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AUTOMORPHISMS OF POSTLIMINAL C^*-ALGEBRAS

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Let $\alpha(\mathfrak{A})$ denote the group of automorphisms of a C^*-algebra $\mathfrak{A}$. The object of this paper is to give an intrinsic algebraic characterization of those elements α of $\alpha(\mathfrak{A})$ which are induced by a unitary operator in the weak closure of $\mathfrak{A}$ in every faithful representation, and it is attained for the class of C^*-algebras known as GCR, or more recently postliminal. The relevant condition is that α should map closed two-sided ideals of $\mathfrak{A}$ into themselves, and the main theorem (Theorem 2) may be thought of as an analogue for C^*-algebras of Kaplansky's theorem for von Neumann algebras, namely that an automorphism of a Type I von Neumann algebra is inner if and only if it leaves the centre elementwise fixed. The proof of Theorem 2 requires the—probably unnecessary—assumption that $\mathfrak{A}$ is separable.

By a C^*-algebra we mean a Banach algebra over the complex numbers, with a conjugate-linear anti-automorphic involution $A \to A^*$ satisfying $\|A^*A\| = \|A^*\|\cdot\|A\|$. The mappings of C^*-algebras which we consider (automorphisms, representations, etc.) will always be assumed to preserve the adjoint operation, and by a homomorphic image of a C^*-algebra $\mathfrak{A}$, we mean the image of a homomorphism from $\mathfrak{A}$ into another C^*-algebra $\mathfrak{B}$ (this is automatically a C^*-subalgebra of $\mathfrak{B}$ [2; 1.8.3]). We shall refer to Dixmier's book [2] for all standard results that we need to quote concerning C^*-algebras. By the theorem of Gelfand-Naĭmark (see, e.g. [2; 2.6.1]), a C^*-algebra has an isometric representation as an algebra of operators on a Hilbert space, and we shall usually think of a given C^*-algebra as being "concretely" represented on some Hilbert space. A *state* of a C^*-algebra $\mathfrak{A}$ is a positive linear functional of norm one. The set $\mathfrak{S}$ of states of $\mathfrak{A}$ is a convex subset of the (Banach) dual space of $\mathfrak{A}$. If $\mathfrak{A}$ has an identity element then $\mathfrak{S}$ is w^*-compact, but in any case $\mathfrak{S}$ contains an abundance of extreme points, which are called *pure states*. The set of pure states of $\mathfrak{A}$ will be denoted by $\mathfrak{P}$.

Given a state ρ of $\mathfrak{A}$, there is a representation ϕ_ρ of $\mathfrak{A}$ on a Hilbert space H_ρ, and a unit vector x_ρ in H_ρ such that $\{\phi_\rho(A)x_\rho: A \in \mathfrak{A}\}$ is dense in H_ρ (i.e. the representation ϕ_ρ is cyclic) and

$$\rho(A) = \langle \phi_\rho(A)x_\rho, x_\rho \rangle$$

for each $A \in \mathfrak{A}$. ϕ_ρ is irreducible if and only if ρ is pure. Given a state ρ of $\mathfrak{A}$, and a representation ϕ of $\mathfrak{A}$ on H, we say that ρ is a *vector state* (in the representation ϕ) if $\rho(A) = \langle \phi(A)x, x \rangle$ for some unit vector x in H; and if ϕ is faithful, we say that ρ is *normal* if the map $A \rightarrow \rho(A)$ is continuous with respect to the topology induced on $\phi(\mathfrak{A})$ by the ultra-weak topology on the algebra $\mathfrak{L}(H)$ of all bounded operators on H. It is clear that a vector state is normal. Let Φ denote the *universal representation* of $\mathfrak{A}$, formed by choosing one element from each unitary equivalence class of cyclic representations of $\mathfrak{A}$ and taking their direct sum; and let Ψ denote the *reduced atomic representation* of $\mathfrak{A}$, formed by choosing one element from each unitary equivalence class of irreducible representations of $\mathfrak{A}$ and taking their direct sum. Both Φ and Ψ are faithful representations, and every state [resp. every pure state] of $\mathfrak{A}$ is a vector state in the representation Φ [resp. Ψ].

