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ON ABELIAN PSEUDO LATTICE ORDERED GROUPS

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Throughout this paper po-group will mean partially ordered abelian group. A subgroup H of a po-group G is an o-ideal if H is a convex, directed subgroup of G. A subgroup M of G is a value of $0 \neq g \in G$ if M is an o-ideal of G that is maximal without g. Let $\mathscr{M}(g) = \{M \subseteqq G \mid M \text{ is a value of } g\}$ and $\mathscr{M}^*(g) = \bigcap \mathscr{M}(g)$. Two positive elements $a, b \in G$ are pseudo disjoint (p-disjoint) if $a \in \mathscr{M}^*(b)$ and $b \in \mathscr{M}^*(a)$, and G is a pseudo-lattice ordered group (pl-group) if each $g \in G$ can be written $g = a - b$ where a and b are p-disjoint.

The main result of § 2 shows that every pl-group G is a Riesz group. That is, G is semiclosed ($ng \geqq 0$ implies $g \geqq 0$ for all $g \in G$ and all positive integers n), and G satisfies the Riesz interpolation property; if, whenever $x_1, \cdots, x_m, y_1, \cdots, y_n$ are elements of G and $x_i \leqq y_j$ for $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, then there is an element $z \in G$ such that $x_i \leqq z \leqq y_j$.

In § 3, we determine which Riesz groups are also pl-groups. In the final section it is shown that each pair of p-disjoint elements a, b determines an o-ideal $H(a, b)$ with the property that if $a - b = x - y$ where x and y are also p-disjoint, then $H(a, b) = H(x, y)$ and $a - x = b - y \in H(a, b)$.

The concept of a pl-group has been introduced by Conrad [1]. For each $g \in G$, $\mathscr{M}^*(g)$ exists by definition, and in particular, $\mathscr{M}^*(0) = G$. In § 2 we list a number of properties of pl-groups that will be used. We adopt the notation $a \| b$ for $a \ngeqq b$ and $b \ngeqq a$. If S is a subset of a po-group G and $a \in G$, the notation $a > S$ means $a > s$ for all $s \in S$. If H is an o-ideal of a po-group G, a natural order is defined in G/H by setting $X \in G/H$ positive if X contains a positive element of G. All quotient structures will be ordered in this manner. Finally, $G^+ = \{x \in G \mid x \geqq 0\}$.

2. Some properties of pl-groups.

We first list a number of properties of pl-groups. The proofs of these may be found in [1]. If G is a pl-group, then

(1) G is semiclosed.

(2) G is directed.

(3) The intersection of o-ideals of G is an o-ideal.

(4) If $g \in G$ and $M \in \mathscr{M}(g)$ and M' is the intersection of all o-ideals of G that properly contain M, then $g \in M'$, M'/M is o-isomorphic to a naturally ordered subgroup of the real numbers and, if $M < X \in G/M \backslash M'/M$, then $X > M'/M$.

(5) If K is an o-ideal of G, then K and G/K are pl-groups.

(6) If K is an o-ideal of G and $g \in G \backslash K$, then there is $M \in \mathscr{M}(g)$ such that $M \supseteqq K$.

(7) If $g = a - b$ where a and b are p-disjoint, then $\mathscr{M}(g) = \mathscr{M}(a) \cup \mathscr{M}(b)$.

(8) A nonzero element $g \in G$ is positive if and only if $g + M > M$ for all $M \in \mathscr{M}(g)$.

(9) If a and b are p-disjoint and $g \leqq a$, $g \leqq b$, then $ng \leqq a$ and $ng \leqq b$ for all $n > 0$.

(10) If a and b are p-disjoint, then no value of a is comparable to a value of b.

The following set of propositions leads to the first theorem which states that every pl-group is a Riesz group.

(2.1) Let G be a po-group and $g \in G$. If $g = a - b$ where a and b are p-disjoint and $z \in G^+$ such that $z \geqq g$, then each value of a is contained in a value of z, and if $a \geqq z$, then z and $z - g$ are p-disjoint.

