v_1(x)|\xi_0|^{p_0} + f(x)v_1(x)\Big\}.\end{aligned} \tag{1.5}$$

In (1.4), (1.5) the numbers p_α are defined by

$$\begin{aligned}&p_\alpha = p \quad \text{for} \quad |\alpha| = m, \quad p_\alpha = q \quad \text{for } |\alpha| = 1, \\ &\frac{1}{p_\alpha} = \frac{|\alpha|-1}{m-1}\cdot\frac{1}{p} + \frac{m-|\alpha|}{m-1}\cdot\frac{1}{q_1} \quad \text{for } 1<|\alpha|<m,\end{aligned} \tag{1.6}$$

and the numbers m, p, q, q_1 are assumed to satisfy

$$m \ge 2, \quad p \ge 2, \quad mp < q_1 < q < n. \tag{1.7}$$

In (1.4), (1.5), $v_\alpha(x)$, $1 \le |\alpha| \le m$, are nonnegative functions which are defined by

$$v_\alpha(x) = v_m(x) \quad \text{for } |\alpha| = m, \quad v_\alpha(x) = v_1(x) \quad \text{for } |\alpha| = 1,$$
$$\begin{aligned} v_\alpha(x) = v_{|\alpha|}(x) = \{[v_m(x)]^{(|\alpha|-1)/(p(m-1))} \\ \times [v_1(x)]^{(m-|\alpha|)/(q_1(m-1))}\}^{p_\alpha} \quad \text{for } 1 < |\alpha| < m, \end{aligned} \tag{1.8}$$

and satisfy the conditions

$$\begin{aligned} &v_m \in L^1(\Omega), \quad v_m^{-1/(p-1)} \in L^1(\Omega), \\ &v_1 \in L^1(\Omega), \quad v_1^{-1/(q-1)} \in L^1(\Omega), \\ &v_1 \in A_q, \qquad v_m(x) \le K_1 v_1(x), \end{aligned} \tag{1.9}$$

where A_q is Muckenhoupt's class defined in [6].

Under this and some additional assumptions on the weight functions $v_1(x)$ and $v_m(x)$ we prove local and global boundedness and Hölder continuity of solutions of equation (1.1).

Conditions on weight functions are connected with imbeddings of Nirenberg–Gagliardo type for weighted spaces. For special weight functions (of the type $|x|^\lambda$) the corresponding imbeddings were proved in [1, 5]. For general weight functions analogous imbeddings are proved in this paper.

All our conditions on weight functions and coefficients are essential as follows from the counterexample in the last section.

2. Formulation of main results

We will assume the following properties for weight functions $v_1(x)$ and $v_m(x)$:

(w) The functions $v_1(x)$ and $v_m(x)$, $x \in \mathbb{R}^n$, are differentiable on $\mathbb{R}^n$ and there exist numbers $\kappa > 1$, $K_2 > 0$ and $R_0 > 0$ such that the function $\widetilde{v}(x)$ defined by

$$\begin{aligned} \widetilde{v}(x) = v_1(x) + [v_1(x)]^{-\varrho/(q_1(m-1))}[v_m(x)]^{\varrho/(p(m-1))} \\ \times \left[\frac{1}{v_m(x)}\left|\frac{\partial v_m(x)}{\partial x}\right| + \frac{1}{v_1(x)}\left|\frac{\partial v_1(x)}{\partial x}\right|\right]^\varrho, \quad \varrho = \frac{(m-1)pq_1}{q_1 - p}, \end{aligned} \tag{2.1}$$

belongs to the class A_∞ and satisfies

$$\frac{R_2}{R_1}\left[\frac{\widetilde{v}(B(x_0,R_2))}{\widetilde{v}(B(x_0,R_1))}\right]^{1/(q\kappa)} \le K_2 \left[\frac{\widetilde{v}_1(B(x_0,R_2))}{\widetilde{v}_1(B(x_0,R_1))}\right]^{1/q} \tag{2.2}$$

for all $x_0 \in \Omega$ and all R_1, R_2 such that $0 < R_2 < R_1 \le R_0$. For every $E \subset \Omega$ we write

$$v_1(E) = \int_E v_1(x)\,dx, \quad \widetilde{v}(E) = \int_E \widetilde{v}(x)\,dx. \tag{2.3}$$

From the condition (1.9) and [6] it follows that

$$v_1 \in A_{\widetilde{q}} \quad \text{for some } \widetilde{q} < q. \tag{2.4}$$

We assume that the number p_0 in (1.4), (1.5) satisfies

$$q \le p_0 < \frac{nq\widetilde{q}}{n\widetilde{q} - q}. \tag{2.5}$$

The nonnegative function $f(x)$ in (1.4), (1.5) satisfies the condition

$$f \in L_r(\Omega), \quad r > n\widetilde{q}/q. \tag{2.6}$$

We will say that a function $u \in W^m_{p,\mathrm{loc}}(\Omega, v_m) \cap W^1_{q,\mathrm{loc}}(\Omega, v_1)$ is a solution of equation (1.1) if for every $\varphi \in \mathring{W}^m_p(\Omega, v_m) \cap \mathring{W}^1_q(\Omega, v_1)$ with compact support in Ω we have the integral identity

$$\sum_{|\alpha| \le m} \int_\Omega A_\alpha(x, u, \ldots, D^m u) D^\alpha \varphi(x)\, dx = 0. \tag{2.7}$$

The left-hand side of (2.7) is finite for the indicated choice of u and φ. This follows from

Theorem 2.1. *Assume that $\partial\Omega$ is of class C^m and condition* (w) *is satisfied. Then there exists a positive constant K such that for every $u \in W^m_p(\Omega, v_m) \cap W^1_{q_1}(\Omega, v_1)$ we have*

$$\begin{aligned} &\|D^k u\|_{L_{p_k}(\Omega, v_k)} \\ &\quad \le K\{\|D^m u\|_{L_p(\Omega, v_m)} + \|D^1 u\|_{L_{q_1}(\Omega, v_1)}\}^{(k-1)/(m-1)} \|D^1 u\|^{(m-k)/(m-1)}_{L_{q_1}(\Omega, v_1)} \end{aligned} \tag{2.8}$$

for $1 < k < m$ with p_k and $v_k(x)$ defined by (1.6) *and* (1.8).

We will give some remarks about the proof of this theorem in Section 7. The inequality (2.8) generalizes the Nirenberg–Gagliardo interpolation inequality to general weight functions.

In estimating the integral on left-hand side of (2.7) we also use the imbedding

$$\mathring{W}^1_q(\Omega, v_1) \subset L_{q\widetilde{\kappa}}(\Omega, v_1), \quad \widetilde{\kappa} = \frac{n\widetilde{q}}{n\widetilde{q} - q}, \tag{2.9}$$

which follows from [2].

For $d > 0$, we define $\Omega_d = \{x \in \Omega : \varrho(x, \partial\Omega) > d\}$, where $\varrho(x, \partial\Omega)$ is the distance from x to the boundary of Ω.

Theorem 2.2. *Assume that the functions $A_\alpha(x, \xi)$, $|\alpha| \le m$, satisfy conditions* (1.4)–(1.7), (2.5), (2.6) *and that the weight functions $v_\alpha(x)$ satisfy conditions* (1.8), (1.9), (w). *Then every solution u of* (1.1) *satisfies the estimate*

$$|u(x)| \le M_d, \quad x \in \Omega_d, \tag{2.10}$$

with a constant M_d *depending only on the known parameters, the norm of* u *in* $W_p^m(\Omega_{d/2}, v_m) \cap W_q^1(\Omega_{d/2}, v_1)$ *and* d.