Let $\hat{\mathfrak{A}}$ denote the structure space of $\mathfrak{A}$, i.e. the set of unitary equivalence classes of irreducible representations of $\mathfrak{A}$, with the Jacobson topology [2; § 3.1]. Following Dixmier, we shall call a C^*-algebra *liminal* if every irreducible representation consists of compact operators, *postliminal* if every nonzero homomorphic image has a nonzero closed two-sided liminal ideal, and *antiliminal* if it possesses no nonzero closed two-sided liminal ideals. If $\mathfrak{A}$ is postliminal then $\hat{\mathfrak{A}}$ is a T_0-space [2; 4.3.7 (ii)], and every representation of $\mathfrak{A}$ has a Type I von Neumann algebra as weak closure [2; 5.5.2]. Also, $\mathfrak{A}$ has a composition series $(I_\rho)_{0 \leqq \rho \leqq \delta}$ (i.e. an increasing nest of closed two-sided ideals of $\mathfrak{A}$ indexed by the ordinals less than or equal to some ordinal δ, such that $I_0 = (0), I_\delta = \mathfrak{A}$ and I_ρ is the closure of $\bigcup_{\rho' < \rho} I_{\rho'}$ for every limit ordinal $\rho \leqq \delta$) such that each difference algebra $I_{\rho+1} - I_\rho$ has Hausdorff structure space [2; 4.5.5 and 4.5.3].

Given a C^*-algebra $\mathfrak{A}$, we denote by $\alpha(\mathfrak{A})$ the group of automorphisms of $\mathfrak{A}$. Each element of $\alpha(\mathfrak{A})$ is an isometric isomorphism of $\mathfrak{A}$ onto itself [2; 1.3.7 and 1.8.1]. If ϕ is a faithful representation of $\mathfrak{A}$ on H, an automorphism α of $\mathfrak{A}$ is said to be *extendable* (in the representation ϕ) if there is an automorphism of the weak closure of $\phi(\mathfrak{A})$ which agrees with $\phi \circ \alpha \circ \phi^{-1}$ on $\phi(\mathfrak{A})$; and *weakly-inner* if $\phi(\alpha(A)) = U^* \phi(A) U$ for each A in $\mathfrak{A}$, where U is a unitary operator in the weak closure of $\phi(\mathfrak{A})$. If $\alpha(A) = U^* A U$ for a unitary operator U in $\mathfrak{A}$, then we say that α is *inner*. Following [6], we denote by $\varepsilon_\phi(\mathfrak{A})$ [resp. $\iota_\phi(\mathfrak{A})$] the set of elements of $\alpha(\mathfrak{A})$ which are extendable [resp. weakly-inner] in the representation ϕ, and by $\pi(\mathfrak{A})$ the intersection of all the sets $\iota_\phi(\mathfrak{A})$ as ϕ ranges through the faithful representations of $\mathfrak{A}$ (the elements of $\pi(\mathfrak{A})$ are called *permanently weakly-inner*, or *π-inner* automorphisms). The sets $\varepsilon_\phi(\mathfrak{A}), \iota_\phi(\mathfrak{A})$ and $\pi(\mathfrak{A})$ are all subgroups of $\alpha(\mathfrak{A})$. According to [6; Lemma 3], $\alpha \in \varepsilon_\phi(\mathfrak{A})$ if $\phi \circ \alpha \circ \phi^{-1}$ is ultra-weakly bicontinuous, equivalently if $\rho \circ \alpha$ is a normal state in the representation ϕ if and only if ρ is. It follows that $\varepsilon_\Phi(\mathfrak{A}) = \alpha(\mathfrak{A})$

since every state is normal in the universal representation.

If $\alpha \in \alpha(\mathfrak{A})$, we shall say that α *preserves ideals* if $\alpha(I) \subseteqq I$ for every closed two-sided ideal I of $\mathfrak{A}$, and that α preserves ideals *carefully* if $\alpha(I) = I$ for each such ideal I. We shall denote by $\tau(\mathfrak{A})$ [resp. $\tau_0(\mathfrak{A})$] the set of elements of $\alpha(\mathfrak{A})$ which preserve ideals [resp. preserve ideals carefully]. It is clear that $\tau_0(\mathfrak{A})$ is a subgroup of $\alpha(\mathfrak{A})$, and that $\tau(\mathfrak{A})$ is a subsemigroup of $\alpha(\mathfrak{A})$, but it is not clear whether $\tau(\mathfrak{A})$ can contain elements not in $\tau_0(\mathfrak{A})$ (*cf.* Corollary 1 of Theorem 1). Since an automorphism preserves the property of being a maximal ideal, an element of $\tau(\mathfrak{A})$ must preserve maximal two-sided ideals carefully, so that $\tau_0(\mathfrak{A}) = \tau(\mathfrak{A})$ if every closed two-sided ideal of $\mathfrak{A}$ is an intersection of maximal ones.