Proof. Let $M \in \mathscr{M}(a)$, then $b \in M$ and $z \geqq g = a - b$ implies $z + b \geqq a \geqq 0$. Hence, $z \notin M$ and there is $M' \in \mathscr{M}(z)$ such that $M' \supseteqq M$.

From $a \geqq z \geqq 0$ it follows that if $M \in \mathscr{M}(z)$, then $a \notin M$. By the above, $M \in \mathscr{M}(a)$ so $b \in M$. Now $a \geqq z \geqq g$ implies $a - g = b \geqq z - g \geqq 0$ so $z - g \in M$. Similarly, if $M \in \mathscr{M}(z - g)$, then $b \notin M$ so $M \in \mathscr{M}(b)$, $a \in M$ and hence, $z \in M$. Thus, z and $z - g$ are p-disjoint.

(2.2) If G is a po-group and $g = a - b = x - y$ where a and b are p-disjoint and x and y are positive, then for each

$$M \in \mathscr{M}(a)[M \in \mathscr{M}(b)]$$

there is $M' \in \mathscr{M}(x)[M' \in \mathscr{M}(y)]$ such that $M' \supseteqq M$. In particular, if x and y are p-disjoint, $\mathscr{M}(a) = \mathscr{M}(x)$, $\mathscr{M}(b) = \mathscr{M}(y)$ and $a - x = b - y \in \mathscr{M}^*(g)$.

Proof. Let $g \in G$ and $g = a - b = x - y$ where a and b are p-disjoint and x and y are positive. Since $y \geqq 0$, we have $x \geqq g$ so for $M \in \mathscr{M}(a)$ there is, by (2.1), $M' \in \mathscr{M}(x)$ such that $M' \supseteqq M$. Similarly for $M \in \mathscr{M}(b)$. If x and y are also p-disjoint then, by interchanging the roles of a and x, y and b we obtain $\mathscr{M}(a) = \mathscr{M}(x)$ and $\mathscr{M}(b) = \mathscr{M}(y)$. Thus, $b, y \in \mathscr{M}^*(a)$ and $a, x \in \mathscr{M}^*(b)$ so

$$a - x = b - y \in \mathscr{M}^*(a) \cap \mathscr{M}^*(b)$$

which is equal to $\mathscr{M}^*(g)$ by property (7).

(2.3) Suppose G is a pl-group, $g \in G, g = a - b$ where a and b are p-disjoint and $z \in G^+$ such that $z \geqq g$. If $M \in \mathscr{M}(a - z)$, then either $M \in \mathscr{M}(z)$ and $z + M > a + M$ or M is properly contained in a value of a.

Proof. If $M \in \mathscr{M}(a - z)$, then by (4),

$$a + M > z + M \quad \text{or} \quad a + M < z + M .$$

For $M \in \mathscr{M}(z)$ and $M \notin \mathscr{M}(a)$, it follows that $z + M > M$ and, from (2.1), that $a \in M$. Hence, $z + M > M = a + M$. For $M \in \mathscr{M}(z)$ and $M \in \mathscr{M}(a)$, we have $a + M = g + M \leqq z + M$ so $a + M < z + M$. Now if $M \notin \mathscr{M}(z)$, then $a \notin M$ so there is $M' \in \mathscr{M}(a)$ such that $M' \supseteqq M$. If $M' = M$, then M is properly contained in $M'' \in \mathscr{M}(z)$ so a and $a - z$ are in M'' and $z \in M''$, a contradiction. Thus M' properly contains M.

LEMMA 2.1. *If G is a pl-group, $g \in G$ and $z \in G^+$ such that $z \geqq g$, then there is $x \in G^+$ such that $z \geqq x$ and $x, x - g$ are p-disjoint. Moreover, if $g = a - b$, with a and b p-disjoint, then there exists such an x with $a \geqq x$.*

Proof. Let G be a pl-group and $g \in G$. Then $g = a - b$ where a and b are p-disjoint. If $z \in G^+$ and $g \leqq z$, take $x = a$ if $z \geqq a$; and take $x = z$ if $z < a$. The result follows from (2.1).

