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DIRECT SUM SUBSET DECOMPOSITIONS OF Z

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Let Z be the set of integers. In this paper it is shown that there is no effective characterization of all direct sum subset decompositions of Z i.e., where $A+B=Z$ and the sums are distinct. Further the result is generalized to include decompositions of a product of sets where Z is a set in the product, and to cases where the number of subsets in the decomposition is greater than two.

The question of characterizing all direct sum subset decompositions for Z, the infinite cyclic group, seems first to have been raised explicitly by de Bruijn [1]. It was mentioned again by de Bruijn [2] in 1956, and Long [5] in 1967. The notation $A \oplus B$ will denote $A + B$ where the sums are distinct. Without loss of generality we will assume 0 is a member of each summand.

For the semi-group Z^+ there is a particularly nice characterization of all direct sum decompositions. The result, which was implicit from the work of de Bruijn [2], was first explicitly by Long [5]. It is the following:

THEOREM 1. *Let $|A| = |B| = \infty$. $A \oplus B = Z^+$ if and only if there exists an infinite sequence of integers $\{m_i\}_{i\geqq 1}$ with $m_i \geqq 2$ for all i, such that A and B are the sets of all finite sums of the form*

$$a = \sum x_{2i} M_{2i}$$
$$b = \sum x_{2i+1} M_{2i+1}$$

respectively, where $0 \leqq x_i < m_{i+1}$ for $i \geqq 0$ where $M_0 = 1$ and $M_i = \prod_{j=1}^{i} m_j$ for $i \geqq 1$.

In case $|A| < \infty$ or $|B| < \infty$, a similar characterization holds with the change that the sequence $\{m_i\}$ will be of finite length r and the only restriction on x_r is that it be nonnegative.

A distinguishing characteristic of decompositions obtained as in Theorem 1 is that either A or B has the property that each of its elements is a multiple of some integer $m \geqq 2$ and it has been conjectured that this property would necessarily hold for any decomposition of Z. The following theorem shows that this is not the case and that the decomposing sets A and B can be quite arbitrary. It follows that there is no real possibility of effectively characterizing A and B. We do obtain a rather weak characterization in Theorem 3.

Throughout this paper, unless otherwise noted, all maximums and minimums will be taken with respect to the following order $0 \prec 1 \prec -1 \prec 2 \prec -2 \prec 3 \prec \cdots$.

THEOREM 2. *Suppose that* A_1 *and* B_1 *are finite, that* $0 \in A_1 \cap B_1$, *and that* $A_1 + B_1 = A_1 \oplus B_1$, *then there exist sets* A *and* B, *both infinite, such that* $A_1 \subset A$, $B_1 \subset B$, *and* $A \oplus B = Z$.

Proof. We first let

$$n_1 = \min(Z \sim (A_1 \oplus B_1))$$

and

$$m_1 = |\max(A_1 \cup B_1 \cup \{n_1\})|$$

where the min and max are taken with respect to the previously mentioned order. We now construct A and B by an inductive procedure. Let

$$A_2 = A_1 \cup \{n_1 + 5m_1\}$$

and

$$B_2 = B_1 \cup \{-5m_1\}$$

then

$$A_2 + B_2 = (A_1 \oplus B_1) \cup (\{n_1 + 5m_1\} + B_1) \cup (\{-5m_1\} + A_1) \cup \{n_1\} .$$

We now claim that $A_2 + B_2 = A_2 \oplus B_2$. Of course, this is immediate if the sets in the above union are mutually disjoint. The fact that they are disjoint is assured by the following inequalities which derive from the definitions of n_1 and m_1 and the fact $m_1 > 0$.

$$\begin{aligned}
&-5m_1 + a \leqq -4m_1 < -2m_1 \leqq a' + b \\
&n_1 + 5m_1 + b \geqq 3m_1 > 2m_1 \geqq a + b' \\
&n_1 + 5m_1 + b > 0 > (-5m_1) + a \\
&n_1 + 5m_1 + b \geqq 3m_1 > |n_1| \\
&|-5m_1 + a| \geqq 4m_1 > |n_1|
\end{aligned}$$

for all $a, a' \in A_1$ and $b, b' \in B_1$.

Note that $n_1 \in A_2 \oplus B_2$ so that we have enlarged the interval about the origin in which all integers are represented. Also it is clear the process can be repeated to obtain sets A_i and B_i for $i \geqq 2$ which are supersets of A_{i-1} and B_{i-1} and such that $A_i + B_i = A_i \oplus B_i$. Setting $A = \bigcup_{i=1}^{\infty} A_i$ and $B = \bigcup_{i=1}^{\infty} B_i$ we have desired decomposition $A \oplus B = Z$, since any given n is an element of $A_{2n+1} \oplus B_{2n+1}$. This completes the proof.