THEOREM 2.3. *Assume that all conditions of Theorem* 2.2 *are satisfied. Then every solution* u *of* (1.1) *satisfies the estimate*

$$|u(x) - u(y)| \leq A_d |x - y|^\alpha, \quad x, y \in \Omega_d, \tag{2.11}$$

with positive constants A_d, α, *where* $\alpha \in (0,1)$ *depends only on the known parameters, and* A_d *depends only on the known parameters, the norm of* u *in* $W_p^m(\Omega_{d/2}, v_m) \cap W_q^1(\Omega_{d/2}, v_1)$ *and* d.

Analogous results on regularity of solutions near the boundary are valid for Dirichlet or Neumann conditions under some regularity of the domain.

We shall say that the domain satisfies *condition* (b) if there exist $\Theta, R_0 > 0$ such that

$$\operatorname{meas}(B(x_0, R) \setminus \Omega) \geq \Theta \operatorname{meas}(B(x_0, R)) \tag{2.12}$$

for all $x_0 \in \partial\Omega$ and $0 < R \leq R_0$.

THEOREM 2.4. *Assume that all conditions of Theorem* 2.2 *on* $A_\alpha(x, \xi)$ *and* $v_\alpha(x)$ *are satisfied. Let* $u \in \mathring{W}_p^m(\Omega, v_m) \cap \mathring{W}_q^1(\Omega, v_1)$ *be a solution of equation* (1.1). *Then:*

1) *there exists a constant* M *depending only on the known parameters and the norm of* u *in* $W_p^m(\Omega, v_m) \cap W_q^1(\Omega, v_1)$ *such that*

$$|u(x)| \leq M, \quad x \in \Omega; \tag{2.13}$$

2) *if* Ω *satisfies* (b) *then there exist* $B, \beta > 0$ *such that*

$$|u(x) - u(y)| \leq B|x - y|^\beta, \quad x, y \in \Omega. \tag{2.14}$$

Moreover, $\beta \in (0,1)$ *and depends only on the known parameters, and* B *depends only on the known parameters and the norm of* u *in* $W_p^m(\Omega, v_m) \cap W_q^1(\Omega, v_1)$.

We shall say that $u \in W_p^m(\Omega, v_m) \cap W_q^1(\Omega, v_1)$ is a solution of the Neumann boundary value problem if the integral identity (2.7) is valid for all $\varphi \in W_p^m(\Omega, v_m) \cap W_q^1(\Omega, v_1)$.

THEOREM 2.5. *Assume that* $\partial\Omega \in C^m$ *and all conditions of Theorem* 2.2 *on* $A_\alpha(x, \xi)$ *and* $v_\alpha(x)$ *are satisfied. Let* $u \in W_p^m(\Omega, v_m) \cap W_q^1(\Omega, v_1)$ *be a solution of the Neumann boundary value problem for* (1.1). *Then the inequalities* (2.13)–(2.14) *hold with* M, B, β *depending on the same parameters as in Theorem* 2.4.

3. Proof of Theorem 2.2

We substitute in (2.7) the test function

$$\varphi(x) = [1 + \lambda_N^2(u(x))]^k u(x)\psi^s(x) \tag{3.1}$$

where $\lambda_N(u) = u$ for $|u| \le N$, $\lambda_N(u) = (N+1)\mathrm{sign}(u)$ for $|u| > N+1$, $d\lambda_N(u)/du \ge 0$, $N \ge 1$, k and s are arbitrary numbers such that $s \ge q$ and $k \ge 0$. The function $\psi(x)$ is a fixed smooth cut-off function equal to one in a ball $B(x_0, d/2)$, to zero outside $B(x_0, 3d/4)$ and such that $|D^\alpha \psi(x)| \le C/d^{|\alpha|}$ for $|\alpha| \le m$ and $x_0 \in \Omega_d$.

We have

$$\begin{aligned} D^\alpha \varphi(x) = \{&[1+\lambda_N^2(u(x))]^k D^\alpha u(x) + 2k[1+\lambda_N^2(u(x))]^{k-1} \\ &\times \lambda_N(u(x))\lambda_N'(u(x))u(x)D^\alpha u(x)\}\psi^s(x) + R_\alpha(x) \end{aligned} \tag{3.2}$$

with the pointwise inequality

$$|R_\alpha| \le C_1(k+s)^m[1+\lambda_N^2(u)]^k \bigg\{ \sum_{|\beta|<|\alpha|} |D^\beta u|^{|\alpha|/|\beta|} + |u| \bigg\} \psi^{s-m}. \tag{3.3}$$

Here and in the sequel the constants C_i depend only on the known parameters and d.

After substitution we obtain

$$\begin{aligned} \int_\Omega \bigg\{ v_m \sum_{|\alpha|=m} |D^\alpha u|^p + v_1 \sum_{|\alpha|=1} |D^\alpha u|^q \bigg\} [1+\lambda_N^2(u)]^k \psi^s\,dx \\ \le C_2(k+s)^{qm} \int_\Omega \bigg\{ \sum_{1<|\alpha|<m} v_\alpha |D^\alpha u|^{p_\alpha} \\ + |u|^{p_0} v_1 + [|f|+1]v_1 \bigg\} [1+\lambda_N^2(u)]^k \psi^{s-m} dx. \end{aligned} \tag{3.4}$$

Now we estimate the terms with derivatives on the right-hand side of (3.4) by using integration by parts. For $|\alpha| = j$, $\alpha = \beta + \gamma$, $|\beta| = j-1$, $|\gamma| = 1$ we have

$$\begin{aligned} &\int_\Omega v_\alpha |D^\alpha u|^{p_\alpha} [1+\lambda_N^2(u)]^k \psi^{s-m} dx \\ &\quad = -\int_\Omega D^\beta u |D^\alpha u|^{p_\alpha - 2} [1+\lambda_N^2(u)]^k \psi^{s-m} v_\alpha \\ &\qquad \times \bigg\{ \frac{1}{v_\alpha} D^\gamma v_\alpha D^\alpha u + (p_\alpha - 1) D^{\alpha+\gamma} u + 2k D^\alpha u \\ &\qquad \times [1+\lambda_N^2(u)]^{-1} \lambda_N(u) \lambda_N'(u) D^\gamma u + (s-m) D^\alpha u \psi^{-1} D^\gamma \psi \bigg\} dx. \end{aligned} \tag{3.5}$$

Let $j > 2$. We estimate the terms on the right-hand side of (3.5) by Young's inequality. For example, for $\psi \neq 0$,

$$\begin{aligned}(3.6)\qquad & v_\alpha |D^\alpha u|^{p_\alpha-2}|D^\beta u|\cdot|D^{\alpha+\gamma}u|\psi^{-m}\\ &= v_\alpha^{(p_\alpha-2)/p_\alpha}|D^\alpha u|^{p_\alpha-2}v_{\alpha+\gamma}^{1/(p_{\alpha+\gamma})}|D^{\alpha+\gamma}u|v_\beta^{1/p_\beta}|D^\beta u|\psi^{-m}\\ &\le \varepsilon v_\alpha|D^\alpha u|^{p_\alpha}+\varepsilon v_{\alpha+\gamma}|D^{\alpha+\gamma}u|^{p_{\alpha+\gamma}}+\varepsilon^{-p_\beta}v_\beta|D^\beta u|^{p_\beta}\psi^{-mp_\beta}.\end{aligned}$$

We have used the equalities

$$(3.7)\qquad \begin{aligned}&\frac{p_\alpha-2}{p_\alpha}+\frac{1}{p_{\alpha+\gamma}}+\frac{1}{p_\beta}=1 \quad \text{for } |\alpha|>2,\ |\beta|=|\alpha|-1,\ |\gamma|=1,\\ &v_\alpha^{2/p_\alpha}=v_{\alpha+\gamma}^{1/p_{\alpha+\gamma}}v_\beta^{1/p_\beta},\end{aligned}$$

which follow from (1.6) and (1.8).

