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DUAL SPACES OF CERTAIN VECTOR SEQUENCE SPACES

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This article is an investigation of certain spaces of sequences with values in a locally convex space analogous to the generalized sequence spaces introduced by Pietsch in his monograph *Verallgemeinerte Volkommene Folgenräume* **(Akademie-Verlag, Berlin, 1962). Pietsch combines a perfect sequence space Λ and a locally convex space E to obtain the space $\Lambda(E)$ of all E valued sequences $x = (x_n)$ such that the scalar sequence $(\langle a, x_n\rangle)$ is in Λ for every $a \in E'$. Define $\Lambda\{E\}$ to be the space of all E valued sequences $x = (x_n)$ such that the scalar sequence $(p(x_n))$ is in Λ for every continuous seminorm p on E. The spaces $\Lambda(E)$ and $\Lambda\{E\}$ are topologized using the topology of E and a certain collection $\mathscr{M}$ of bounded subsets of Λ^x, the $\alpha -$ dual of Λ.**

The criteria for bounded sets, compact sets, and completeness are similar for both spaces. The significant difference lies in the duality theory. The dual of $\Lambda(E)_{\mathscr{M}}$ is difficult to represent, but the dual of $\Lambda\{E\}_{\mathscr{M}}$ is shown to be easily representable for general Λ and E. For many special cases of Λ and E the dual of $\Lambda\{E\}_{\mathscr{M}}$ is of the form $\Lambda^x\{E'\}$ where Λ^x is the $\alpha -$ dual of Λ and E' is the strong dual of E.

We begin by recalling basic definitions and elementary facts about sequence spaces and establishing some notation. After defining the space $[\Lambda\{E\}_{\mathscr{M}}]$ and deriving some elementary properties, we proceed to a description of its dual space. We show that the notion of a "fundamentally Λ-bounded" space E provides sufficient conditions for the duality relationship $\Lambda\{E\}' = \Lambda^x\{E\}$. We next show that there are large classes of Λ and E satisfying these conditions and we conclude by applying our results to the case $\Lambda = l^p$ obtain, for example, that the strong dual of $l^p\{E\}$ is $l^q\{E'\}$ for E a normed, Frechet, or (DF)-space, $1 \leqq p < \infty$, $p^{-1} + q^{-1} = 1$.

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2. Definitions and notations. A sequence space Λ is a vector space of real or complex sequences with the usual coordinatewise operations. To each sequence space Λ there corresponds another sequence space Λ^x, called the $\alpha -$ dual of of Λ, consisting of all $\alpha = (\alpha_n)$, such that the scalar products $\langle \alpha, \beta\rangle = \sum \alpha_n\beta_n$ converge absolutely, that is $\sum |\alpha_n\beta_n| < \infty$, for all β in Λ. Letting Λ^{xx} denote the $\alpha -$ dual of

Λ^x, we have $\Lambda \subset \Lambda^{xx}$. If $\Lambda^{xx} = \Lambda$, then Λ is called a perfect sequence space.

Every perfect sequence space Λ satisfies $\phi \subset \Lambda \subset \omega$, where ϕ is the space of all sequences with only a finite number of nonzero coordinates and ω is the space of all scalar sequences. Henceforth we shall consider only perfect spaces Λ.

A subset B of Λ is called bounded if for every α in Λ^x there exists a positive constant ρ such that $\sum |\alpha_n \beta_n| \leqq \rho$ for all β in B. A subset M of Λ is called normal if whenever M contains α it also contains all β satisfying $|\beta_n| \leqq |\alpha_n|$ for all n. The normal hull $N(M)$ of a set M is the set of all sequences β such that $|\beta_n| \leqq |\alpha_n|$ for all n, for some α in M. A simple consequence of these definitions is that the normal hull of a bounded set is bounded. Also every perfect sequence space is normal.

The bilinear form $\langle \alpha, \beta \rangle = \sum \alpha_n \beta_n$ on $\Lambda^x \times \Lambda$ places Λ^x and Λ in duality with each other. If M is any bounded subset of Λ^x, then $M^0 = \{\beta \in \Lambda \,|\, |\langle \alpha, \beta \rangle| = |\sum \alpha_n \beta_n| \leqq 1 \text{ for all } \alpha \in M\}$ is an absorbing absolutely convex subset of Λ. A family $\mathscr{M}$, consisting of bounded subsets of Λ^x, is called a normal topologizing system for Λ if $\mathscr{M}$ has the following properties: (i) if $M_1, M_2 \in \mathscr{M}$, then there exists $M \in \mathscr{M}$ such that $M_1 \cup M_2 \subset M$. (ii) if $M \in \mathscr{M}$ and $\rho > 0$, then $\rho M \in \mathscr{M}$. (iii) if $\alpha \in \Lambda^x$, then $\alpha \in M$ for some $M \in \mathscr{M}$. (iv) the normal hull of every set in $\mathscr{M}$ is in $\mathscr{M}$.

(1) *If $\mathscr{M}$ is a normal topologizing system for Λ, then the collection of all M^0, $M \in \mathscr{M}$, forms a neighborhood base at* 0 *for a locally convex topology on Λ. A base of seminorms for this $\mathscr{M}$-topology on Λ is given by the seminorms*

$$\begin{aligned} p_{M^0}(\beta) &= \sup\{|\textstyle\sum \alpha_n \beta_n| \,|\, \alpha \in M\} \\ &= \sup\{\textstyle\sum |\alpha_n \beta_n| \,|\, \alpha \in M\} \end{aligned}$$

where M ranges over the normal sets in $\mathscr{M}$.

It is the normality of M that allows the absolute value to be brought inside the summation above.

The two extreme cases of $\mathscr{M}$ are the class $\mathscr{B} = \mathscr{B}(\Lambda^x)$ consisting of all normal bounded subsets of Λ^x and the class $\mathscr{N} = \mathscr{N}(\Lambda^x)$ consisting of all normal hulls $N(\alpha)$ of single elements of Λ^x. The $\mathscr{B}$-topology on Λ is the so called strong or $T_b(\Lambda^x)$-topology on Λ and the $\mathscr{N}$-topology on Λ is the normal topology on Λ in the sense of Köthe, [1. §30]. Note that we always have $\mathscr{N} \subset \mathscr{M} \subset \mathscr{B}$.