If $z - a \parallel 0$, then $z - a = p - q$ where p and q are p-disjoint. We first show $\mathscr{M}(q) = \{M \in \mathscr{M}(z - a) \mid z + M < a + M\}$. Let $M \in \mathscr{M}(q)$, then $M \in \mathscr{M}(z - a)$ and $(z - a) + M = -q + M < M$ so $z + M < a + M$. Conversely, if $M \in \mathscr{M}(z - a)$ and $z + M < a + M$, then $M \in \mathscr{M}(p)$ or $M \in \mathscr{M}(q)$. If $M \in \mathscr{M}(p)$, then $q \in M$ so $(z - a) + M = p + M > M$. This implies $z + M > a + M$, a contradiction. Thus, $M \in \mathscr{M}(q)$.

Now let $x = a - q = z - p$, then $x < a$ and $x < z$. If $M \in \mathscr{M}(x)$, then $q \in M$. For if $q \notin M$, then $M \subseteqq M' \in \mathscr{M}(q), M' \in \mathscr{M}(z - a)$ and $z + M' < a + M'$. By (2.3), M' is properly contained in $M'' \in \mathscr{M}(a)$. Thus, $x \in M'', q \in M''$ so $a \in M''$ a contradiction. Therefore, $q \in M$ and hence $a \notin M$. We now have $M \neq a + M = x + q + M$ so $M < a + M = x + M$ for all $M \in \mathscr{M}(x)$. By (8), $x \geqq 0$.

To complete the proof we need only show $x \geqq g$, for then the result follows by (2.1). To accomplish this we show $(b - q) + M > M$ for all $M \in \mathscr{M}(b - q)$. Thus, let $M \in \mathscr{M}(b - q)$. *If* $M \in \mathscr{M}(q)$, then $M \in \mathscr{M}(z - a)$ and $z + M < a + M$, so $b \notin M$. By (2.3) and (10) there must exist $M' \in \mathscr{M}(b)$ such that M' properly contains M. But M' properly containing M implies $b - q, q$ and hence $b \in M'$, a contradiction. Thus, $M \notin \mathscr{M}(q)$.

Now since $b \notin M$, there is $M'' \in \mathscr{M}(b)$ such that $M'' \supseteqq M$. If

$M'' \neq M$, then $b - q \in M''$ so $M'' < b + M'' = q + M''$ and $q \notin M''$. By (2.3), every value of q is contained in a value of a so M'' is contained in a value of a, a contradiction. Thus $M'' = M \in \mathscr{M}(b)$, and as above, it follows that $q \in M$. Consequently, $b - q + M = b + M > M$ so by (8), $b > q$ and $x > g$. This completes the proof.

With Lemma 2.1 we are now able to prove the following.

THEOREM 2.1. *Every pl-group is a Riesz group.*

Proof. Since by (1), a *pl*-group is semiclosed, we need only show a *pl*-group G satisfies the Riesz interpolation property. Without loss of generality, we may assume, $g, u, z \in G$ and $u \geqq 0, z \geqq 0, u \geqq g, z \geqq g$. There exists, by Lemma 2.1, an element $a \in G^+$ such that $u \geqq a$ with $a, a - g$ p-disjoint. Also, there is $x \in G^+$ such that $a \geqq x, z \geqq x$ with $x, x - g$ p-disjoint. Hence, $u \geqq x \geqq 0, z \geqq x \geqq g$ and G is a Riesz group.

We note that the above theorem and Theorem 4.8 in [1] answer affirmitively the open question posed at the end of [2].