LEMMA 1. *For any C^*-algebra $\mathfrak{A}$, $\varepsilon_\Psi(\mathfrak{A}) = \alpha(\mathfrak{A})$.*

Proof. To save writing Ψ constantly, we shall suppose that $\mathfrak{A}$ is given in its reduced atomic representation. Let $\mathfrak{N}$ denote the closure in the norm topology on $\mathfrak{S}$ of the convex hull of $\mathfrak{P}$. Let $\alpha \in \alpha(\mathfrak{A})$, then it is easy to see that α preserves pure states, i.e. $\rho \in \mathfrak{P} \Leftrightarrow \rho \circ \alpha \in \mathfrak{P}$. Also, for any bounded linear functional f on $\mathfrak{A}$, $\|f \circ \alpha\| = \|f\|$. It follows that $\sigma \in \mathfrak{N} \Leftrightarrow \sigma \circ \alpha \in \mathfrak{N}$.

Let $\mathfrak{N}_0$ denote the set of normal states of $\mathfrak{A}$. We shall show that $\mathfrak{N}_0 = \mathfrak{N}$ from which it follows that α and α^{-1} preserve normal states and by [6; Lemma 3] the lemma will be proved. Now $\mathfrak{N}_0$ is norm-closed and convex, and contains $\mathfrak{P}$ since every pure state is a vector state in the given representation, hence $\mathfrak{N}_0 \supseteqq \mathfrak{N}$. Conversely, if $\rho \in \mathfrak{N}_0$, then ρ is a norm limit of convex combinations of vector states [1; Chap. I § 4 Théorème 1] so it will suffice to show that each vector state is in $\mathfrak{N}$.

Denote by ω_x the state $A \to \langle Ax, x\rangle$ where x is a unit vector in the space H on which $\mathfrak{A}$ acts. Since $\mathfrak{A}$ is given in the reduced atomic representation we can write $H = \bigoplus_{\gamma \in \Gamma} H_\gamma$ where each H_γ is a subspace of H invariant under $\mathfrak{A}$, and the restriction $\mathfrak{A}|_{H_\gamma}$ is irreducible. Write $x = \sum_{\gamma \in \Gamma} x_\gamma$, with $x_\gamma \in H_\gamma$. Then

$$\begin{aligned} A \in \mathfrak{A} &\Longrightarrow Ax_\gamma \in H_\gamma \quad \text{for each } \gamma \in \Gamma \\ &\Longrightarrow \langle Ax, x\rangle = \sum_{\gamma \in \Gamma} \langle Ax_\gamma, x_\gamma\rangle , \end{aligned}$$

so that

$$\omega_x = \sum_{\gamma \in \Gamma} \omega_{x_\gamma} , \quad \text{where} \quad \sum_{\gamma \in \Gamma} \| x_\gamma \|^2 = 1 . \tag{1}$$

But ω_{x_γ} is either zero (if $x_\gamma = 0$) or a multiple $\|x_\gamma\|^{-2}$ of a vector state of an irreducible representation, which is pure. It follows from

(1) that $\omega_x \in \mathfrak{N}$, showing that $\mathfrak{N}_0 \subseteqq \mathfrak{N}$.

LEMMA 2. *For any C^*-algebra $\mathfrak{A}$, $\iota_\Psi(\mathfrak{A}) \subseteqq \tau_0(\mathfrak{A})$.*

Proof. We shall again suppose that $\mathfrak{A}$ is given in its reduced atomic representation with weak closure $\mathfrak{A}^-$. Writing $H = \bigoplus_{\gamma\in\Gamma} H_\gamma$ as in Lemma 1, we have ([3]) $\mathfrak{A}^- = \bigoplus_{\gamma\in\Gamma} \mathfrak{L}(H_\gamma)$. If $\alpha \in \iota_\Psi(\mathfrak{A})$, let $U = \sum U_\gamma$ be a unitary in $\mathfrak{A}^-$ which induces α, where U_γ is a unitary operator on $H_\gamma(\gamma \in \Gamma)$. Let π_γ be the irreducible representation of $\mathfrak{A}$ on H_γ defined by $A \rightarrow A|_{H_\gamma}$ (for some $\gamma \in \Gamma$), and suppose $\pi_\gamma(A) = 0$. Then

$$\begin{aligned}\pi_\gamma(\alpha(A)) &= U^* A U|_{H_\gamma}\\ &= U_\gamma^* A U_\gamma\\ &= 0 .\end{aligned}$$