We shall need the following result due to Pietsch [2. Satz 1.4].

(2) *A subset A of Λ is bounded if, and only if, it is bounded for*

some (every) $\mathscr{M}$-topology on Λ.

Let α be any scalar sequence. We denote by $\alpha(\leqq i)$ the ith finite section of α, that is the sequence with coordinates α_n for $n = 1, 2, \cdots i$ and 0 for $n > i$. $\alpha(\leqq i) = (\alpha_1, \alpha_2, \cdots \alpha_i, 0 \cdots)$. Now let $\Lambda_{\mathscr{M}}$ denote Λ equipped with an $\mathscr{M}$-topology and define $[\Lambda_{\mathscr{M}}]$ to be that subspace of $\Lambda_{\mathscr{M}}$ consisting of all sequences α which are the $\mathscr{M}$-limit of their finite sections.

(3) *For any normal topologizing system $\mathscr{M}$, $\Lambda_{\mathscr{M}}$ is complete. $[\Lambda_{\mathscr{M}}]$ is a closed subspace of $\Lambda_{\mathscr{M}}$ and hence also complete.*

(4) (a) *$[\Lambda_{\mathscr{N}}] = \Lambda_{\mathscr{N}}$ for every perfect space Λ.*
(b) *If $\Lambda_{\mathscr{B}}$ is reflexive, then $[\Lambda_{\mathscr{B}}] = \Lambda_{\mathscr{B}}$.*

The proof of (3) is in Pietsch [2. Satz 1.13, 1.14]. The proofs of (4) are in Köthe [1. § 30.5(8) and § 30.7(1), (5)].

Our terminology for locally convex spaces will be that of Köthe [1]. E will always denote a locally convex Hausdorff space. E has a fundamental system of absolutely convex closed neighborhoods of zero which we denote by $\mathscr{U}(E)$. For every $U \in \mathscr{U}(E)$ there is a continuous seminorm on E denoted by p_U and defined by the formula

$$p_U(x) = \sup \{|\langle u, x\rangle| \mid u \in U^o\} .$$

We shall always consider E', the topological dual of E, to be equipped with the strong topology, that is, the topology defined by the neighborhoods B^o or seminorms

$$p_{B^o}(u) = \sup \{|\langle u, x\rangle| \mid x \in B\}$$

where B ranges over the bounded subsets of E.

Let $U \in \mathscr{U}(E)$ and p_U be the corresponding seminorm. Let $N(U)$ denote the kernel of p_U and let $E_U = E/N(U)$ be the normed quotient space formed by equipping $E/N(U)$ with the quotient norm induced by p_U. Dually, let B be a closed absolutely convex bounded subset of E and let $E_B = \bigcup_{n=1}^{\infty} nB$. Then E_B is a linear subspace of E and the Minkowski functional q_B of B is a norm on E_B. In particular we may perform these constructions in the dual space E'. If B is bounded in E then B^o is an absolutely convex closed neighborhood of o in E' and we can form the quotient space E'_{B^o} which is a normed space with norm $p_{B^o}(a) = \sup \{|\langle a, x\rangle| \mid x \in B\}$. Dually if $U \in \mathscr{U}(E)$ then U^o is an absolutely convex closed bounded (weakly compact) subset of E' and we can form the subspace E'_{U^o} which is a (B)-space with norm $q_{U^o}(a) = \sup \{|\langle a, x\rangle| \mid x \in U\}$. The next proposition is an easy conseqence of these definitions

(5) (a) *E'_{U^o} is a (B)-space with norm q_{U^o} and can be identified with the dual space of E_U, p_U.*

(b) *E_B is a norm space with norm q_B and can be identified with a linear subspace of the dual space of E'_{B^o}, p_{B^o}.*

3. The space $\Lambda\{E\}_{\mathscr{M}}$. Let Λ be a perfect sequence space and let E be a locally convex space. $\Lambda\{E\}$ is the vector space of all E-valued sequences $x = (x_n)$ such that the sequence of scalars $p_U(x_n)$ is in Λ for every $U \in \mathscr{U}(E)$. If $\mathscr{M}$ is a normal topologizing system for Λ, $\Lambda\{E\}_{\mathscr{M}}$ will denote $\Lambda\{E\}$ equipped with the locally convex Hausdorff $\mathscr{M}$-topology defined by the family of seminorms

(1) $\pi_{M,U}(x) = \sup\{\Sigma|\alpha_n|p_U(x_n)|\alpha \in M\}$ where $M \in \mathscr{M}$, $U \in \mathscr{U}(E)$.

The following two statements are simple consequences of these definitions.

(2) *I_n: $\Lambda\{E\}_{\mathscr{M}} \to E$ defined by $I_n(x) = x_n$ is a continuous linear map for every $n = 1, 2, \cdots$.*

(3) *I_U: $\Lambda\{E\}_{\mathscr{M}} \to \Lambda_{\mathscr{M}}$ defined by $I_U(x) = (p_U(x_n))$ is uniformly continuous for every $U \in \mathscr{U}(E)$.*

A subset A of $\Lambda\{E\}$ is called bounded if for every $\alpha \in \Lambda^x$ and $U \in \mathscr{U}(E)$ there exists a constant ρ such that $\Sigma|\alpha_n|p_U(x_n) \leqq \rho$ for all $x \in A$. For each $x \in \Lambda\{E\}$, define $N(x) = \{(\lambda_n x_n) \mid |\lambda_n| \leqq 1 \text{ all } n\}$. A subset A of $\Lambda\{E\}$ is called normal if $x \in A$ implies $N(x) \subset A$. The set $N(A) = \bigcup_{x \in A} N(x)$ is called normal hull of A. We observe that $\Lambda\{E\}$ is itself normal since Λ is normal.