3. Sufficient conditions for pseudo-lattice ordering. As a consequence of § 2 we have that every *pl*-group G is a Riesz group that satisfies

(*) for each $g \in G$, there is $a \in G^+$ such that $g \leqq a$ and whenever $0 \leqq x, g \leqq x$ then $a \leqq x + h$ for some $h \in \mathscr{M}^*(a) \cap \mathscr{M}^*(a - g)$.

To see this let $g \in G$, then g can be written $g = a - b$ where a and b are p-disjoint, so $a \in G^+$ and $g \leqq a$. If $x \in G^+$ and $x \geqq g$, then, since G is a Riesz group, there is $u \in G$ such that $a \geqq u \geqq 0$ and $x \geqq u \geqq g$. By (2.1), u and $u - g$ are p-disjoint and by (2.2) and (7), $a - u \in \mathscr{M}^*(a) \cap \mathscr{M}^*(a - g)$. By setting $a - u = h$ we have $u = a - h$ so $x \geqq u = a - h$ which implies $x + h \geqq a$.

In this section we show that every Riesz group that satisfies (*) is a *pl*-group. For the remainder of this section we assume G is a Riesz group that satisfies (*).

LEMMA 3.1. *The intersection of o-ideals of G is again an o-ideal.*

Proof. Let $M_\alpha, \alpha \in J$ be o-ideals of G and $M = \bigcap_{\alpha \in J} M_\alpha$. Clearly, M is a convex subgroup of G. To show M is directed let $g \in M$. By (*) there is $a \in G$ such that $0 \leqq a, g \leqq a$. Now for each $\alpha \in J, M_\alpha$ is directed so M_α is a Riesz group. Thus, there are elements $y_\alpha \in M_\alpha$, $x_\alpha \in G$ such that $y_\alpha \geqq 0, y_\alpha \geqq g, a \geqq x_\alpha \geqq g$ and $y_\alpha \geqq x_\alpha \geqq 0$. Thus, $x_\alpha \in M_\alpha$ and $a \leqq x_\alpha + h_\alpha$ for some $h_\alpha \in \mathscr{M}^*(a) \cap \mathscr{M}^*(a - g)$.

Now $x_\alpha \in M_\alpha$ and $x_\alpha + h_\alpha \geqq a \geqq x_\alpha$ implies $a - x_\alpha \in \mathscr{M}^*(a)$. Thus, if $a \notin M_\alpha$ then there is $M' \in \mathscr{M}(a)$ such that $M' \supseteqq M_\alpha$. But then x_α,

$a - x_\alpha$ and hence $a \in M'$, a contradiction. Thus $a \in M_\alpha$ for all α, M is directed and M is an o-ideal of G.

We note that in the above we have proved that if a satisfies (*) for g and $a \geqq x \geqq 0, x \geqq g$ then $a - x \in \mathscr{M}^*(a) \cap \mathscr{M}^*(a - g)$.

LEMMA 3.2. *If M is an o-ideal of G, then M and G/M are Riesz groups satisfying* (*).

Proof. If M is an o-ideal of G, then M and G/M are Riesz groups by [2, p. 1393]. If $g \in M$, then let $a \in G$ such that a satisfies (*) for g. There then are elements $m \in M^+$ and $x \in G$ such that $m \geqq g$, $a \geqq x \geqq g$ and $m \geqq x \geqq 0$, which implies $x \in M$ and $a - x \in \mathscr{M}^*(a)$. As a consequence of this latter part, $a \in M$. Now if $0 \leqq y \in M$ and $g \leqq y$ then there is $u \in M$ such that $y \geqq u \geqq 0, a \geqq u \geqq g$. Thus, by the remark preceding this lemma, $u = a + h$ where

$$h \in \mathscr{M}^*(a) \cap \mathscr{M}^*(a - g)$$

and hence $u - a = h \in M$. By Lemma 3.1, every o-ideal M' of M that is maximal without a $[a - g]$ can be written $M' = M \cap \bar{M}$ where $\bar{M} \in \mathscr{M}(a)[\bar{M} \in \mathscr{M}(a - g)]$. Thus, it follows that h belongs to every value of a and every value of $a - g$ in M and M satisfies (*).