Thus a preserves the primitive ideal $\pi_\gamma^{-1}(0)$. But every primitive ideal is of this form, and every closed two-sided ideal in $\mathfrak{A}$ is an intersection of primitive ideals, hence α preserves ideals.

Since $\iota_\Psi(\mathfrak{A})$ is a group, α^{-1} also preserves ideals, and so α preserves ideals carefully.

As an immediate corollary to the above lemma, we have $\pi(\mathfrak{A}) \subseteqq \tau_0(\mathfrak{A})$ for any C^*-algebra $\mathfrak{A}$, a fact which has previously been noted by R. V. Kadison (private communication).

THEOREM 1. *If $\mathfrak{A}$ is a postliminal C^*-algebra, then $\tau(\mathfrak{A}) = \iota_\Psi(\mathfrak{A})$.*

Proof. We continue to assume that $\mathfrak{A}$ is given in the reduced atomic representation, and we shall use the notation established in Lemma 2. By that lemma, we have only to prove that $\tau(\mathfrak{A}) \subseteqq \iota_\Psi(\mathfrak{A})$.

For each closed two-sided ideal I of $\mathfrak{A}$, define subsets $\mathfrak{U}(I)$ and $\mathfrak{V}(I)$ of the structure space $\hat{\mathfrak{A}}$ by

$$\begin{aligned}\mathfrak{U}(I) &= \{\pi \in \hat{\mathfrak{A}}: \pi(I) = (0)\} ,\\ \mathfrak{V}(I) &= \{\pi \in \hat{\mathfrak{A}}: \pi(I) \neq (0)\} .\end{aligned}$$

These sets are, respectively, closed and open in $\hat{\mathfrak{A}}$ [2; 3.2.1].

Suppose that $\alpha \in \tau(\mathfrak{A})$. By Lemma 1, α has an extension to an automorphism $\bar{\alpha}$ of $\mathfrak{A}^- = \bigoplus_{\gamma\in\Gamma} \mathfrak{L}(H_\gamma)$. Given $\pi \in \hat{\mathfrak{A}}$ there is a unique subspace H_γ of H such that π is unitarily equivalent to π_γ. Let $E_\pi \in \mathfrak{A}^-$ denote the projection from H onto H_γ. The elements $\{E_\pi: \pi \in \hat{\mathfrak{A}}\}$ are precisely the minimal central projections of $\mathfrak{A}^-$, and they generate the centre of $\mathfrak{A}^-$ (as a von Neumann algebra). An automorphism preserves the property of being a minimal central projection, so $\bar{\alpha}$ permutes the E_π.

Let $(I_\rho)_{0 \leq \rho \leq \delta}$ be a composition series for $\mathfrak{A}$ such that each difference algebra $I_{\rho+1} - I_\rho$ has Hausdorff structure space. Suppose that σ is an ordinal $(0 < \sigma \leq \delta)$ and that for $\rho < \sigma$ we have shown that

$$\bar{\alpha}(E_\pi) = E_\pi \quad \text{for all} \quad \pi \in \mathfrak{V}(I_\rho) . \tag{2}$$

Clearly (2) is (vacuously) satisfied for $\sigma = 1$. If σ is a limit ordinal then $\mathfrak{V}(I_\sigma) = \bigcup_{\rho<\sigma} \mathfrak{V}(I_\rho)$ so that (2) holds with $\rho = \sigma$. Suppose that σ is not a limit ordinal, and let $\theta \in \mathfrak{V}(I_\sigma)$. Let $\bar{\alpha}(E_\theta) = E_\phi$. We shall suppose $\phi \neq \theta$ and obtain a contradiction.