For $j = 2$ instead of (3.6) we have

$$\begin{aligned}(3.8)\qquad & v_\alpha|D^\alpha u|^{p_\alpha-2}|D^\beta u|\cdot|D^{\alpha+\gamma}u|^{p_{\alpha+\gamma}}\psi^{-m}\\ &\le \varepsilon\{v_\alpha|D^\alpha u|^{p_\alpha}+v_{\alpha+\gamma}|D^{\alpha+\gamma}u|^{p_{\alpha+\gamma}}+v_1|D^\beta u|^q\}\\ &\quad+\varepsilon^{-qq_1/(q-q_1)}v_1\psi^{-qq_1/(q-q_1)}.\end{aligned}$$

Analogously we estimate the other summands on the right-hand side of (3.5):

$$\begin{aligned}(3.9)\qquad & kv_\alpha|D^\beta u|\cdot|D^\alpha u|^{p_\alpha-1}|D^\gamma u|\psi^{-m}\\ &\le \varepsilon\{v_\alpha|D^\alpha u|^{p_\alpha}+v_\beta|D^\beta u|^{p_\beta}+v_1|D^\gamma u|^q\}+C_3k^{a_1}\varepsilon^{-a_1}v_1\psi^{-ma_1}.\end{aligned}$$

Here and in the sequel we denote by a_i positive numbers depending only on m, p, q, q_1.

In the same way we have the pointwise inequalities

$$\begin{aligned}(3.10)\qquad & (s-m)v_\alpha|D^\beta u|\cdot|D^\alpha u|^{p_\alpha-1}|D^\gamma\psi|\psi^{-m-1}\\ &\le \varepsilon\{v_\alpha|D^\alpha u|^{p_\alpha}+v_\beta|D^\beta u|^{p_\beta}\}+C_4\varepsilon^{-a_1}v_1\psi^{-(m+1)a_1}(s-m)^{a_1},\\ (3.11)\qquad & |D^\beta u|\cdot|D^\alpha u|^{p_\alpha-1}|D^\gamma v_\alpha|\psi^{-m}\\ &\le \varepsilon\{v_\alpha|D^\alpha u|^{p_\alpha}+v_\beta|D^\beta u|^{p_\beta}\}+\varepsilon^{-\varrho}\psi^{-\varrho m}\widetilde{v},\end{aligned}$$

where $\widetilde{v}(x)$ and ϱ are defined in (2.1).

Using inequalities (3.6)–(3.11) we obtain for

$$(3.12)\qquad I_j(s)=\sum_{|\alpha|=j}\int_\Omega v_\alpha|D^\alpha u|^{p_\alpha}[1+\lambda_N^2(u)]^k\psi^s\,dx$$

the estimate

$$\begin{aligned}(3.13)\qquad I_j(s-m) &\le \varepsilon I_{j+1}(s)+\varepsilon I_j(s)+C_5\varepsilon^{-a_2}I_{j-1}(s-ma_2)\\ &\quad+C_5(s+k)^{a_2}\varepsilon^{-a_2}\int_\Omega[1+\lambda_N^2(u)]^k\psi^{s-ma_2}\widetilde{v}\,dx\end{aligned}$$

for $j > 2$.

For $j = 2$ we obtain instead of (3.13) the estimate

$$I_2(s-m) \le \varepsilon\{I_3(s) + I_2(s) + I_1(s)\} + C_6(s+k)^{a_3}\varepsilon^{-a_3}\int_\Omega [1+\lambda_N^2(u)]^k \psi^{s-a_3}\widetilde{v}\,dx. \tag{3.14}$$

Using estimates (3.13), (3.14) we obtain by induction the inequality

$$\sum_{j=2}^{m-1} I_j(s-m) \le \varepsilon\{I_m(s) + I_1(s)\} + C_7\varepsilon^{-a_4}(s+k)^{a_4}\int_\Omega [1+\lambda_N^2(u)]^k \psi^{s-a_4}\widetilde{v}\,dx. \tag{3.15}$$

From (3.4) and (3.15) we get the estimate

$$\int_\Omega \Big\{ v_m \sum_{|\alpha|=m} |D^\alpha u|^{p_\alpha} + v_1 \sum_{|\alpha|=1} |D^\alpha u|^q \Big\} [1+\lambda_N^2(u)]^k \psi^s\,dx \le C_8(k+s)^{a_5}\int_\Omega [1+\lambda_N^2(u)z]^k\{|u|^{p_0}v_1 + [f+1]v_1 + \widetilde{v}\}\psi^{s-a_5}\,dx. \tag{3.16}$$

Further, we estimate the summands of the right-hand side of (3.16) by imbedding theorems. Using the imbedding (2.9) we have

$$\int_\Omega [1+\lambda_N^2(u)]^k |u|^{p_0}\psi^s v_1\,dx \le C_9(k+s)^{\widetilde{\kappa}}\Big\{\int_\Omega ([1+\lambda_N^2(u)]^{k/\widetilde{\kappa}}|\partial u/\partial x|^q + [1+\lambda_N^2(u)]^{k/\widetilde{\kappa}}(|u|^{p_0}+1)\psi^{s/\widetilde{\kappa}-q}v_1)\,dx\Big\}^{\widetilde{\kappa}}. \tag{3.17}$$

Using the Hölder inequality and the imbedding (2.9) we have, with $r' = r/(r-1)$,

$$\int_\Omega [1+\lambda_N^2(u)]^k \psi^s [f+1] v_1\,dx \le C_{10}(k+s)^{q\widetilde{\kappa}}\Big\{\int_\Omega [1+\lambda_N^2(u)]^{r'k/\widetilde{\kappa}}(|\partial u/\partial x|^q \psi^{r's/\widetilde{\kappa}} + \psi^{r's/\widetilde{\kappa}-q})v_1\,dx\Big\}^{\widetilde{\kappa}/r'} \tag{3.18}$$

with the constant C_{10} depending on the norm of f.

From the condition (w) the imbedding

$$\overset{\circ}{W}{}^1_q(\Omega, v_1) \subset L_{q\kappa}(\Omega, \widetilde{v}) \tag{3.19}$$

follows [2] with $\kappa > 1$.