Now let $g + M \in G/M$, and let $a \in G$ such that a satisfies (*) for g. Then $a + M \geqq M$ and $a + M \geqq g + M$. If $M \leqq x + M \in G/M$ and $x + M \geqq g + M$, then there are elements $m_1, m_2 \in M$ such that $x + m_1 \geqq 0$ and $x + m_2 \geqq g$. Since M is directed, there is $m \in M$ such that $m \geqq m_1, m \geqq m_2$.

By (*), $a \leqq (x + m) + h$ so $a + M \leqq (x + M) + (h + M)$ where

$$h \in \mathscr{M}^*(a) \cap \mathscr{M}^*(a - g) .$$

Now let X be a value of $a + M$ in G/M. Then $X = M'/M$ where M' is an o-ideal of G and $a \notin M'$. It follows that $M' \in \mathscr{M}(a)$ so $h \in M'$ and $h + M \in X$. In a similar manner, $h + M$ belongs to every value of $(a - g) + M$ in G/M. The proof is complete.

LEMMA 3.3. *Let H be the intersection of all nonzero o-ideals of G. If $x \in H^+$, $g \in G \backslash H$ and $g < x$, then $g < 0$.*

Proof. Suppose H is the intersection of all nonzero o-ideals of G. If $x \in H^+, g \in G\backslash H$ and $g < x$, then $a \leqq x + h$ where a satisfies (*) for g and $h \in \mathscr{M}^*(a) \cap \mathscr{M}^*(a - g)$. If $a \neq 0$ and $M \in \mathscr{M}(a)$, then $M \neq 0$ so $H \subseteqq M$ and $x + h \in M$. This implies $a \in M$ since $0 \leqq a \leqq x + h$, a contradiction. Thus, $a = 0$ and $g < 0$.

COROLLARY. *If H is the intersection of all nonzero o-ideals of G, then every positive element of $G\backslash H$ exceeds every element of H.*

Proof. Let $0 < g \in G\backslash H$ and $h \in H$. By Lemma 3.1, H is an o-ideal of G so there is $h' \in H^+$ such that $h' \geqq h$. Now $h' - g \in G\backslash H$ and $h' - g < h'$ so $h' - g < 0$, $h \leqq h' < g$ and the corollary follows.

As a final observation before we turn to the main proof of this section, we note that if G has no proper o-ideals then G is a subgroup of the naturally ordered real numbers. This is a special case of 4.6 in [1].

THEOREM 3.1. *A Riesz group G is a pl-group if and only if G satisfies.*

(*) *for each $g \in G$, there is $a \in G^+$ such that $g \leqq a$ and whenever $0 \leqq x, g \leqq x$ then $a \leqq x + h$ for some $h \in \mathscr{M}^*(a) \cap \mathscr{M}^*(a - g)$.*

Proof. Let $g \in G$ and a satisfy (*) for g. We show a and $a - g$ are p-disjoint. If $a = 0$ or $a = g$, the result easily follows so we assume $g \parallel 0$. Let $M \in \mathscr{M}(a)$ and let M' be the intersection of all o-ideals of G that properly contain M. Then M' is an o-ideal of G, $a \in M'$, M'/M is o-isomorphic to a subgroup of the naturally ordered real numbers and if $M < X \in (G/M)\backslash(M'/M)$, then $X > M'/M$.