(4) *The following statements are equivalent for a subset A of $\Lambda\{E\}$.*

(a) *A is bounded.*

(b) *The normal hull of A is bounded.*

(c) *A is $\mathscr{M}$-bounded for some (every) $\mathscr{M}$-topology on $\Lambda\{E\}$.*

(d) *For every $U \in \mathscr{U}(E)$, $I_U(A)$ is bounded in Λ.*

(e) *For every $U \in \mathscr{U}(E)$, $I_U(A)$ is $\mathscr{M}$-bounded in Λ for some (every) $\mathscr{M}$-topology on Λ.*

Proof. The equivalences (a) $\Leftrightarrow$ (b), (a) $\Leftrightarrow$ (d), and (c) $\Leftrightarrow$ (e) follow directly from the definitions. (d) $\Leftrightarrow$ (e) is a consequence of 2.(2).

(5) *If E is complete, then $\Lambda\{E\}_{\mathscr{M}}$ is complete.*

Proof. Let $x^{(\nu)}$ be a Cauchy net in $\Lambda\{E\}_{\mathscr{M}}$. Continuity of the linear map I_n implies $x_n^{(\nu)}$ is a Cauchy net in E for each fixed n and hence must converge to some x_n in E. Uniform continuity of the map I_U implies $(p_U(x_n^{(\nu)}))$ is a Cauchy net in $\Lambda_{\mathscr{M}}$ and hence must converge to some $\alpha^{(U)} = (\alpha_n^{(U)})$ in $\Lambda_{\mathscr{M}}$. Because of the coordinatewise convergence of $x^{(\nu)}$ to $x = (x_n)$ we have $p_U(x_n) = \alpha_n^{(U)}$. Thus $(p_U(x_n))$ is in Λ and x is therefore in $\Lambda\{E\}$. Finally $x^{(\nu)}$ converges to x in the $\mathscr{M}$-topology for if $\varepsilon > 0$ is given and ν_0 is such that

$$\pi_{M,U}(x^{(\nu)} - x^{(\mu)}) = \sup\{\Sigma|\alpha_n|p_U(x_n^{(\nu)} - x_n^{(\mu)})|\alpha \in M\} < \varepsilon$$

for all $\nu, \mu \geqq \nu_0$, then

$$\pi_{M,U}(x^{(\nu)} - x) \leqq \varepsilon \quad \text{for all} \quad \nu \geqq \nu_0 .$$

We denote by $x(\leqq n) = (x_1, \cdots, x_n, 0 \cdots)$ the nth finite section of a sequence x in $\Lambda\{E\}$. Let $[\Lambda\{E\}_{\mathscr{M}}]$ be the subspace of $\Lambda\{E\}_{\mathscr{M}}$ consisting of all those x in $\Lambda\{E\}_{\mathscr{M}}$ which are the $\mathscr{M}$-limit of their finite sections; that is $[\Lambda\{E\}_{\mathscr{M}}]$ consists of those x for which $\pi_{M,U}(x - x(\leqq n))$ converges to zero for every $M \in \mathscr{M}$ and $U \in \mathscr{U}(E)$.

(6) *A sequence x in $\Lambda\{E\}$ is in $[\Lambda\{E\}_{\mathscr{M}}]$ if, and only if, for every $U \in \mathscr{U}(E)$, $I_U(x) = (p_U(x_n))$ is in $[\Lambda_{\mathscr{M}}]$.*

In general $[\Lambda\{E\}_{\mathscr{M}}]$ will be a proper subspace of $\Lambda\{E\}_{\mathscr{M}}$, but using (6) and 2.(4) we obtain

(7) (a) $[\Lambda\{E\}_{\mathscr{N}}] = \Lambda\{E\}_{\mathscr{N}}$.
(b) *If $\Lambda_{\mathscr{B}}$ is reflexive then $[\Lambda\{E\}_{\mathscr{B}}] = \Lambda\{E\}_{\mathscr{B}}$.*

(8) *$[\Lambda\{E\}_{\mathscr{M}}]$ is a closed subspace of $\Lambda\{E\}_{\mathscr{M}}$ and hence complete if E is complete.*

Proof. If $x \in \Lambda\{E\}$ is the limit of a net $x^{(\nu)}$ in $[\Lambda\{E\}_{\mathscr{M}}]$, then for each $U \in \mathscr{U}(E)$ $I_U(x) = \lim_\nu I_U(x^{(\nu)})$ is in $[\Lambda_{\mathscr{M}}]$ since $[\Lambda_{\mathscr{M}}]$ is closed in $\Lambda_{\mathscr{M}}$. But then by (6) x is in $[\Lambda\{E\}_{\mathscr{M}}]$.

4. The dual space of $[\Lambda\{E\}_{\mathscr{M}}]$. The α – dual of $\Lambda\{E\}$, denoted $\Lambda\{E\}^x$, is the vector space of all E'-valued sequences $a = (a_n)$ such that $\Sigma|\langle a_n, x_n\rangle| < \infty$ for all $x = (x_n)$ in $\Lambda\{E\}$.

(1) *For every a in $\Lambda\{E\}^x$ and for every bounded set B in E, $(p_{B^0}(a_n))$ is in Λ^x. That is $\Lambda\{E\}^x \subset \Lambda^x\{E'\}$.*

Proof. Let $B \in \Lambda$ be arbitrary. For each n, there exists $x_n \in B$ such that

$$|\beta_n|p_{B^0}(a_n) = p_{B^0}(\beta_n a_n) \leqq |\langle \beta_n a_n, x_n\rangle| + 2^{-n} .$$

Since (x_n) is a bounded sequence in E, $(\beta_n x_n)$ is in $\Lambda\{E\}$ and therefore

$$\Sigma |\beta_n| p_{B^o}(a_n) \leqq \Sigma |\langle a_n, \beta_n x_n\rangle| + 2^{-n} < \infty .$$

Since $\beta \in \Lambda$ was arbitrary, $p_{B^o}(a_n)$ is in Λ^x.

(2) *If* $x \in \Lambda\{E\}$ *and* $\gamma \in c_0$ (c_0 = *scalar sequences convergent to zero*), *then* $\gamma x = (\gamma_n x_n)$ *is in* $[\Lambda\{E\}_{\mathscr{M}}]$.

Proof. It follows easily from the definition of the seminorms $\pi_{M,U}$ that

$$\pi_{M,U}(\gamma x(> i)) \leqq \sup_{n>i} |\gamma_n| \pi_{M,U}(x)$$

and the right side converges to zero as $i \to \infty$, so γx is the limit of its finite sections.