Using the imbedding (3.19) we have

$$\begin{aligned}(3.20)\quad &\int_\Omega [1+\lambda_N^2(u)]^k \psi^s \widetilde{v}\,dx \\ &\le C_{11}(k+s)^{q\kappa}\left\{\int_\Omega [1+\lambda_N^2(u)]^{k/\kappa}(|\partial u/\partial x|^q \psi^{s/\kappa} + \psi^{s/\kappa - q})v_1\,dx\right\}^\kappa.\end{aligned}$$

From inequalities (3.16)–(3.20), for

$$(3.21)\qquad I_N(k,s) = \int_\Omega [1+\lambda_N^2(u)]^k \psi^s \{|u|^{p_0} v_1 + [f+1]v_1 + \widetilde{v}\}\,dx$$

we obtain the estimate

$$(3.22)\qquad I_N(k,s) \le C_{12}(k+s)^{a_6}[I_N(k/\overline{\kappa}, s/\overline{\kappa} - a_6)]^{\overline{\kappa}},$$

where

$$(3.23)\qquad \overline{\kappa} = \min\{\widetilde{\kappa}, \widetilde{\kappa}/r', \kappa\} > 1.$$

Using Moser's iteration process we obtain from (3.22) the boundedness of $u(x)$ in $B(x_0, d/2)$ provided for some positive k_0 and s_0,

$$(3.24)\qquad \sup_{N>0} I_N(k_0, s_0) < \infty.$$

We know that $u \in L_{q\kappa,\mathrm{loc}}(\Omega, \widetilde{v}) \cap L_{q\widetilde{\kappa},\mathrm{loc}}(\Omega, v_1)$. Thus (3.24) is valid if

$$(3.25)\qquad 2k_0 + p_0 \le q\widetilde{\kappa}, \quad k_0 r_0' \le q\widetilde{\kappa}, \quad k_0 \le q\kappa.$$

We can satisfy inequalities (3.25) by a suitable choice of k_0. In this way we proved Theorem 2.2.

4. Proof of Theorem 2.3

Let x_0 be an arbitrary point in Ω_d. For $0 < R < d$ we define

$$(4.1)\qquad \begin{aligned}\omega_1(R) &= \operatorname{ess\,inf}\{u(x) : x \in B(x_0, R)\},\\ \omega_2(R) &= \operatorname{ess\,sup}\{u(x) : x \in B(x_o, R)\},\end{aligned}$$

$$(4.2)\qquad \omega(R) = \omega_2(R) - \omega_1(R).$$

For given x_0 and R we shall consider two posibilities:

$$(4.3)\qquad \operatorname{meas} E(R) \ge \tfrac{1}{2}\operatorname{meas} B(x_0, R)$$

and

$$(4.4)\qquad \operatorname{meas}\{B(x_0,R) \setminus E(R)\} > \tfrac{1}{2}\operatorname{meas} B(x_0, R),$$

where

$$(4.5)\qquad E(R) = \{x \in B(x_0, R) : u(x) \ge (\omega_1(R) + \omega_2(R))/2\}.$$

If (4.3) holds, we will prove that an auxiliary function

$$\ln \frac{e\omega(R)}{z(x)}, \quad z(x) = u(x) - \omega_1(R) + R^\delta, \tag{4.6}$$

is estimated in the ball $B(x_0, R/2)$ by a constant independent of R. In the case (4.4) it is sufficient to repeat the whole discussion for another auxiliary function. The number δ in (4.6) will be chosen later, and e is a natural number.

We substitute in the integral identity (2.7) the test function

$$\varphi(x) = \frac{1}{z^{q-1}(x)} \left[\ln \frac{e\omega(R)}{z(x)} \right]^k \chi^s(x), \tag{4.7}$$

where $\chi(x)$ is a smooth function such that

$$\chi(x) = \begin{cases} 1 & \text{for } x \in B(x_0, R/2), \\ 0 & \text{for } x \notin B(x_0, R/2), \end{cases} \quad |D^\alpha \chi(x)| \le C/R^{|\alpha|} \quad \text{for } |\alpha| \le m. \tag{4.8}$$

We will assume that

$$\omega(R) \ge R^\delta. \tag{4.9}$$

We have

$$\begin{aligned} D^\alpha \varphi(x) = & -\left\{ (q-1) \left[\ln \frac{e\omega(R)}{z(x)} \right]^k + k \left[\ln \frac{e\omega(R)}{z(x)} \right]^{k-1} \right\} \\ & \times \frac{1}{z^q(x)} D^\alpha u(x) \chi^s(x) + \widetilde{R}_\alpha(x) \end{aligned} \tag{4.10}$$

with the pointwise estimate

$$\begin{aligned} |\widetilde{R}_\alpha| \le C_{13}(k+s)^m & \frac{1}{z^{q-1}} \left[\ln \frac{e\omega(R)}{z} \right]^k \\ & \times \left\{ \sum_{1 \le |\beta| < |\alpha|} \frac{|D^\beta u|^{|\alpha|/|\beta|}}{|z|^{|\alpha|/|\beta|}} + \frac{1}{R^{|\alpha|}} \right\} \chi^{s-m}. \end{aligned} \tag{4.11}$$

After the substitution of $\varphi(x)$ from (4.7) in (2.7) and using (4.10), (4.11), and conditions (1.4), (1.5) we obtain

$$\begin{aligned} \int_\Omega & \frac{1}{z^q} \left[\ln \frac{e\omega(R)}{z} \right]^k \{ v_m |D^m u|^p + v_1 |D^1 u|^q \} \chi^s \, dx \\ & \le C_{14}(k+s)^{a_7} \int_\Omega \left[\ln \frac{e\omega(R)}{z(x)} \right]^k \\ & \quad \times \left\{ \frac{1}{z^q} \left[\sum_{1<|\beta|<m} |D^\beta u|^{p_\beta} v_\beta + [1+f] v_1 \right] + \frac{1}{R^q} v_1 \right\} \chi^{s-a_7} \, dx. \end{aligned} \tag{4.12}$$

Now we transform and estimate the terms on the right-hand side of (4.12) with derivatives of $u(x)$. As in Section 3 we use integration by parts and Young's inequality. For $|\alpha| = j > 2$, $\alpha = \beta + \gamma$, $|\beta| = j - 1$, $|\gamma| = 1$, we have

$$\begin{aligned}(4.13)\qquad &\int_\Omega \frac{1}{z^q}\left[\ln\frac{e\omega(R)}{z}\right]^k |D^\alpha u|^{p_\alpha} v_\alpha \chi^{s-a_7}\,dx \\ &= -\int_\Omega D^\beta u \frac{1}{z^q}\left[\ln\frac{e\omega(R)}{z}\right]^k |D^\alpha u|^{p_\alpha-2} v_\alpha \chi^{s-a_7} \\ &\quad\times\left\{(p_\alpha-1)D^{\alpha+\gamma}u + \frac{D^\alpha u}{v_\alpha}D^\gamma v_\alpha + (s-a_7)D^\alpha u\cdot\frac{1}{\chi}D^\gamma\chi\right. \\ &\quad\left. - k\left[\ln\frac{e\omega(R)}{z}\right]^{-1}\frac{D^\alpha u}{z}D^\gamma u - q\frac{D^\alpha u}{z}D^\gamma u\right\}dx.\end{aligned}$$

We estimate the terms on the right-hand side of (4.13). We have, for $\chi \neq 0$,

$$\begin{aligned}(4.14)\qquad &k v_\alpha |D^\beta u|\cdot|D^\alpha u|^{p_\alpha-1}\frac{1}{z}|D^\gamma u|\cdot\left[\ln\frac{e\omega(R)}{z}\right]^{-1}\cdot\chi^{-a_7} \\ &\le \varepsilon\{v_\beta|D^\beta u|^{p_\beta} + v_\alpha|D^\alpha u|^{p_\alpha} + v_1|D^1 u|^q\} + C_{15}\kappa^{a_8}\varepsilon^{-a_8}v_1\frac{1}{R^{\delta a_8}}\chi^{-a_8}.\end{aligned}$$

We formulate the first assumption on δ:

$$(4.15)\qquad \delta q + \delta\left(\frac{|\alpha|}{|\beta|}-1\right)p_\alpha\frac{|\beta|p_\beta}{|\beta|p_\beta - |\alpha|p_\alpha} \le q, \quad \delta q + \delta a_8 \le q.$$

Note that the first inequality of (4.15) was used in the proof of the inequality (4.12).