Let $\{\phi\}^-$ denote the closure of $\{\phi\}$ in the Jacobson topology. We shall first show that $\theta \notin \{\phi\}^-$. To see this, note that

$$\hat{\mathfrak{A}} = \mathfrak{V}(I_{\sigma-1}) \cup (\mathfrak{V}(I_\sigma) \cap \mathfrak{U}(I_{\sigma-1})) \cup \mathfrak{U}(I_\sigma) ,$$

so that ϕ must belong to one of these three sets.

(i) for $\pi \in \mathfrak{V}(I_{\sigma-1})$ we have by (2), $\bar{\alpha}(E_\pi) = E_\pi$, so that all the elements $E_\pi (\pi \in \mathfrak{V}(I_{\sigma-1}))$ are already bespoken as values for the (injective) mapping $\bar{\alpha}$, hence it is not possible that $\phi \in \mathfrak{V}(I_{\sigma-1})$ unless $\theta = \phi$. Thus $\phi \notin \mathfrak{V}(I_{\sigma-1})$ and also $\theta \notin \mathfrak{V}(I_{\sigma-1})$.

(ii) $\mathfrak{V}(I_\sigma) \cap \mathfrak{U}(I_{\sigma-1})$ is homeomorphic with the structure space of $I_\sigma - I_{\sigma-1}$ [2; 3.2.1], and this is Hausdorff (and hence a T_1-space) so that if $\phi \in \mathfrak{V}(I_\sigma) \cap \mathfrak{U}(I_{\sigma-1})$, $\theta \notin \{\phi\}^-$ since by (i) θ is also in $\mathfrak{V}(I_\sigma) \cap \mathfrak{U}(I_{\sigma-1})$.

(iii) $\mathfrak{U}(I_\sigma)$ is closed, and $\theta \notin \mathfrak{U}(I_\sigma)$. Thus if $\phi \in \mathfrak{U}(I_\sigma)$, it follows that $\{\phi\}^- \subseteqq \mathfrak{U}(I_\sigma)$ and $\theta \notin \{\phi\}^-$.

Thus in any case $\theta \notin \{\phi\}^-$, i.e. $\mathrm{Ker}\,(\phi) \nsubseteqq \mathrm{Ker}\,(\theta)$. Choose $A \in \mathfrak{A}$ such that $\phi(A) = 0$, $\theta(A) \neq 0$. Then

$$\begin{aligned} \theta(A) \neq 0 &\Longrightarrow AE_\theta \neq 0 \\ &\Longrightarrow \bar{\alpha}(AE_\theta) \neq 0 \\ &\Longrightarrow \bar{\alpha}(A)\cdot\bar{\alpha}(E_\theta) \neq 0 \\ &\Longrightarrow \alpha(A)\cdot E_\phi \neq 0 . \end{aligned}$$

On the other hand, $\alpha \in \tau(\mathfrak{A})$ so α preserves $\mathrm{Ker}\,(\phi)$, hence

$$\begin{aligned} \phi(A) = 0 &\Longrightarrow A \in \mathrm{Ker}\,(\phi) \\ &\Longrightarrow \alpha(A) \in \mathrm{Ker}\,(\phi) \\ &\Longrightarrow \alpha(A)\cdot E_\phi = 0 . \end{aligned}$$

We have arrived at a contradiction, thus showing that $\bar{\alpha}(E_\theta) = E_\theta$ for $\theta \in \mathfrak{V}(I_\sigma)$, i.e. (2) holds for $\rho = \sigma$.

By transfinite induction, $\bar{\alpha}(E_\pi) = E_\pi$ for all $\pi \in \hat{\mathfrak{A}} (= \mathfrak{V}(I_\delta))$. Since the centre of $\mathfrak{A}^-$ is generated as a von Neumann algebra by the E_π and $\bar{\alpha}$ is ultra-weakly continuous (*cf.* Lemma 1), $\bar{\alpha}$ leaves the centre elementwise fixed. But $\mathfrak{A}^-$ is Type I, so by Kaplansky's theorem [7] $\bar{\alpha}$ is inner, which proves the theorem.