(3) *Every continuous linear form* F *on* $[\Lambda\{E\}_{\mathscr{M}}]$ *has a unique representation of the form*

$$\langle F, x\rangle = \langle a, x\rangle = \Sigma\langle a_n, x_n\rangle$$

with $a = (a_n)$ *in* $\Lambda\{E\}^x$.

Proof. Define linear forms on E by $\langle a_n, x\rangle = \langle F, e_n x\rangle$, $x \in E$, e_n is the nth unit coordinate vector in Λ. Continuity of F implies $|\langle F, x\rangle| \leqq \pi_{M,U}(x)$ for some seminorm $\pi_{M,U}$ and for every x in $[\Lambda\{E\}_{\mathscr{M}}]$. Since M is bounded, we have for each n, $\rho_n = \sup\{|\alpha_n| \,|\, \alpha \in M\} < \infty$. For every x in E we have therefore $|\langle a_n, x\rangle| = |\langle F, e_n x\rangle| \leqq \pi_{M,U}(e_n x) = \sup\{|\alpha_n| p_U(x) \,|\, \alpha \in M\} = \rho_n p_U(x)$ and the continuity of a_n is established.

Clearly $a = (a_n)$ represents F since $\langle F, x\rangle = \lim_{i\to\infty}\langle F, x(\leqq i)\rangle = \lim_{i\to\infty}\langle F, \sum_{n=1}^{i} e_n x_n\rangle = \lim_{i\to\infty}\sum_{n=1}^{i}\langle a_n, x_n\rangle = \sum\langle a_n, x_n\rangle$.

Finally we show $a \in \Lambda\{E\}^x$. Let $x \in \Lambda\{E\}^x$ be arbitrary. For every $\gamma \in c_0$, we can choose $\lambda = (\lambda_n)$ with $|\lambda_n| = 1$ so that $|\gamma_n\langle a_n, x_n\rangle| = \lambda_n\gamma_n\langle a_n, x_n\rangle$. By (2), $\lambda\gamma x = (\lambda_n\gamma_n x_n)$ is in $[\Lambda\{E\}_{\mathscr{M}}]$ and hence $\sum|\gamma_n|\,|\langle a_n, x_n\rangle| = \sum\lambda_n\gamma_n\langle a_n, x_n\rangle = \langle F, \lambda\gamma x\rangle < \infty$. Since $\gamma \in c_0$ was arbitrary, this shows that $\sum|\langle a_n, x_n\rangle| < \infty$ and hence that $a \in \Lambda\{E\}^x$.

REMARKS. Combining (1) and (3) yields $[\Lambda\{E\}_{\mathscr{M}}]' \subset \Lambda\{E\}^x \subset \Lambda^x\{E'\}$. Conditions sufficient for the equality of these spaces are given in the next section. We now proceed to an explicit characterization of $[\Lambda\{E\}_{\mathscr{M}}]'$.

(4) *If* $a \in \Lambda\{E\}^x$ *defines a continuous linear form on* $[\Lambda\{E\}_{\mathscr{M}}]$, *then there exists* $U \in \mathscr{U}(E)$ *such that* $a_n \in E'_{U^o}$ *for all* n *and moreover* $(q_{U^o}(a_n)) \in \Lambda^x$.

Proof. Continuity of a implies $|\langle a, x\rangle| \leqq \pi_{M,U}(x)$ for some seminorm $\pi_{M,U}$ and for all $x \in [\Lambda\{E\}_{\mathscr{M}}]$. As in the proof of (3), we obtain that for every n, and for every $u \in E$, $|\langle a_n, u\rangle| \leqq \rho_n p_U(u)$ from which it follows that $a_n \in E'_{U^\circ}$ and $q_{U^\circ}(a_n) \leqq \rho_n$. We must show that $(q_{U^\circ}(a_n)) \in \Lambda^x$.

Let $\beta \in \Lambda$ be arbitrary and set $\rho = \sup\{\sum |\alpha_n\beta_n| \mid \alpha \in M\}$. For each n, there exists $y_n \in U$ such that $q_{U^\circ}(\beta_n a_n) \leqq \langle \beta_n a_n, y_n\rangle + 2^{-n}$. For each i, the finite section $\beta y(\leqq i)$ of the sequence $(\beta_n y_n)$ is in $[\Lambda\{E\}_{\mathscr{M}}]$ and therefore

$$\sum_{n=1}^{i} \langle \beta_n a_n, y_n\rangle = \langle a, \beta y(\leqq i)\rangle \leqq \pi_{M,U}(\beta y(\leqq i))$$
$$= \sup\left\{\sum_{n=1}^{i} |\alpha_n| p_U(\beta_n y_n) \mid \alpha \in M\right\}$$
$$\leqq \sup\left\{\sum_{n=1}^{i} |\alpha_n\beta_n| \mid \alpha \in M\right\} \leqq \rho .$$

Since i was arbitrary, $\sum\langle \beta_n a_n, y_n\rangle < \infty$. It follows that $\sum|\beta_n| q_{U^0}(a_n) = \sum q_{U^0}(\beta_n a_n) < \infty$ and therefore that $(q_{U^0}(a_n)) \in \Lambda^x$ since $\beta \in \Lambda$ was arbitrary.

(5) *The dual space of $[\Lambda\{E\}_{\mathscr{M}}]$ is the space of all E'-valued sequences $a = (a_n)$ which have a representation of the form $a = \alpha u = (\alpha_n u_n)$ with $\alpha \in \Lambda^x$ and (u_n) an equicontinuous sequence in E'.*

Proof. If we set $\alpha_n = q_{U^0}(a_n)$ and $u_n = (1/\alpha_n)a_n$, ($u_n = 0$ if $\alpha_n = 0$), then (4) says that every element in the dual of $[\Lambda\{E\}_{\mathscr{M}}]$ has the given form.

Conversely, if $a = \alpha u = (\alpha_n u_n)$ with $\alpha \in \Lambda^x$ and (u_n) equicontinuous, then, choosing M with $\alpha \in M$ and $U \in \mathscr{U}(E)$ with $(u_n) \subset U^\circ$, we obtain

$$|\langle a, x\rangle| \leqq \sum |\alpha_n| \, |\langle u_n, x_n\rangle| \leqq \pi_{M,U}(x)$$

for all x in $[\Lambda\{E\}_{\mathscr{M}}]$ and hence a is continuous.