If $(a - g) + M \geqq a + M$, then there is $m \in M^+$ such that $a - g + m \geqq a$, so $m \geqq g$. By (*), $0 < a \leqq m + h$ where $h \in \mathscr{M}^*(a) \cap \mathscr{M}^*(a - g)$.
Thus, $m + h \in M$ and $a \in M$, a contradiction. Since $(a - g) + M$ is comparable to $a + M$, we must have $(a - g) + M < a + M$, so there is $m \in M$ such that $a > (a - g) + m$. Let $m' \in M$ such that $m' < m$, $m' < 0$, then $g - m' > g$ and $g - m' > 0$. Thus, by (*), $a \leqq (g - m') + h'$ where $h' \in \mathscr{M}^*(a) \cap \mathscr{M}(a - g)$, and $0 < a - g \leqq -m' + h' \in M$. By convexity $a - g \in M$ so $a - g \in \mathscr{M}^*(a)$.

By interchanging the roles of a and $a - g$ in the above we are led to the conclusion that $a + M < (a - g) + M$ where $M \in \mathscr{M}(a - g)$. There then is $m \in M^+$ such that $a < (a - g) + m$ so $g < m$. As always, $a \leqq m + h$ with $h \in \mathscr{M}^*(a) \cap \mathscr{M}^*(a - g)$ so $a \in M$. Thus, a and $a - g$ are p-disjoint and G is a pl-group.

The necessity follows from the remarks at the beginning of this section.

4. Pseudo-disjoint elements.

Throughout this section we assume G is a pl-group. We have shown if $g \in G$ and $g = a - b = x - y$ where a, b and x, y are pairs of p-disjoint elements then

$$a - x = b - y \in \mathscr{M}^*(a) \cap \mathscr{M}^*(b) .$$

However, the converse of this is not true. For if $K = R_1 + R_2 + R_3$ (the cardinal sum) where each R_i is the real numbers, $i = 1, 2, 3$; then K is an l-group so, of course, a pl-group. Clearly, $(1, -1, 0) = (1, 0, 0) - (0, 1, 0)$ where $(1, 0, 0), (0, 1, 0)$ are p-disjoint. Now $(1, 0, 0)$ has exactly one value namely $M_1 = R_2 + R_3$ and $(0, 1, 0)$ has the value $M_2 = R_1 + R_3$. Thus, $R_3 = M_1 \cap M_2$ and if $0 \neq h \in R_3$ it is clear that $(1, 0, 0) + (0, 0, h) = (1, 0, h)$ and $(0, 1, 0) + (0, 0, h) = (0, 1, h)$ are not p-disjoint but $(1, -1, 0) = (1, 0, h) - (0, 1, h)$.

We now show how pairs of p-disjoint elements a, b and x, y are related, when $g = a - b = x - y$. Assume a and b are p-disjoint and let $K = \{0 \leqq m \in G \mid m \leqq a, m \leqq b\}$. Clearly, K is convex. If $m_1, m_2 \in K$, then by the Riesz interpolation property, there is an element $m \in G$ such that $m_1 \leqq m \leqq a$ and $m_2 \leqq m \leqq b$. Moreover, $2m \geqq m_1 + m_2 \geqq 0$ and by (9), $2m \leqq a$, $2m \leqq b$ since a and b are p-disjoint. Thus, $2m \in K$ so $m_1 + m_2 \in K$ and K is a convex subsemigroup of G^+ that contains 0. Let H be the o-ideal of G that is generated by K. It is well known that $H^+ = K$ and any $x \in H$ can be written $x = h_1 - h_2$ where $h_1, h_2 \in K$. Thus $H < a$ and $H < b$. We denote by $H(a, b)$ the o-ideal generated by $\{0 \leqq m \in G \mid m \leqq a, m \leqq b\}$ for p-disjoint elements a, b.

LEMMA 4.1. *If a and b are p-disjoint and $m \in H(a, b)$, then $\mathscr{M}(a) = \mathscr{M}(a + m)$ and $\mathscr{M}(b) = \mathscr{M}(b + m)$.*

Proof. We first consider $0 \leqq m \in H(a, b)$. Since $a \geqq a - m \geqq 0$ and $a-m \geqq a - b$ (2.1) implies $a - m$ and $b - m$ are p-disjoint, so $\mathscr{M}(a) = \mathscr{M}(a - m)$, $\mathscr{M}(b) = \mathscr{M}(b - m)$ by (2.2).