We estimate another term of (4.13):

$$\begin{aligned}(4.16)\qquad &(s-a_7)v_\alpha|D^\beta u|\cdot|D^\alpha u|^{p_\alpha-1}\chi^{-1-a_7}|D^\gamma\chi| \\ &\le \varepsilon\{v_\beta|D^\beta u|^{p_\beta} + v_\alpha|D^\alpha u|^{p_\alpha}\} + C_{16}s^{r_\alpha}\varepsilon^{-r_\alpha}v_1(1/R)^{r_\alpha}\chi^{-(1+a_7)r_\alpha},\end{aligned}$$

where r_α is determined by the condition

$$\frac{1}{p_\beta} + \frac{p_\alpha - 1}{p_\alpha} + \frac{1}{r_\alpha} = 1.$$

This r_α satisfies the inequality

$$(4.17)\qquad r_\alpha < q_1 < q$$

and we formulate the second assumption on δ:

$$(4.18)\qquad r_\alpha + \delta q \le q \quad \text{for } 1 < |\alpha| < m.$$

By using estimates (4.14), (4.16), (3.8) and (3.11) we obtain from (4.13) the inequality

$$
\begin{aligned}
(4.19)\quad J_j(s-a_7) \le \varepsilon(J_{j+1}(s)+J_j(s)) &+ C_{17}\varepsilon^{-a_9}J_{j-1}(s-a_9)\\
&+ C_{17}(k+s)^{a_9}\varepsilon^{-a_9}\int_\Omega \left[\ln\frac{e\omega(R)}{z}\right]^k\left\{\frac{\widetilde{v}}{z^q}+\frac{v_1}{R^q}\right\}\chi^{s-a_9}\,dx
\end{aligned}
$$

for $j > 2$, where

$$
(4.20)\qquad J_j(s) = \int_\Omega \frac{1}{z^q}\left[\ln\frac{e\omega(R)}{z}\right]^k \sum_{|\alpha|=j}|D^\alpha u|^{p_\alpha} v_\alpha \chi^s\,dx.
$$

For $j = 2$ we obtain instead of (4.19) the inequality

$$
\begin{aligned}
(4.21)\quad & J_2(s-a_7)\\
&\le \varepsilon(J_1(s)+J_2(s)+J_3(s))\\
&\quad + C_{18}(k+s)^{a_{10}}\varepsilon^{-a_{10}}\int_\Omega \left[\ln\frac{e\omega(R)}{z}\right]^k\left\{\frac{\widetilde{v}}{z^q}+\frac{v_1}{R^q}\right\}\chi^{s-a_{10}}\,dx.
\end{aligned}
$$

Using estimates (4.19) and (4.21) we obtain by induction the inequality

$$
\begin{aligned}
(4.22)\quad & \sum_{j=2}^{m-1} J_j(s-a_7)\\
&\le \varepsilon\{J_m(s)+J_1(s)\}\\
&\quad + C_{19}(k+s)^{a_{11}}\varepsilon^{-a_{11}}\int_\Omega \left[\ln\frac{e\omega(R)}{z}\right]^k\left\{\frac{\widetilde{v}}{z^q}+\frac{v_1}{R^q}\right\}\chi^{s-a_{11}}\,dx.
\end{aligned}
$$

From (4.12) and (4.22) we get

$$
\begin{aligned}
(4.23)\quad & \int_\Omega \frac{1}{z^q}\left[\ln\frac{e\omega(R)}{z}\right]^k\{v_m|D^m u|^p + v_1|D^1 u|^q\}\chi^s\,dx\\
&\le C_{20}(k+s)^{a_{12}}\int_\Omega \left[\ln\frac{e\omega(R)}{z}\right]^k\left\{\frac{[1+f]v_1+\widetilde{v}}{z^q}+\frac{v_1}{R^q}\right\}\chi^{s-a_{12}}\,dx.
\end{aligned}
$$

We introduce

$$
(4.24)\quad J_R(k,s) = \frac{R^q}{v_1(B(x_0,R))}\int_\Omega \left[\ln\frac{e\omega(R)}{z}\right]^k\left\{\frac{[1+f]v_1+\widetilde{v}}{R^{\delta q}}+\frac{v_1}{R^q}\right\}\chi^s\,dx
$$

and we prove that

$$
(4.25)\qquad J_R(q,s) \le B_1
$$

for some $s_1 > 0$, with a constant B_1 depending only on the known parameters and the norm of u, and independent of R.

Note that from the definition of the class A_∞,

$$\widetilde{v}(B(x_0,R)) \le K_3 R^{\widetilde{\lambda}}, \quad v(B(x_0,R)) \le K_4 R^{\lambda_1} \tag{4.26}$$

with constants K_3 and K_4 independent of R.

Introduce the function

$$g(x) = \ln \frac{e\omega(R)}{u(x) - \omega_1(R) + R^\delta} = \ln \frac{e\omega(R)}{z(x)} \tag{4.27}$$

for $x \in B(x_0,R)$. By condition (4.3) we have, for $x \in E(R)$,

$$g(x) \le \ln \frac{e\omega(R)}{\omega(R)/2 + R^\delta} \le \ln 2e. \tag{4.28}$$

Then by the Hölder inequality and Poincaré inequality [2] we obtain the estimate

$$\begin{aligned}
\frac{1}{v_1(B(x_0,R))} \int_{B(x_0,R)} |g|^q v_1\,dx
&\le C_{21}\Big\{1 + \frac{1}{v_1(B(x_0,R))} \int_{B(x_0,R)} [g - \ln 2e]_+^q v_1\,dx\Big\} \\
&\le C_{21}\Big\{1 + \Big[\frac{1}{v_1(B(x_0,R))} \int_{B(x_0,R)} [g - \ln 2e]_+^{q\widetilde{\kappa}} v_1\,dx\Big]^{1/\widetilde{\kappa}}\Big\} \\
&\le C_{22}\Big\{1 + \frac{R^q}{v_1(B(x_0,R))} \int_{B(x_0,R)} \frac{1}{z^q}\Big|\frac{\partial u}{\partial x}\Big|^q v_1\,dx\Big\}.
\end{aligned} \tag{4.29}$$

From (4.23) with $k=0$ and (4.29) we have, for $s_1 = a_{12}+1$,

$$\begin{aligned}
J_R(q,s_1) &\le C_{23}\Big\{\frac{R^q}{v_1(B(x_0,R))} \int_\Omega \Big(\Big[\ln\frac{e\omega(R)}{z}\Big]^q + 1\Big) \\
&\qquad \times \frac{[1+f]v_1 + \widetilde{v}}{R^{\delta q}} \chi\,dx + 1\Big\} \\
&\le C_{24}\Big\{\frac{R^q}{v_1(B(x_0,R))} \Big(\Big[\ln\frac{2eM_{d/2}}{R^\delta}\Big]^q + 1\Big) \\
&\qquad \times \frac{\|1+f\|_{L_r(\Omega,v_1)}[v_1(B(x_0,R))]^{1-1/r} + \widetilde{v}(B(x_0,R))}{R^{\delta q}} + 1\Big\}.
\end{aligned} \tag{4.30}$$

The right-hand side of the last inequality is bounded by a constant independent of R if we choose δ satisfying

$$R^{q-2\delta q}[v_1(B(x_0,R))]^{-1/r} \le C_{25}, \tag{4.31}$$

$$R^{q-2\delta q}\widetilde{v}(B(x_0,R))[v_1(B(x_0,R))]^{-1} \le C_{25}. \tag{4.32}$$

Now (2.4) yields

$$R^{n\widetilde{q}} \le K_4 v_1(B(x_0,R)) \tag{4.33}$$

and hence (4.31) is satisfied provided

$$q - 2\delta q - n\widetilde{q}/r > 0. \tag{4.34}$$

The possibility of choosing a positive value of δ is guaranteed by (2.6).