Using the methods of the proofs of (4) and (5), one can show

(6) *The equicontinuous subsets of $[\Lambda\{E\}_{\mathscr{M}}]'$ are the sets of the form*

$$\{\alpha u \mid \alpha = (\alpha_n) \in M, u = (u_n) \subset U^\circ\}$$

where $M \in \mathscr{M}$ and $U \in \mathscr{U}(E)$.

5. **Fundamentally Λ-bounded spaces.** In the previous section, we saw that $[\Lambda\{E\}_{\mathscr{M}}]' \subset \Lambda\{E\}^x \subset \Lambda^x\{E'\}$. In this section we establish conditions sufficient for the equality $\Lambda\{E\}^x = \Lambda^x\{E'\}$ and for the more interesting equality $[\Lambda\{E\}_{\mathscr{M}}]' = \Lambda^x\{E'\}$. We also give conditions which insure the strong dual of $[\Lambda\{E\}_{\mathscr{M}}]$ is $\Lambda^x\{E'\}_{\mathscr{B}}$. Finally we give suffi-

cient conditions for $\Lambda\{E\}_{\mathscr{B}}$ to be reflexive.

The important concept in all these conditions is that of a "fundamantally Λ-bounded" space E. A locally convex space E is fundamentally Λ-bounded if all the bounded subsets of $\Lambda\{E\}$ can be obtained in a natural way from the bounded subsets of Λ and E.

Let R be a normal bounded subset of Λ and let B be a closed absolutely convex bounded subset of E. Define $[R, B] = \{x \in \Lambda\{E\} \mid x_n \in E_B$ and $(q_B(x_n)) \in R\}$.

The following are simple consequences of this definition.

(1) *$[R, B]$ is a bounded subset of $\Lambda\{E\}$.*

(2) *If $R \subset R'$ and $B \subset B'$, then $[R, B] \subset [R', B']$.*

Let V be a vector space in which the notion of a bounded set has been defined. A collection $\mathscr{B}$ of subsets of V is called a fundamental system of bounded sets for V if every bounded set in V is contained in some set in $\mathscr{B}$.

We shall say that a locally convex space E is fundamentally Λ-bounded if the collection of all sets of the form $[R, B]$ form a fundamental system of bounded sets for $\Lambda\{E\}$, where R and B run through a fundamental system of bounded sets for Λ and E respectively.

(3) *If E is fundamentally Λ-bounded, then $\Lambda\{E\}^x = \Lambda^x\{E'\}$.*

Proof. We need only show the inclusion $\Lambda^x\{E'\} \subset \Lambda\{E\}^x$. Let $a \in \Lambda^x\{E'\}$ and let $x \in \Lambda\{E\}$. Then there exist R and B with $x \in [R, B]$ and hence $(q_B(x_n)) \in \Lambda$. But $(p_{B^0}(a_n)) \in \Lambda^x$, and therefore

$$\sum |\langle a_n, x_n \rangle| \leqq \sum p_{B^0}(a_n) q_B(x_n) < \infty .$$

Since x was arbitrary, this shows $a \in \Lambda\{E\}^x$.

Recall that a locally convex space E is called σ-infrabarreled if every countable strongly bounded subset of E' is equicontinuous. Clearly every infrabarreled space is σ-infrabarreled.

The next theorem is the main result of this section.

(4) *Let E be a σ-infrabarreled space and let Λ be a perfect sequence space.*

(a) *If E' is fundamentally Λ^x-bounded, then the dual of $[\Lambda\{E\}_{\mathscr{M}}]$ is $\Lambda^x\{E'\}$.*

(b) *If moreover E is fundamentally Λ-bounded, then the strong dual of $[\Lambda\{E\}_{\mathscr{M}}]$ is $\Lambda^x\{E'\}_{\mathscr{B}}$.*

Proof. (a) We need only show the inclusion $\Lambda^x\{E'\} \subset [\Lambda\{E\}_{\mathscr{M}}]$. Let $a \in \Lambda^x\{E'\}$. By hypothesis there exists a bounded set D in E' such that $(q_D(a_n)) \in \Lambda^x$. For each n, set $u_n = q_D(a_n)^{-1} a_n$ ($u_n = 0$ if $q_D(a_n) = 0$).

Then u_n is in D for each n. Since E is σ-infrabarreled, $\{u_n \mid n = 1, 2, \cdots\}$ is equicontinuous and hence $a = (a_n) = (q_D(a_n)u_n)$ is in $[\Lambda\{E\}_{\mathscr{M}}]'$ by 4.(5).

(b) If E is fundamentally Λ-bounded, then the strong topology on $[\Lambda\{E\}_{\mathscr{M}}]' = \Lambda^x\{E'\}$ is defined by the seminorms

$$\sigma_{[R,B]}(a) = \sup |\sum \langle a_n, x_n\rangle| = \sup \sum |\langle a_n, x_n\rangle|$$

where the sup is taken over x in $[R, B] \cap [\Lambda\{E\}_{\mathscr{M}}]$. The topology on $\Lambda^x\{E'\}_{\mathscr{B}}$ is defined by the seminorms

$$\pi_{R,B^o}(a) = \sup \{\sum |\alpha_n| p_{B^o}(a_n) \mid \alpha \in R\} .$$

In both cases, R ranges over all normal bounded subsets of Λ and B over all absolutely convex bounded subsets of E. We show these seminorms coincide.