If $M \in \mathscr{M}(a + m)$, then $a - m \notin M$ so there is $M' \supseteqq M$ such that $M' \in \mathscr{M}(a - m) = \mathscr{M}(a)$. Since $0 \leqq m \leqq b \in M'$, $m \in M'$ so $M = M' \in \mathscr{M}$(a). Conversely, if $M \in \mathscr{M}(a)$ then $0 \leqq m \leqq b \in M$ implies $m \in M$ so $a + m \notin M$ and $M \in \mathscr{M}(a + m)$. Hence, $\mathscr{M}(a) = \mathscr{M}(a + m)$. Similarly, $\mathscr{M}(b) = \mathscr{M}(b + m)$.

For an arbitrary element $m \in H(a, b)$ there are elements m_1, $m_2 \in H(a, b)$ such that $m_1 \leqq 0$ and $m_1 \leqq m$, $0 \leqq m_2 \leqq a$, $m \leqq m_2 \leqq b$. Hence, $0 \leqq a + m_1 \leqq a + m$ and $0 \leqq a + m \leqq a + m_2$. By the above, $\mathscr{M}(a) = \mathscr{M}(a + m_1) = \mathscr{M}(a + m_2)$. If $M \in \mathscr{M}(a + m)$, then $a + m_2 \notin M$ so $M \in \mathscr{M}(a + m_2) = \mathscr{M}(a)$. Conversely, if $M \in \mathscr{M}(a)$, then $m \in M$ and $M \in \mathscr{M}(a + m_1)$ so a $+ m \notin M$ and $M \in \mathscr{M}(a + m)$. Thus, for any $m \in H(a, b)$, $\mathscr{M}(a) = \mathscr{M}(a + m)$. In a similar manner $\mathscr{M}(b) = \mathscr{M}(b + m)$.

We note at this point that if $0 \leqq m \in H(a, b)$, then $0 \leqq m \leqq a$ implies $m \in \mathscr{M}^*(b)$ and $0 \leqq m \leqq b$ implies $m \in \mathscr{M}^*(a)$. Consequently, $H(a, b) \subset \mathscr{M}^*(a) \cap \mathscr{M}^*(b)$.

LEMMA 4.2. *If a and b are p-disjoint in G, then $a + m$ and*

$b + m$ are p-disjoint if and only if $m \in H(a, b)$.

Proof. Let a and b be p-disjoint and $m \in H(a, b)$, since $\mathscr{M}(a) = \mathscr{M}(a + m)$, b, m and hence $b + m \in \mathscr{M}^*(a + m)$. Dually, $a + m \in \mathscr{M}^*(b + m)$ so $a + m$ and $b + m$ are p-disjoint.

Conversely, if $a + m$ and $b + m$ are p-disjoint, then $a \geqq - m$, $b \geqq - m$ so there is $h \in G$ such that $a \geqq h \geqq 0$ and $b \geqq h \geqq - m$. This implies $h \in H(a, b)$. Since $\mathscr{M}(a) = \mathscr{M}(a + m)$ and $\mathscr{M}(b) = \mathscr{M}(b + m)$ we have $m \in \mathscr{M}^*(a) \cap \mathscr{M}^*(b)$. Now if $M \in \mathscr{M}(a - m)$ and $a + m \in M$, then $a \notin M$, so $M \in \mathscr{M}(a) = \mathscr{M}(a + m)$ and $a + m \notin M$, a contradiction. Thus, $a + m \notin M$ so $M \in \mathscr{M}(a + m) = \mathscr{M}(a)$, $a \notin M$, $b \in M$. Therefore $M < a + M = (a - m) + M$. By (8), $a - m > 0$. A similar argument shows $b > m$. Finally, by the Riesz interpolation property, there is an element $h' \in G$ such that $a \geqq h' \geqq 0$ and $b \geqq h' \geqq m$. Thus, $h' \in H(a, b)$ and we have $h' \geqq m \geqq - h$ so $m \in H(a, b)$.