Theorem 2 shows that any two finite sets A and B with $A \subset Z$, $B \subset Z$ and $A + B = A \oplus B$ can be extended to two infinite sets which are a direct sum decomposition of Z. This shows that it is certainly not necessarily the case that every element of A or B has a multiple of some integer $m \geqq 2$. It also shows that no condition can be placed on the size or location of the two elements which sum to a given n. Thus, a best possible type of characterization of decomposition of Z will be of the nature of Theorem 3. Let $A(k) = \{a \in A \mid |a| \leqq |k|\}$.

THEOREM 3. *$A \oplus B = Z$ if and only if for each $n \in Z$ there exists $k \in Z$ such that*

$$A(k) \oplus B(k) \supset Z(n) .$$

Proof. Suppose first that $A \oplus B = Z$ and let $n \in Z$. For each $i \in Z(n)$, set

$$k_i = \max\{a_i, b_i \mid a_i + b_i = i, a_i \in A, b_i \in B\} .$$

Also, set $k = \max\{k_i \mid i \in Z(n)\}$. Then, since $i \in A(k_i) + B(k_i)$, $A(k_i) \subset A(k)$ and $B(k_i) \subset B(k)$ for all $i \in Z(n)$, it follows that

$$Z(n) \subset A(k) + B(k) = A(k) \oplus B(k) .$$

The last equality follows from the fact that $A(k) \subset A$, $B(k) \subset B$ and $A + B = A \oplus B$.

Conversely, suppose that for each $n \in Z$ there exists k_n such that

$$A(k_n) \oplus B(k_n) \supset Z(n) .$$

If we set $A = \bigcup_{n=1}^{\infty} A(k_n)$ and $B = \bigcup_{n=1}^{\infty} B(k_n)$, then clearly $n \in A + B$ for any $n \in Z$; i.e., $Z \subset A + B$. Since $A + B \subset Z$ is trivially true, it follows that $Z = A + B$. If $A + B \neq A \oplus B$, then there exist positive integers i and j and an integer n such that

$$n = a + b = a_i + b_j$$

with $a \in A(k_n)$, $b \in B(k_n)$, $a_i \in A(k_i)$, $b_j \in B(k_j)$, $a \neq a_i$, and $b \neq b_j$. But then, if $k' = \max\{k_i, k_j, k_n\}$, it is clear that

$$A(k_p) + B(k_q) \subset A(k') + B(k')$$

for p and $q \in \{i, j, n\}$, and this implies that n has two representations in $A(k') + B(k')$ in violation to the fact

$$A(k') + B(k') = A(k') \oplus B(k') .$$

This completes the proof.

We now consider the remaining case for $A \oplus B = Z$ when one of A and B is finite. We assume without loss of generality that $|A| < \infty$.

THEOREM 4. *Let* $|A| < \infty$, *then* $A \oplus B = Z$ *if and only if there exists an* n *such that* $B = nZ \oplus C$ *where* $C \subset \{0, 1, \cdots, n-1\}$ *and* $A \oplus C$ *is a complete residue system modulo* n.

Proof. Since $A \oplus C$ is a complete residue system modulo n, it is clear that

$$Z = (A \oplus C) \oplus nZ = A \oplus (C \oplus nZ) = A \oplus B .$$

Conversely suppose $A \oplus B = Z$ with A of finite order. Under these conditions, Hajós [3] proved that B is periodic, i.e., there exists an $n \neq 0$ such that $n + B = B$. Since $0 \in B$ we have $nZ \subset B$. Letting $C = B \cap \{0, 1, \cdots, n-1\}$ we have $B = nZ \oplus C$. Since $Z = A \oplus B = (A \oplus C) + nZ$, it is clear that $A \oplus C$ must be a complete residue system modulo n.

2. Generalizations. Consider subsets $D_i \subset Z$ with $0 \in D_i$ for $1 \leqq i \leqq n$, then a generalization of Theorem 2 is obtained by replacing Z with $Z \times D_1 \times \cdots \times D_k$. The method of proof is similar except that the order is replaced by $(x_0, \cdots, x_k) \succ (y_0, y_1, \cdots, y_k)$ if $\sum |x_i| > \sum |y_i|$ or, in case $\sum |x_i| = \sum |y_i|$, $(x_0, \cdots, x_k) \succ (y_0, \cdots, y_k)$ if $x_i > y_i$ for the least i such that $x_i \neq y_i$. The least element in the ordering for which there is no representation by $A_1 \oplus B_1$ is n_1. The m_1 is the maximum over any entry in any element of A_1, B_1 or $\{n_1\}$. In particular, this shows that no strong characterization such as that exhibited for $Z^+ \times Z^+$ by Hanson [4] and Niven [6] can exist for $Z \times Z$ or $Z \times Z^+$.

The preceding theorems all generalize from the case of two summands to the case of any number of summands. Further, since the construction of Theorem 2 is by a single element at a time, the summands can be created with any given order.

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