One inequality is easy:

$$\begin{aligned} \sigma_{[R,B]}(a) &= \sup \{\sum |\langle a_n, x_n\rangle| \mid x \in [R, B] \cap [\Lambda\{E\}_{\mathscr{M}}]\} \\ &\leqq \sup \{\sum p_{B^o}(a_n) p_B(x_n) \mid x \in [R, B] \cap [\Lambda\{E\}_{\mathscr{M}}]\} \\ &\leqq \sup \{\sum |\alpha_n| p_{B^o}(a_n) \mid \alpha \in R\} \\ &= \pi_{R,B^o}(a) . \end{aligned}$$

Now the reverse inequality. Let $a \in \Lambda^x\{E'\}$ and let $\varepsilon > 0$. By definition of π_{R,B^o} there exists $\alpha \in R$ with $\pi_{R,B^o}(a) \leqq \varepsilon + \sum |\alpha_n| p_{B^o}(a_n)$. For each n there exists $y_n \in B$ such that $p_{B^o}(a_n) \leqq |\langle a_n, y_n\rangle| + \varepsilon 2^{-n} |\alpha_n|^{-1}$. (If a_n or α_n is zero, let y_n be any element in B.) Let $z_n = \alpha_n y_n$. Then $z \in [R, B]$ and

$$\begin{aligned} \pi_{R,B^o}(a) &\leqq \varepsilon + \sum |\alpha_n| p_{B^o}(a_n) \\ &\leqq \varepsilon + \sum |\alpha_n| \, |\langle a_n, y_n\rangle| + \varepsilon 2^{-n} \\ &= 2\varepsilon + \sum |\langle a_n, z_n\rangle| \\ &= 2\varepsilon + \sup_{\gamma} \{\sum |\gamma_n| \, |\langle a_n, z_n\rangle| \mid \gamma \in c_0, \|\gamma\|_\infty \leqq 1\} \\ &= 2\varepsilon + \sup_{\gamma} \{\sum |\langle a_n, \gamma_n z_n\rangle| \mid \gamma \in c_0, \|\gamma\|_\infty \leqq 1\} \\ &\leqq 2\varepsilon + \sigma_{[R,B]}(a) . \end{aligned}$$

The last inequality follows from the fact that $\gamma z \in [R, B] \cap [\Lambda\{E\}_{\mathscr{M}}]$. Since ε was arbitrary the theorem is proved.

(5) *Let E be locally convex and let Λ be a perfect sequence space such that*

(i) *$\Lambda_{\mathscr{B}}$ and E are both reflexive, and*

(ii) *E is fundamentally Λ-bounded and E' is fundamentally Λ^x-bounded. Then both $\Lambda\{E\}_{\mathscr{B}}$ and its strong dual $\Lambda^x\{E'\}_{\mathscr{B}}$ are reflexive.*

Proof. Since E is reflexive, both E and E' are σ-infrabarreled.

Also E'' is fundamentally Λ^{xx}-bounded since $E = E''$ and $\Lambda = \Lambda^{xx}$. Since $\Lambda_{\mathscr{B}}$ is reflexive, so also is its strong dual $\Lambda^x_{\mathscr{B}}$. It follows from 2.(7)(b) that $[\Lambda\{E\}_{\mathscr{B}}] = \Lambda\{E\}_{\mathscr{B}}$ and $[\Lambda^x\{E'\}_{\mathscr{B}}] = \Lambda^x\{E'\}_{\mathscr{B}}$. This theorem now follows by applying (4) twice, first to $[\Lambda\{E\}_{\mathscr{B}}]$ and then to $[\Lambda^x\{E'\}_{\mathscr{B}}]$.

6. **Examples of fundamentally Λ-bounded spaces.** In this section, we show that there exist nontrivial classes of spaces E and Λ for which E is fundamentally Λ-bounded.

(1) *Every normed space E is fundamentally Λ-bounded for every perfect sequence space Λ.*

Proof. Let A be any bounded subset of $\Lambda\{E\}$, and let B denote the unit ball of E. Then $I_B(A) = \{(\|x_n\|) \mid x \in A\}$ is a bounded subset of Λ and hence contained in some normal bounded set R. Thus $A \subset [R, B]$.

(2) (a) *If E is normed and if Λ is any perfect sequence space, then the strong dual of $[\Lambda\{E\}]_{\mathscr{A}}]$ is $\Lambda^x\{E'\}_{\mathscr{B}}$.*

(b) *If E is reflexive (B)-space and if $\Lambda_{\mathscr{B}}$ is reflexive, then $\Lambda\{E\}_{\mathscr{B}}$ and its strong dual $\Lambda^x\{E'\}_{\mathscr{B}}$ are reflexive.*

This follows from (1) above and 5.(4), (5).

The next lemma is due to Pietsch [3. Satz 1.5.8].

(3) *Every metrizable locally convex space E is fundamentally l^1-bounded.*

We shall also use the following well-known fact. (See e.g. [1. §29.1.(5)].)

(4) *If E is a metrizable locally convex space, and if B_k is a sequence of bounded subsets of E, then there always exist positive scalars λ_k such that $B = \bigcup_{k=1}^{\infty} \lambda_k B_k$ is also bounded.*

(5) *Let Λ and Λ^x be perfect sequences spaces which are α — dual to one another. Suppose Λ^x has a countable fundamental system of bounded sets $N_1 \subset N_2 \subset N_3 \subset \cdots$. Then:*

(a) *Every metrizable locally convex space is fundamentally Λ-bounded.*

(b) *Every (DF)-space is fundamentally Λ^x-bounded.*

See [1. §29], for example, for the definition and basic properties of (DF)-spaces.

Proof. (a) Let E be metrizable and let A be a bounded subset of $\Lambda\{E\}$. Then by A is $\mathscr{B}$-bounded in $\Lambda\{E\}$. Thus for each k and each $U \in \mathscr{U}(E)$, there exists a constant $\rho_{k,U}$ such that for all $x \in A$,

$$\pi_{N_k,U}(x) = \sup\{\sum |\alpha_n| p_U(x_n) | \alpha \in N_k\} \leqq \rho_{k,U} .$$