In order to check (4.32) we remark that (2.2) implies

$$R\widetilde{v}(B(x_0,R))^{1/(q\kappa)} \le K_5 v_1(B(x_0,R))^{1/q}. \tag{4.35}$$

Using (4.26) and (4.35) we obtain the estimate (4.32) if

$$-2\delta q + (1 - 1/\kappa)\widetilde{\lambda} > 0. \tag{4.36}$$

So we have proved the inequality (4.25) by a suitable choice of δ and s_1.

Now we will organize Moser's iteration process for $J_R(k,s)$. For this we estimate various summands in $J_R(k,s)$ by imbedding theorems. Using the imbeddings (2.9), (3.19) and the inequalities (4.23), (4.32), (4.33) one can prove the estimate

$$J_R(k,s) \le C_{26}(k+s)^{a_{13}} J_R(k/\overline{\kappa}, s/\overline{\kappa} - a_{13})^{\overline{\kappa}} \tag{4.37}$$

with $\overline{\kappa}$ defined by (3.23).

Using (4.25) and (4.37) in Moser's iteration process we see that for

$$k_i = q\overline{\kappa}^i, \quad s_i = a_{13}\frac{\overline{\kappa}}{\overline{\kappa}-1} + s_1\overline{\kappa}^i$$

the inequality

$$J_R(k_i,s_i)^{1/k_i} \le C_{27}$$

holds and consequently

$$\frac{\omega(R)}{u(x) - \omega_1(R) + R^\delta} \le C_{28} \quad \text{for } x \in B(x_0, R/2).$$

From the last estimate we obtain

$$\omega(R/2) \le \omega(R)[1 - 1/C_{28}] + R^\delta. \tag{4.38}$$

So we have proved that for each $R \in (0,d]$, either (4.38) holds or $\omega(R) < R^\delta$ (if (4.9) fails). Now, the proof of Theorem 2.3 is completed in a standard way.

5. Proof of Theorem 2.4

The proof of (2.13) is analogous to the proof of Theorem 2.2. Now we substitute in (2.7) the test function

$$\varphi = [1 + \lambda_N^2(u)]^k u, \tag{5.1}$$

where $\lambda_N(u)$ is the same as in (3.1). We repeat the argument of Section 3 and prove the boundedness of $u(x)$.

The proof of Hölder continuity near the boundary is analogous to the proof in Section 4. Let $x_0 \in \partial\Omega$ and $R \in (0, R_0)$, where R_0 is the number from condition (b).

We introduce

$$\begin{aligned} \omega_1'(R) &= \operatorname{ess\,inf}\{u(x) : x \in B(x_0, R) \cap \Omega\}, \\ \omega_2'(R) &= \operatorname{ess\,sup}\{u(x) : x \in B(x_0, R) \cap \Omega\}, \\ \omega'(R) &= \omega_2'(R) - \omega_1'(R). \end{aligned} \tag{5.2}$$

Since $u(x) = 0$ on $\partial\Omega$ we have $\omega_1'(R) \leq 0$ and $\omega_2'(R) \geq 0$. Analogously to (4.9) we will assume that

$$\omega'(R) \geq R^{\delta'} \tag{5.3}$$

with some δ' depending only on the known parameters.

Consider two possibilities:

$$\omega_2'(R) \geq \frac{\omega'(R)}{2}, \quad -\omega_1'(R) > \frac{\omega'(R)}{2}. \tag{5.4}$$

One of these inequalities, say the second, holds. In this case we substitute in (2.7) the test function

$$\varphi(x) = \left\{ \frac{1}{[z'(x)]^{q-1}} - \frac{1}{[-\omega_1'(R) + R^{\delta'}]^{q-1}} \right\} \left[\ln \frac{e\omega'(R)}{z'(x)} \right]^k \chi^s(x), \tag{5.5}$$

where $z'(x) = u(x) - \omega_1'(R) + R^{\delta'}$. If the first inequality of (5.4) is valid we use a different test function. In (5.5), the numbers k, s and the function $\chi(x)$ are the same as in (4.7).

Using the reasonings of Section 4 we prove Hölder continuity near the boundary. We only make two remarks:

1) When applying the Poincaré inequality as in deriving (4.29), we use condition (b).

2) In the considered case (the second inequality of (5.4) valid) we have the estimate

$$\left| \frac{1}{[z'(x)]^{q-1}} - \frac{1}{[-\omega_1'(R) + R^{\delta'}]^{q-1}} \right| \leq \frac{2^{q-1}}{[z'(x)]^{q-1}}. \tag{5.6}$$

Indeed, this is trivial for x with $u(x) \leq 0$. If $u(x) > 0$ we have from (5.4),

$$\begin{aligned} \left| \frac{1}{[z'(x)]^{q-1}} - \frac{1}{[-\omega_1'(R) + R^{\delta'}]^{q-1}} \right| &\leq \frac{1}{[-\omega_1'(R) + R^{\delta'}]^{q-1}} \leq \frac{2^{q-1}}{[\omega'(R) + R^{\delta'}]^{q-1}} \\ &\leq \frac{2^{q-1}}{[z'(x)]^{q-1}}. \end{aligned}$$

Repeating the argument of Section 4 we complete the proof of Theorem 2.4.

6. Proof of Theorem 2.5

Under the conditions of Theorem 2.5 we can make substitutions of the type (5.1) or (4.7) (for $x_0 \in \partial\Omega$), but in this case the corresponding transformation of the integral with derivatives (as the integral on the left-hand side of (3.5)) is nontrivial. If we transform this integral by using integration by parts and if $\psi(x)$ is not equal to zero on $\partial\Omega$ then an integral on $\partial\Omega$ arises which is difficult to estimate.

We use another way connected with extension of functions outside Ω. We explain this approach by the example of the integral on the left-hand side of (3.5).

Let $x_0 \in \partial\Omega$, and let $\psi(x)$ be equal to one in $B(x_0, \overline{R})$, and zero outside $B(x_0, 2\overline{R}_0)$, where $\overline{R}_0$ is some fixed number. So we will estimate the derivatives of $\psi(x)$ by constants. We assume that the integral on the left-hand side of (3.5) is transformed into local coordinates such that

$$\Omega \cap B(x_0, 2\overline{R}_0) = B_+ = B_+(x_0, 2\overline{R}_0) = \{x \in B(x_0, 2\overline{R}_0) : x_n > 0\}.$$

Let $B_- = B_-(x_0, 2\overline{R}_0) = \{x \in B(x_0, 2\overline{R}_0) : x_n < 0\}$.