COROLLARY 1. *If $\mathfrak{A}$ is postliminal, then $\tau_0(\mathfrak{A}) = \tau(\mathfrak{A})$.*

Proof. By Lemma 2 and Theorem 1 we have

$$\tau_0(\mathfrak{A}) \subseteqq \tau(\mathfrak{A}) = \iota_\psi(\mathfrak{A}) \subseteqq \tau_0(\mathfrak{A}) .$$

COROLLARY 2. *If $\mathfrak{A}$ is postliminal, $\alpha \in \tau(\mathfrak{A})$ and ϕ is an irreducible representation of $\mathfrak{A}$, then α induces a weakly-inner automorphism α_ϕ of $\phi(\mathfrak{A})$.*

Proof. Suppose that $\mathfrak{A}$ is given in its reduced atomic representation. ϕ is unitarily equivalent to the map $A \to AE_\pi$ (for some $\pi \in \hat{\mathfrak{A}}$). By Theorem 1, $\alpha(A) = U^* AU$ (for all $A \in \mathfrak{A}$) for some $U \in \mathfrak{A}^-$. The map $AE_\pi \to (UE_\pi)^* AE_\pi(UE_\pi)$ is then unitarily equivalent to the required automorphism of $\phi(\mathfrak{A})$.

Our results so far have mirrored those of Miles [8] on derivations. In the case of derivations, it is now known ([5] and [9]) that every derivation of a C^*-algebra is permanently weakly-inner. We shall now show that the analogous result holds for ideal-preserving automorphisms of (separable) postliminal C^*-algebras, by making use of the decomposition of a representation of such an algebra as a direct integral of irreducible representations. For an account of this decomposition, see [1; Chap. II] and [2; § 8].

LEMMA 3. *If $\mathfrak{A}$ is a C^*-algebra, $\alpha \in \tau_0(\mathfrak{A})$ and $\mathfrak{B}$ is any homomorphic image of $\mathfrak{A}$, then α induces an automorphism in $\tau_0(\mathfrak{B})$.*

Proof. Let ψ be a homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, with kernel I. Define a map $\tilde{\alpha}$ on $\mathfrak{B}$ by $\tilde{\alpha}(\psi(A)) = \psi(\alpha(A))$. $\tilde{\alpha}$ is well-defined since α preserves I. It is clearly a homomorphism, with range the whole of $\mathfrak{B}$, and since α preserves I carefully it is injective. Thus it is an automorphism.

If J is a closed two-sided ideal in $\mathfrak{B}$ then $\psi^{-1}(J)$ is a closed two-sided ideal in $\mathfrak{A}$ containing I and is carefully preserved by α, from which it follows that $\tilde{\alpha}$ carefully preserves J. Thus $\tilde{\alpha} \in \tau_0(\mathfrak{B})$.

THEOREM 2. *If $\mathfrak{A}$ is a separable postliminal C^*-algebra then $\pi(\mathfrak{A}) = \tau(\mathfrak{A})$.*

Proof. We have already noted that $\pi(\mathfrak{A}) \subseteqq \tau(\mathfrak{A})$. Suppose $\alpha \in \tau(\mathfrak{A})$, and let ϕ be any faithful representation of $\mathfrak{A}$. We have to show that α is weakly-inner in the representation ϕ. Since $\mathfrak{A}$ is postliminal, the weak closure $\overline{\phi(\mathfrak{A})}$ is a Type I von Neumann algebra, so is isomorphic to an algebra with abelian commutant, i.e. ϕ is quasi-equivalent to a multiplicity-free representation (*cf.* [2; 5.4.1]). Since the property of being weakly-inner is preserved by quasi-equivalence, we may suppose that ϕ is multiplicity-free and $\phi(\mathfrak{A})'$ is abelian (we use a prime to denote the commutant of a set of operators). Since we are assuming that $\mathfrak{A}$ is separable, $\overline{\phi(\mathfrak{A})}$ is generated (as a von Neumann algebra) by a countable set of operators.

Let E be a cyclic projection in $\phi(\mathfrak{A})'$ (which is the centre of $\overline{\phi(\mathfrak{A})}$). The restriction of $\phi(\mathfrak{A})$ to E is a homomorphic image of $\mathfrak{A}$, so by Lemma 3 α induces an ideal-preserving automorphism on it. If the automorphism so induced on each cyclic portion of the centre of $\overline{\phi(\mathfrak{A})}$ is weakly-inner, then (taking a maximal orthogonal family of cyclic central projections) it follows that α is weakly-inner. We may thus restrict to a cyclic central projection and we can therefore assume that ϕ acts on a separable Hilbert space H.