This implies that the set $A_k = \{\alpha x = (\alpha_n x_n) | \alpha \in N_k,\ x \in A\}$ is a bounded subset of $l^1\{E\}$. By Lemma (3), there exists a bounded set B_k in E such that $A_k \subset [R_1, B_k]$ where R_1 denotes the unit ball of l^1, or equivalently

$$(*) \qquad \sum |\alpha_n| q_{B_k}(x_n) = \sum q_{B_k}(\alpha_n x_n) = ||(q_{B_k}(\alpha_n x_n))||_{l^1} \leqq 1$$

for all $\alpha \in N_k$, $x \in A$. By (4) there exist positive scalars λ_k such that $B = \bigcup_{k=1}^{\infty} \lambda_k B_k$ is bounded. Since $B_k \subset \lambda_k^{-1} B$ we have for all $x \in E_{B_k}$ that $q_B(x) \leqq \lambda_k^{-1} q_{B_k}(x)$. Thus for every k and for all $x \in A$, we have

$$\begin{aligned} p_{N_k^\circ}(q_B(x_n)) &= \sup\{\sum |\alpha_n| q_B(x_n) | \alpha \in N_k\} \\ &\leqq \sup\{\sum |\alpha_n| \lambda_k q_{B_k}(x_n) | \alpha \in N_k\} \\ &\leqq \lambda_k \end{aligned}$$

by (*). This implies that the set $\{(q_B(x_n)) | x \in A\}$ is $\mathscr{B}$-bounded and hence bounded in Λ, and is therefore contained in some normal bounded subset R of Λ. Thus $A \subset [R, B]$ and (a) is proved.

(b) Let E be a (DF)-space. Then E has a countable fundamental system of bounded sets $B_1 \subset B_2 \subset B_3 \subset \cdots$.

Suppose E is not fundamentally Λ^x-bounded, then there exists a bounded subset A in $\Lambda^x\{E\}$ such that A is not contained in any of the sets $[N_k, B_k]$, $k = 1, 2, \cdots$. We show this leads to a contradiction.

For every index k, A not a subset of $[N_k, B_k]$ implies that there exists $x^{(k)} \in A$ such that $(q_{B_k}(x_n^{(k)})) \notin N_k$. Thus there exists $\beta^{(k)} \in N_k^\circ$ such that $\sum \beta_n^{(k)} q_{B_k}(x_n^{(k)}) > 1$. In fact for each k, there exists a finite set $\{u_n^{(k)}\} \subset B_k^\circ$, $n = 1, 2, \cdots, f_k$, such that

$$\sum_{n=1}^{f_k} \beta_n^{(k)} |\langle u_n^{(k)}, x_n^{(k)} \rangle| > 1 .$$

Let $G = \{u_n^{(k)} | k = 1, 2, \cdots,$ and $n = 1, 2, \cdots, f_k\}$. Then G is a countable subset of E'. If G is strongly bounded in E', then G is equicontinuous since E is a (DF)-space. We show G is strongly bounded. Fix m. Since $\{u_n^{(k)} | k = 1, 2, \cdots, m,\ n = 1, 2, \cdots, f_k$ is finite, there exists a positive constant $\rho_m \geqq 1$ with $u_n^{(k)} \in \rho_m B_m^\circ$ for $k = 1, \cdots, m$ and $n = 1, \cdots, f_k$, since B_m° is an absorbing subset of E'. For $k > m$, $B_k \supset B_m$ and hence $B_k^\circ \subset B_m^\circ$ so $u_n^{(k)} \in B_k^\circ \subset B_m^\circ$ for all $k > m$ and $n = 1, 2, \cdots, f_k$. Thus for every m, there exists a positive constant ρ_m with $G \subset \rho_m B_m^\circ$. The sets $B_1^\circ \supset B_2^\circ \supset B_3^\circ \supset \cdots$ form a neighborhood base for the strong topology on E', so G is strongly bounded and hence equi-continuous.

Let $U \in \mathscr{U}(E)$ be such that $G \subset U^\circ$. Since A is bounded in $\Lambda^x\{E\}$, the set $\{p_U(x_n)) \mid x \in A\}$ is bounded in Λ^x and hence contained in some N_k. Since $\beta^{(k)} \in N_k^0$, this implies $\sum \beta_n^{(k)} p_U(x_n) \leqq 1$ for all $x \in A$. But taking $x = x^{(k)}$, we obtain $\sum \beta_n^{(k)} p_U(x_n^{(k)}) > \sum_{n=1}^{f_k} \beta_n^{(k)} |\langle u_n^{(k)}, x_n^{(k)}\rangle| > 1$ which is a contradiction.

As in theorem (5), let Λ and Λ^x be α – dual perfect sequence spaces such that Λ^x has a countable fundamental system of bounded sets. The results of (5) cannot be improved to include either of the following assertions.

(a) Every (DF)-space is fundamentally Λ-bounded.

(b) Every metrizable locally convex space is fundamentally Λ^x-bonnded.

Counterexamples are provided by (9) and (8) below.

Recall that ω is the space of all scalar sequences and ϕ is the space of all scalar sequences with only finitely many nonzero coordinates. ϕ and ω are perfect and α – dual to each other. Moreover ϕ has a countable fundamental system of bounded sets $N_1 \subset N_2 \subset \cdots$, where $N_k = \{\alpha \in \phi \mid |\alpha_n| \leqq k \text{ if } n \leqq k \text{ and } \alpha_n = 0 \text{ if } n > k\}$. The following lemma is due to Pietsch [2, Satz 3.19].

(6) *Let E be a metrizable locally convex space which has no continuous norm. Then there exists $x \in \phi\{E\}$ such that for every index n, $x_n \neq 0$.*

Proof. Let $p_1 \leqq p_2 \leqq \cdots$ be a fundamental system of seminorms for E. No p_k is a norm. Thus for each integer k there exists $x_k \in E$ with $x_k \neq 0$ but $p_k(x_k) = 0$. *Set* $x = (x_n)$. Fix k. For all $n \geqq k$ we have $p_n(x_n) = 0$ but $p_k \leqq p_n$, so $p_k(x_n) = 0$ for all $n \geqq k$. Thus $(p_k(x_n)) \in \phi$ for each seminorm p_k.

(7) *For any locally convex space E, $\omega\{E\}$ is the space of all E-valued sequences.*

(8) *There exist metrizable locally convex spaces E such that E is not fundamentally ϕ-bounded.*

Proof. Let E be a metrizable space with no continuous norm. By (6) there exists $x \in \phi\{E\}$ with $x_n \neq 0$ for all n. Therefore there exist $a_n \in E'$ with $\langle a_n, x_n\rangle = 1$. But by (7), $a = (a_n) \in \omega\{E'\}$. Since $\langle a, x\rangle = \sum \langle a_n, x_n\rangle = \infty$, we conclude $\phi\{E\}^x \neq \omega\{E'\} = \phi^x\{E'\}$. By 5.(3) this implies E is not fundamentally ϕ-bounded.