We have

$$\begin{aligned} I_+(\alpha, s) &= \int_{B_+} v_\alpha |D^\alpha u|^{p_\alpha} [1 + \lambda_N^2(u)]^k \psi^s \, dx \\ &\le \int_{B(x_0, 2\overline{R})} F_\alpha^{p_\alpha} H^{1-p_\alpha} \, dx = I(\alpha, s), \end{aligned} \tag{6.1}$$

where

$$F_\alpha(x) = \begin{cases} v_\alpha(x) |D^\alpha u(x)| \cdot [1 + \lambda_N^2(u(x))]^k \psi^s(x), & x \in B_+, \\ d_1 F_\alpha(x_1^*) + d_2 F_\alpha(x_2^*), & x \in B_-, \end{cases} \tag{6.2}$$

$$H(x) = \begin{cases} H_1(x) H_m(x) \widetilde{H}(x), & x \in B_+, \\ \frac{1}{8}[H_1(x_1^*) + H_1(x_2^*)] & \\ \quad \times [H_m(x_1^*) + H_m(x_2^*)][\widetilde{H}(x_1^*) + \widetilde{H}(x_2^*)], & x \in B_-, \end{cases} \tag{6.3}$$

$$\begin{aligned} H_1(x) &= [v_1(x)]^{(m-|\alpha|)p_\alpha/(q_1(m-1))}, \\ H_m(x) &= [v_m(x)]^{(|\alpha|-1)p_\alpha/(p(m-1))}, \\ \widetilde{H}(x) &= [1 + \lambda_N^2(u(x))]^k \psi^s(x). \end{aligned}$$

Here $x_1^* = (x_1, \dots, x_{n-1}, -x_n)$, $x_2^* = (x_1, \dots, x_{n-1}, -2x_n)$ and

$$d_1 = -3, \quad d_2 = 4. \tag{6.4}$$

We assume further that $\alpha = \beta + \gamma$, $|\gamma| = 1$, $\gamma = (0, \dots, 0, 1) = e_n$. If $\gamma \ne e_n$ with $|\gamma| = 1$ it is possible to repeat all the discussion of Section 3.

We have

$$F_\alpha(x) = D^\gamma \widetilde{F}_{\alpha\beta}(x) + G_{\beta\gamma}(x), \tag{6.5}$$

where

$$\begin{aligned} &\widetilde{F}_{\alpha\beta}(x) = \begin{cases} v_\alpha(x) D^\beta u(x)[1+\lambda_N^2(u(x))]^k \psi^s(x), & x \in B_+, \\ -d_1 \widetilde{F}_{\alpha\beta}(x_1^*) - \frac{1}{2} d_2 \widetilde{F}_{\alpha\beta}(x_2^*), & x \in B_-, \end{cases} \\ &G_{\beta\gamma}(x) = \begin{cases} -D^\beta u(x) D^\gamma \{v_\alpha(x)[1+\lambda_N^2(u(x))]^k \psi^s(x)\}, & x \in B_+, \\ d_1 G_{\beta\gamma}(x_1^*) + d_2 G_{\beta\gamma}(x_2^*), & x \in B_-. \end{cases} \end{aligned} \tag{6.6}$$

Now we can transform $I(\alpha, s)$ defined by the right-hand side of (6.1) using integration by parts:

$$\begin{aligned} I(\alpha, s) = &\int_{B(x_0,2R)} F_\alpha^{p_\alpha - 2} F_\alpha \{D^\gamma \widetilde{F}_{\alpha\beta} + G_{\beta\gamma}\} H^{1-p_\alpha}\, dx \\ &= -\int_{B(x_0,2R)} \widetilde{F}_{\alpha\beta} D^\gamma \{F_\alpha^{p_\alpha-2} H^{1-p_\alpha} F_\alpha\}\, dx \\ &\quad + \int_{B(x_0,2R)} F_\alpha^{p_\alpha-2} F_\alpha H^{1-p_\alpha} G_{\beta\gamma}\, dx. \end{aligned} \tag{6.7}$$

Now we have to estimate the integral on the right-hand side of (6.7) corresponding to $B_-(x_0, 2R)$. We consider one typical term:

$$\begin{aligned} |I_-^{(1)}(\alpha, s)| &= \left| \int_{B_-(x_0,2R)} \widetilde{F}_{\alpha,\beta}(x) D^\gamma \{F_\alpha(x)^{p_\alpha-2} F_\alpha(x)\} H(x)^{1-p_\alpha}\, dx \right| \\ &\le C_{29}(k+s) \sum_{i=1}^{2} \int_{B_-(x_0,2R)} \widetilde{F}_{\alpha\beta}(x) F_\alpha(x)^{p_\alpha-2} H(x)^{1-p_\alpha} \\ &\quad \times \{|D^{\alpha+\gamma} u(x_i^*)| v_\alpha(x_i^*)[1+\lambda_N^2(u(x_i^*))]^k \psi^s(x_i^*) \\ &\quad + |D^\alpha u(x_i^*)| v_\alpha(x_i^*)[1+\lambda_N^2(u(x_i^*))]^k D^\gamma u(x_i^*) \psi^s(x_i^*) \\ &\quad + |D^\alpha u(x_i^*)| D^\gamma v_\alpha(x_i^*)[1+\lambda_N^2(u(x_i^*))]^k \psi^s(x_i^*) \\ &\quad + |D^\alpha u(x_i^*)| v_\alpha(x_i^*)[1+\lambda_N^2(u(x_i^*))]^k \psi^{s-1}(x_i^*)\}\, dx. \end{aligned} \tag{6.8}$$

Here and below the constants C_i depend only on the known parameters.

We demonstrate the estimation of the right-hand side of (6.8) on one typical term. For $|\alpha| > 2$ we have

$$
\begin{aligned}
(6.9)\quad & I_-^{(2)}(\alpha,s) \\
&= \sum_{i=1}^{2}\int_{B_-(x_0,2R)} \widetilde{F}_{\alpha\beta}(x)F_\alpha(x)^{p_\alpha-2}H(x)^{1-p_\alpha} \\
&\quad\times |D^{\alpha+\gamma}u(x_i^*)|v_\alpha(x_i^*)[1+\lambda_N^2(u(x_i^*))]^k\psi^s(x_i^*)\,dx \\
&\le C_{30}\sum_{i,j,l=1}^{2}\int_{B_-(x_0,2R)}\{v_\beta(x_j^*)|D^\beta u(x_j^*)|^{p_\beta}[1+\lambda_N^2(u(x_j^*))]^k\psi^s(x_j^*)\}^{1/p_\beta} \\
&\quad\times\{|D^\alpha u(x_l^*)|^{p_\alpha}v_\alpha(x_l^*)[1+\lambda_N^2(u(x_l^*))]^k\psi^s(x_l^*)\}^{(p_\alpha-2)/p_\alpha} \\
&\quad\times\{|D^{\alpha+\gamma}u(x_i^*)|^{p_{\alpha+\gamma}}v_{\alpha+\gamma}(x_i^*)[1+\lambda_N^2(u(x_i^*))]^k\psi^s(x_i^*)\}^{1/p_{\alpha+\gamma}}R_{ijl}(x)\,dx,
\end{aligned}
$$

where

$$
\begin{aligned}
R_{ijl}(x) &= v_\alpha(x_j^*)[v_\beta(x_j^*)]^{-1/p_\beta}v_\alpha(x_i^*)[v_{\alpha+\gamma}(x_i^*)]^{-1/p_{\alpha+\gamma}} \\
&\quad\times[v_\alpha(x_l^*)]^{(p_\alpha-2)(1-1/p_\alpha)}\{[H_1(x_1^*)+H_1(x_2^*)][H_m(x_1^*)+H_m(x_2^*)]\}^{1-p_\alpha}.
\end{aligned}
$$