There exist [2; 8.3.2] a standard Borel space Z, a bounded positive measure μ on Z, a measurable field $\zeta \to H_\zeta$ of Hilbert spaces on Z, a measurable field of representations $\zeta \to \pi_\zeta$ of $\mathfrak{A}$ on the field (H_ζ) and an isometry from H onto $\int^{\oplus} H_\zeta d\mu(\zeta)$, which transforms $\phi(\mathfrak{A})'$ into the diagonal operators and ϕ into $\int^{\oplus} \pi_\zeta d\mu(\zeta)$. We shall equate H, $\phi(\mathfrak{A})$, &c. with their transforms under this equivalence. Since $\phi(\mathfrak{A})'$ consists of diagonal operators, almost every π_ζ is irreducible [2; 8.5.1]. For almost all $\zeta \in Z$, α induces an automorphism α_ζ of $\pi_\zeta(\mathfrak{A})$, which by Corollary 2 of Theorem 1 is weakly-inner, and so in particular extends to an automorphism (which we still call α_ζ) of $\mathfrak{L}(H_\zeta)$. Define $\alpha_\zeta = 0$ on the exceptional null set. α_ζ is ultra-weakly continuous, hence strongly continuous on bounded sets. Thus we have a field (which we do not yet know to be measurable) of automorphisms α_ζ, such that for each $A \in \mathfrak{A}$, $\phi(\alpha(A)) = \int^{\oplus} \alpha_\zeta(\pi_\zeta(A)) d\mu(\zeta)$.

We now show that α is weakly continuous on the unit ball of $\mathfrak{A}$ (in the representation ϕ). To do this it suffices, by [4; Remark 2.2.3], to show that α is weakly continuous at zero on the set of positive operators in the unit ball of $\mathfrak{A}$. Since H is separable, the unit ball is metrizable in the weak topology, and we need only deal with sequences. Suppose that $I \geqq A_n \geqq 0$ and $\phi(A_n) \to 0$ weakly. Then $\phi(A_n^{1/2}) \to 0$ strongly and by [1; Chap. II § 2 Prop. 4 (i)] there is a subsequence (n_k) such that, locally almost everywhere, $\pi_\zeta(A_{n_k}^{1/2}) \to 0$ strongly. Since α_ζ is strongly continuous on bounded sets, we have locally almost everywhere, $\pi_\zeta(\alpha(A_{n_k}^{1/2})) = \alpha_\zeta(\pi_\zeta(A_{n_k}^{1/2})) \to 0$ strongly. Since the sequence (A_{n_k}) is bounded, it follows from [1; Chap. II § 2 Prop. 4 (ii)] that $\alpha(A_{n_k}^{1/2}) \rightarrow 0$ strongly and so $\alpha(A_{n_k}) \rightarrow 0$ weakly. Thus α (and similarly α^{-1}) is weakly continuous on bounded sets in the representation ϕ, hence ultra-weakly continuous, and so α is extendable to an automorphism $\bar{\alpha}$ of $\overline{\phi(\mathfrak{A})}$.

We shall next show that the field of automorphisms $\zeta \rightarrow \alpha_\zeta$ induces $\bar{\alpha}$ on $\overline{\phi(\mathfrak{A})}$ (and so is measurable). Let A be a fixed element of $\overline{\phi(\mathfrak{A})}$, and let $\zeta \rightarrow A_\zeta$ be a measurable operator field representing A. Let $\zeta \rightarrow B_\zeta$ be a measurable operator field representing $\bar{\alpha}(A)$. By metrizability of the strong topology [1; p. 33] and Kaplansky's Density Theorem [1; Chap. I § 3 Th. 3], we can choose a sequence (A_n) in $\mathfrak{A}$ such that $\|A_n\| \leqq \|A\|$ and $\phi(A_n) \rightarrow A$ strongly. By passing to a subsequence and using [1; Chap. II § 2 Prop. 4(i)] again, we can even suppose that $\pi_\zeta(A_n) \rightarrow A_\zeta$ strongly, locally almost everywhere. Since $\bar{\alpha}$ is strongly continuous on bounded sets, $\phi(\alpha(A_n)) \rightarrow \bar{\alpha}(A) = \int^{\oplus} B_\zeta d\mu(\zeta)$ strongly, and there is a subsequence (A_{n_k}) of (A_n) such that $\pi_\zeta(\alpha(A_{n_k})) \rightarrow B_\zeta$ strongly, locally almost everywhere. But since α_ζ is strongly continuous on bounded sets, we have $\pi_\zeta(\alpha(A_{n_k})) = \alpha_\zeta(\pi_\zeta(A_{n_k})) \rightarrow \alpha_\zeta(A_\zeta)$ strongly, locally a.e. Hence, locally almost everywhere, we have $B_\zeta = \alpha_\zeta(A_\zeta)$. Thus $\bar{\alpha}(A) = \int^{\oplus} \alpha_\zeta(A_\zeta) d\mu(\zeta)$, as required.