(9) *There exist (DF)-spaces E such that E is not fundamentally ω-bounded.*

Proof. Let E be a (DF)-space whose strong dual E' is an (F)-space with no continuous norm. By (6) there exists $a \in \phi\{E'\}$ such that $a_n \neq 0$ for all n. Let $x_n \in E$ be such that $\langle a_n, x_n \rangle = 1$. Then $x = (x_n) \in \omega\{E\}$ but $\langle a, x \rangle = \sum \langle a_n, x_n \rangle = \infty$ so we conclude $\omega\{E\} \neq \phi\{E'\} = \omega^\alpha\{E'\}$. By 5.(3) this implies E is not fundamentally ω-bounded.

The space ω may be viewed as a topological product of countably many copies of the scalar field. With the product topology it is a (F)-space with no continuous norm. It is the strong dual of the (DF)-space ϕ viewed as a locally convex direct sum of countably many copies of the scalar field. Thus the examples in (8) and (9) can be made more explicit by taking $E = \omega$ in (8) and $E = \phi$ in (9).

7. The spaces $l^p\{E\}$ $1 \leqq p \leqq \infty$. It is well known that for $1 \leqq p \leqq \infty$ the α – dual of l^p is l^q where $p^{-1} + q^{-1} = 1$. The bounded subsets of l^p are easily seen to be the sets which are bounded in l^p-norm $\|\alpha\|_p = (\sum |\alpha_n|p)^{1/p}$. Thus every l^p space has a countable fundamental system of bounded sets consisting of positive integer multiples of the unit ball.

A sequence $x = (x_n)$ in a locally convex space E is called absolutely p-summable, $1 \leqq p < \infty$, if for every continuous seminorm p_U on E, $\sum p_U(x_n)^p < \infty$.

(1) *$l^p\{E\}, 1 \leqq p < \infty$, is the vector space of all absolutely p-summable sequences in E. $l^\infty\{E\}$ is the vector space of all bounded sequences in E.*

The seminorms defining the $\mathscr{B} = \mathscr{B}(l^q)$ topology on $l^p\{E\}$, $1 \leqq p < \infty$, are given by

$$\begin{aligned}\pi_{kB,U}(x) &= \sup\{\textstyle\sum |\alpha_n| p_U(x_n) \mid \alpha \in kB\} \\ &= \sup\{\textstyle\sum |\alpha_n| p_{k^{-1}U}(x_n) \mid \alpha \in B\} \\ &= (\textstyle\sum p_{k^{-1}U}(x_n)^p)^{1/p}\end{aligned}$$

where k is a positive integer, B is the unit ball in l^q, and U is any absolutely convex neighborhood of 0. Since $k^{-1}U$ is also such a neighborhood, we have

(2) $1 \leqq p < \infty$. *A base of seminorms for $l^p\{E\}_{\mathscr{B}}$ is given by the family of seminorms*

$$\pi_U^{(p)}(x) = (\textstyle\sum p_U(x_n)^p)^{1/p} \qquad U \in \mathscr{U}(E)\,.$$

A similar argument for the case $p = \infty$ yields

(3) *A base of seminorms for $l^\infty\{E\}_{\mathscr{B}}$ is given by the family of*

seminorms

$$\pi_U^{(\infty)}(x) = \sup\{p_U(x_n) \mid n = 1, 2, \cdots\} .$$

It follows that an element x in $l^\infty\{E\}_{\mathscr{B}}$ will be the limit of its finite sections if and only if $p_U(x_n)$ converges to 0 for every $U \in \mathscr{U}(E)$. Clearly every element of $l^p\{E\}_{\mathscr{B}}$ is the limit of its finite sections.

(4) $[l^p\{E\}_{\mathscr{B}}] = l^p\{E\}_{\mathscr{B}}$ for $1 \leqq p < \infty$

$[l^\infty\{E\}_{\mathscr{B}}] = c_0\{E\}_{\mathscr{B}} =$ *vector space of all sequences in E converging to* 0.

We now show how the results of the previous sections can be applied to the duality theory of the $l^p\{E\}$ spaces.

(5) *Every metrizable locally convex space and every (DF)-space is fundamentally l^p-bounded for every $p, 1 \leqq p \leqq \infty$.*

Proof. Since every $l^q, 1 \leqq q \leqq \infty$, has a countable fundamental system of bounded sets, and since $(l^p)^x = l^q$ with $p^{-1} + q^{-1} = 1$, this result follows immediately from 6.(5).

(6) *Let E be a metrizable locally convex space or a (DF)-space. For $1 \leqq p < \infty$, the strong dual of $l^p\{E\}_{\mathscr{B}}$ is $l^q\{E'\}_{\mathscr{B}}$, and the strong dual of $[l^\infty\{E\}_{\mathscr{B}}] = c_0\{E\}_{\mathscr{B}}$ is $l^1\{E'\}_{\mathscr{B}}$.*

Proof. This is a direct application of (5) above and 5.(4). (We are also using the facts that the dual of a metrizable space is a (DF)-space and the dual of a (DF)-space is metrizable.)

(7) *If E is a reflexive (B)-, (F)-, or (DF)-space, then for $1 < p < \infty$, $l^p\{E\}_{\mathscr{B}}$ is a reflexive (B)-, (F)-, or (DF)-space respectively.*

Proof. By (6) above and 5.(5), $l^p\{E\}_{\mathscr{B}}$ is reflexive. If E is a (B)- or (F)-space, then it is clear from the fact that the seminorms $\pi_U^{(p)}$, $U \in \mathscr{U}(E)$, define the $\mathscr{B}$-topology on $l^p\{E\}$, that $l^p\{E\}$ is a (B)- or (F)-space respectively. If E is a reflexive (DF)-space, then E' is an (F)-space and $l^p\{E\}_{\mathscr{B}}$ as the strong dual of the (F)-space $l^q\{E'\}_{\mathscr{B}}$ must be a (DF)-space.

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