Now we check that $R_{ijl}(x)\le 1$. We have

$$
\begin{aligned}
(6.10)\quad R_{ijl}(x) &\le v_\alpha(x_j^*)[v_\beta(x_j^*)]^{-1/p_\beta}v_\alpha(x_i^*)[v_{\alpha+\gamma}(x_i^*)]^{-1/p_{\alpha+\gamma}} \\
&\quad\times\{[H_1(x_1^*)+H_1(x_2^*)][H_m(x_1^*)+H_m(x_2^*)]\}^{(p_\alpha-2)(1-1/p_\alpha)+1-p_\alpha} \\
&= [v_m(x_j^*)]^{\frac{|\alpha|-1}{p(m-1)}p_\alpha-\frac{|\beta|-1}{p(m-1)}}[v_1(x_j^*)]^{\frac{m-|\alpha|}{q_1(m-1)}p_\alpha-\frac{m-|\beta|}{q_1(m-1)}} \\
&\quad\times[v_m(x_i^*)]^{\frac{|\alpha|-1}{p(m-1)}p_\alpha-\frac{|\alpha+\gamma|-1}{p(m-1)}}[v_1(x_i^*)]^{\frac{m-|\alpha|}{q_1(m-1)}p_\alpha-\frac{m-|\alpha+\gamma|}{q_1(m-1)}} \\
&\quad\times\{[H_1(x_1^*)+H_1(x_2^*)][H_m(x_1^*)+H_m(x_2^*)]\}^{2/p_\alpha-2}.
\end{aligned}
$$

Note that

$$
(|\alpha|-1)p_\alpha-(|\alpha+\gamma|-1)\ge 0,\quad (m-|\alpha|)p_\alpha-(m-|\beta|)\ge 0.
$$

So the right-hand side of (6.10) is not greater than

$$
\begin{aligned}
&[H_1(x_1^*)+H_1(x_2^*)]^{2-2/p_\alpha}[H_m(x_1^*)+H_m(x_2^*)]^{2-2/p_\alpha} \\
&\qquad\times\{[H_1(x_1^*)+H_1(x_2^*)][H_m(x_1^*)+H_m(x_2^*)]\}^{2/p_\alpha-2}=1.
\end{aligned}
$$

Using the above estimates and applying Young's inequality we obtain

$$
(6.11)\qquad I_-^{(2)}(\alpha,s)\le\varepsilon\{I_+(\alpha,s)+I_+(\alpha+\gamma,s)\}+C_{31}\varepsilon^{-p_\beta}I_+(\beta,s).
$$

We also use the transformation of variables of the type $x_j^*=y$.

The estimate (6.11) is analogous to the estimate for the corresponding term of the right-hand side of (3.5) which follows from (3.6). In that way it is possible to estimate the other terms of the right-hand side of (6.7). Thus using the described method which is based on prolongation of functions outside Ω and the discussions of Sections 3, 4 we get the assertion of Theorem 2.5.

7. Sketch of proof of Theorem 2.1

The proof of the estimate (2.8) is based on arguments analogous to those of the preceding sections. Using a partition of unity we reduce the estimation of the left-hand side of (2.8) to that of the integral

$$\int_\Omega |D^k u|^{p_k} v_k \varphi^q \, dx \tag{7.1}$$

with a smooth cut-off function $\varphi(x)$.

If $\operatorname{supp}\varphi \cap \partial\Omega = \emptyset$ we repeat the reasoning of Section 3. We transform the integral in (7.1) using integration by parts (analogously to the equality (3.5)). Then we estimate the resulting terms by Hölder's inequality.

If $\operatorname{supp}\varphi \cap \partial\Omega \neq \emptyset$ the transformation of the integral (7.1) and its estimation are based on extension of functions outside Ω and the arguments of Section 6. In that way we establish the inequality

$$I_k \le C I_{k-1}^{1/2}(I_{k+1}^{1/2} + I_{k-1}^{1/2}) \quad \text{for } I_k = \sum_{|\alpha|=k} \int_\Omega |D^\alpha u|^{p_k} v_k \, dx \tag{7.2}$$

with some constant C depending only on the known parameters. From (7.2) we obtain the estimate (2.8) by induction.

8. Example and counterexample

Take a weight function of the type

$$v_\alpha(x) = \{\operatorname{dist}(x, E)\}^{\lambda_\alpha},$$

where E is some subset of $\overline{\Omega}$. For instance we can take $E = \{x_0\}$, $x_0 \in \Omega$,

$$v_1(x) = |x - x_0|^{\lambda_1}, \quad v_m(x) = |x - x_0|^{\lambda_m}. \tag{8.1}$$

For this choice, conditions (1.9) are satisfied if we assume that

$$-n < \lambda_1 < n(q-1), \quad -n < \lambda_m < n(p-1), \quad \lambda_m \ge \lambda_1. \tag{8.2}$$

Condition (w) is satisfied for example if

$$\lambda_m q_1 - \lambda_1 p > (m-1)pq_1 - (q_1 - p)n. \tag{8.3}$$

So under assumptions (8.2), (8.3) for the weight functions defined by (8.1) all the preceding results are valid.

Now we construct a counterexample to show that our conditions are essential. We cannot weaken conditions (1.7) because [8] gives an example, for $q = mp$, of an equation of the considered structure with an unbounded solution. An analogous example shows that the condition

$$v_m(x) \le K_1 v_1(x) \tag{8.4}$$

in (1.9) is essential.

Consider the equation

$$\begin{aligned}\sum_{k,l=1}^{n} \frac{\partial^2 u}{\partial x_k \partial x_l}\bigg\{|D^2u|^{p-2}|x|^{\lambda_2}\bigg[\sigma_1 \frac{\partial^2 u}{\partial x_k \partial x_l} \\ + \sum_{i,j=1}^{n}\left(\frac{x_i x_j}{|x|^2} + \sigma_2 \delta_i^j\right)\left(\frac{x_l x_k}{|x|^2} + \sigma_2 \delta_k^l\right)\frac{\partial^2 u}{\partial x_i \partial x_j}\bigg]\bigg\} \\ - \sigma_3 \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left\{|D^1 u|^{q-2}|x|^{\lambda_1}\frac{\partial u}{\partial x_i}\right\} = 0.\end{aligned} \tag{8.5}$$

A calculation shows that for a suitable choice of σ_1, σ_2, σ_3 and under the condition

$$\lambda_2 - 2p = \lambda_1 - q > -n \tag{8.6}$$

the equation (8.5) has a solution $u(x) = \ln|x| \in W_p^2(B, v_2) \cap W_q^1(B, v_1)$ in $B = B(0,1)$ with $v_2(x) = |x|^{\lambda_2}$ and $v_1(x) = |x|^{\lambda_1}$. In fact, σ_1 and σ_3 can be chosen to be positive.

Let now the inequality (8.4) be not valid, so $\lambda_2 < \lambda_1$. If we now choose

$$q = 2p + (\lambda_1 - \lambda_2) > 2p \tag{8.7}$$

we can satisfy all conditions on $v_1(x)$ and $v_2(x)$ in our paper except the condition (8.4). But in this case we have an unbounded solution $u(x) = \ln|x|$. This shows that the condition (8.4) is essential.

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Manuscript received October 2, 1995

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