Now since $\bar{\alpha}$ is induced by the field $\zeta \rightarrow \alpha_\zeta$, it is clear that $\bar{\alpha}$ leaves each diagonal operator fixed, i.e. $\bar{\alpha}$ leaves the centre of $\overline{\phi(\mathfrak{A})}$ elementwise fixed. Hence by Kaplansky's Theorem $\bar{\alpha}$ is inner (since $\overline{\phi(\mathfrak{A})}$ is Type I), and the proof is complete.

It is possible for an automorphism of a postliminal algebra to be weakly-inner in some representation without being π-inner, as the following example shows. Let ν denote Lebesgue measure on the interval $[0, 1]$, and let $H = L_2([0, 1], \nu)$. Let $\mathfrak{K}$ denote the set of compact operators on H. For $f \in C([0, 1])$ let T_f denote the operator defined by

$$T_f x(t) = f(t)x(t) ,$$

and let $\mathfrak{T} = \{T_f : f \in C([0, 1])\} \subseteqq \mathfrak{L}(H)$. Then $\mathfrak{A} = \mathfrak{K} + \mathfrak{T}$ is a C^*-algebra [2; 1.8.4] and is postliminal since $\{(0), \mathfrak{K}, \mathfrak{A}\}$ is a composition series for which each difference algebra has Hausdorff structure space (because $\mathfrak{A} - \mathfrak{K} \cong \mathfrak{T}$). Let $U \in \mathfrak{L}(H)$ be the unitary operator defined by

$$U x(t) = x(1 - t) ,$$

then U induces an automorphism of $\mathfrak{A}$: for if $K \in \mathfrak{K}$, $T_f \in \mathfrak{T}$ then $U^*(K + T_f) U = U^* K U + T_g$ (where $g(t) = f(1 - t)$). Let

$$I_0 = \{T_f \in \mathfrak{T}: f(t) = 0 \quad \text{for} \quad 0 \leqq t \leqq \tfrac{1}{2}\}$$

and let $I_1 = \mathfrak{K} + I_0$, then it is easy to see that $U^* \cdot U$ does not preserve I_1, so by Theorem 2, $U^* \cdot U$ is not π-inner. (In fact, it is not weakly-inner in the representation of $\mathfrak{A}$ on $H \oplus H$ defined by $K + T \rightarrow (K + T) \oplus T$.) But it is clearly weakly-inner in the given representation, since this is irreducible.

This example also shows that an automorphism of a postliminal C^*-algebra can leave the centre elementwise fixed and yet not be π-inner: for the centre of $\mathfrak{K} + \mathfrak{T}$ consists just of scalar multiples of the identity.

We conclude with a few remarks about the antiliminal case. Let $\mathfrak{A}$ be a factor of Type II_1. Then $\mathfrak{A}$ has no nonzero proper closed two-sided ideals, so that $\tau_0(\mathfrak{A}) = \tau(\mathfrak{A}) = \alpha(\mathfrak{A})$ in this case. On the other hand, there are many outer automorphisms of $\mathfrak{A}$. Thus the sets $\tau_0(\mathfrak{A})$ and $\tau(\mathfrak{A})$ are probably not of great interest when $\mathfrak{A}$ is antiliminal.

Let $\mathfrak{A}$ be an antiliminal algebra with a faithful irreducible representation. Then $\mathfrak{A}$ has uncountably many such representations, all inequivalent [2; 4.7.2]. Intuitively, it seems unlikely that an automorphism would be weakly-inner in all these representations without actually being inner. In [6; Ex. a] an example is given of such an algebra (the Fermion algebra $\mathfrak{F}$) together with an automorphism of $\mathfrak{F}$ which is weakly-inner in one representation, but not π-inner. It would be interesting to have an example of an automorphism of $\mathfrak{F}$ which is π-inner but not inner.

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