COROLLARY. *If a and b are p-disjoint in G, then $a \wedge b = 0$ if and only if $H(a, b) = 0$.*

As a consequence of Lemma 4.2 we can associate with $g = a - b$, a and b p-disjoint, the o-ideal $H(a, b)$. Moreover, $H(a, b)$ depends only on g and is independent of the representation of g as the difference of p-disjoint elements. To show this, let $g = x - y$ where x and y are also p-disjoint. Then by (2.2) $\mathscr{M}(a) = \mathscr{M}(x)$ and $\mathscr{M}(b) = \mathscr{M}(y)$. If $0 \leqq k \in H(x, y)$ then $k \in \mathscr{M}^*(a) \cap \mathscr{M}^*(b)$ and $a + k$, $b + k$ are p-disjoint so $k \in H(a, b)$ and $H(x, y) \subseteqq H(a, b)$. Dually, we can show $H(a, b) \subseteqq H(x, y)$ so $H(a, b) = H(x, y)$.

Using the above we can easily show a pl-group G satisfies

(**) for each $g \in G$, there is $a \in G^+$ such that $g \leqq a$ and whenever $0 \leqq x$, and $g \leqq x$, then $a \leqq x + h$ for some $h \in H(a, a - g)$.

To see this, let $g \in G$ and a satisfy (*) for g. If $0 \leqq x$, $g \leqq x$ there is $z \in G$ such that $a \geqq z \geqq 0$ and $x \geqq z \geqq g$ since every pl-group is a Riesz group. By (2.1), z and $z - g$ are p-disjoint and since $a = z + (a - z)$ and $a - g = (z - g) + (a - z)$ we have $a - z \in H(z, z - g) = H(a, a - g)$. Therefore, $x \geqq z = a - (a - z)$ so $x + (a - z) \geqq a$.

We have shown, that in a pl-group G, $H(a, b)$ is the o-ideal generated by $K = \{0 \leqq m \in G \mid m \leqq a, m \leqq b\}$ for a and b p-disjoint, and $H(a, b)^+ = K$. If we now let $H(x, y)$ be the o-ideal generated by $K = \{0 \leqq m \in G \mid m \leqq x, m \leqq y\}$ for arbitrary positive elements x and y, it may happen that $H(x, y)^+ \neq K$ and the following example shows (**) is not sufficient for a Riesz group G to be a pl-group.

Let R be the naturally ordered real numbers and $G = R + R$. Let $(u, v) \in G$ be positive if $v > 0$ or $v = 0$ and $u = 0$. Then G is a Riesz group but G is not a pl-group. If $g = (g_1, g_2) \in G$ and $g_2 > 0$

let $a = g$; if $g_2 < 0$ let $a = 0$. In either case $H(a, a - g) = 0$ and a satisfies (**) for g. If $g_2 = 0$ and $g_1 = 0$ take $a = 0$. If $g_2 = 0$ and $g_1 \neq 0$ let $a = (a_1, a_2)$ where $a_2 > 0$. Then $a > 0$, $a > g$ and $H(a, a - g) = G$. For any $b = (b_1, b_2) \geqq (0, 0)$ and $(b_1, b_2) \geqq (g_1, g_2)$ we must have $b_2 > 0$. If $h = (0, a_2)$, then $(a_1, a_2) < (b_1, b_2) + (0, a_2)$ and $h \in H(a, a - g)$. Thus (**) holds.

REFERENCES

1. P. Conrad, *Representations of partially ordered abelian groups as groups of real-valued functions*, Acta. Math. **116** (1966), 199-221.
2. J. R. Teller, *On partially ordered groups satisfying the Riesz interpolation property*, Proc. Amer. Math. Soc. **16** (1965), 1392-1400.

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