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ON THE FROBENIUS RECIPROCITY THEOREM FOR SQUARE-INTEGRABLE REPRESENTATIONS

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In this paper, a global version of the Frobenius reciprocity theorem is established for irreducible square-integrable representations of locally compact unimodular groups. As in the classical compact case, it asserts that certain intertwining spaces are canonically and isometrically isomorphic. The proof is elementary, and the appropriate isomorphism is exhibited explicitely. The essential point is that square-integrability implies the continuity of functions in certain subspaces of L^2 spaces on which the group acts and leads to a characterization of the subspaces in terms of reproducing kernels.

The preliminary results on reproducing kernels are contained in Theorems 1 and 2 in § 2. Our main result on reciprocity, Theorem 3 in § 3, does not require direct integral decomposition theory as in [2] and [4] and is formally similar to the version of the reciprocity theorem proved by C. C. Moore in [5]; however, we only consider unitary representations, and do not need to formulate the result in terms of summable induced representations on L^1-spaces.

After this paper was initially submitted, we learned that A. Wawrzyńczyk [6] had already proved a result, similar but not identical to our Theorem 3. His proof is based on a general duality theorem for automorphic forms due to K. and L. Maurin [3], and he does not prove results corresponding to our Theorems 1 and 2.

Let G be a locally compact unimodular group and S a continuous irreducible square-integrable unitary representation of G on a complex Hilbert space $\mathscr{H}$. We recall that this implies

$$x \longrightarrow (S(x)\varphi \mid \psi),\ x \in G$$

is square-integrable on G for all φ and ψ in $\mathscr{H}$ and the existence of a positive constant d (the formal degree) such that

$$\int_G (S(x)\varphi | \alpha)\, \overline{(S(x)\psi \mid \beta)} dx = d^{-1}(\varphi \mid \psi)\overline{(\alpha \mid \beta)} \tag{1.1}$$

for all φ, α, ψ, β in $\mathscr{H}$.

Let K be a compact subgroup of G and λ a continuous irreducible unitary representation of K on a complex-Hilbert space $\mathscr{K}$. Let $T = T(\cdot, \lambda)$ be the continuous unitary representation of G induced by λ. By definition, $T(y)(y \in G)$ is right translation by y on the space $L^2(G, \lambda)$ of all square-integrable maps $f: G \rightarrow \mathscr{K}$ such that

$$f(kx) = \lambda(k)f(x) \tag{1.2}$$

for all (k, x) in $K \times G$.

Now let $\mathscr{I}(S, T)$ denote the Banach space of bounded linear maps $U: \mathscr{H} \to L^2(G, \lambda)$ which intertwine S and T, i.e., are such that

$$US(x) = T(x)U \tag{1.3}$$

for all x in G. Similarly, let $\mathscr{I}(S_K, \lambda)$ denote the space of operators intertwining S_K (the restriction of S to K) and λ.

In § 2, we obtain certain properties of the spaces $U(\mathscr{H})$ for U in $\mathscr{I}(S, T)$, and using these properties, we then show in § 3 that there is a canonical isometric isomorphism of $\mathscr{I}(S_K, \lambda)$ onto $\mathscr{I}(S, T)$. From this we conclude that T contains S (discretely) exactly as many times as S_K contains λ.

2. The spaces $U(\mathscr{H})$, $U \in \mathscr{I}(S, T)$. Because S is irreducible, it is easy to see that each U in $\mathscr{I}(S, T)$ is a scalar multiple of an isometry (cf. the argument proving (3.3)). Hence, $U(\mathscr{H})$ is a closed subspace (possibly 0) of $L^2(G, \lambda)$. Less obvious and much more important is the fact that each function class $U\varphi(U \in \mathscr{I}(S, T), \varphi \in \mathscr{H})$ contains a unique continuous function.

THEOREM 1. *Let U be any operator intertwining S and T. Then $U(\mathscr{H})$ is a closed subspace of $L^2(G, \lambda)$ consisting of continuous functions in which point evaluations*

$$f \to f(x) , \qquad f \in U(\mathscr{H})$$

are continuous linear maps of $U(\mathscr{H})$ into $\mathscr{K}$ for every x in G.

Proof. Let $\varphi \in \mathscr{H}$ and f any function in the class $U\varphi$. Set

$$e(y) = d(\varphi \mid S(y)\varphi) , \qquad y \in G .$$

Then, because G is unimodular and in view of (1.1), it follows that

$$\int_G |e(y)| \, ||f(xy)|| \, dy \leqq \left(\int_G |e(y)|^2 dy\right)^{1/2} \left(\int_G ||f(xy)||^2 dy\right)^{1/2}$$
$$= d^{1/2} ||\varphi||^2 ||f||_2 .$$

Thus, we can define a bounded function $g: G \to \mathscr{K}$ which satisfies (1.2) by setting

$$g(x) = \int_G e(y)f(xy)dy . \tag{2.1}$$

Moreover, g is continuous. For

$$\|g(x) - g(zx)\| \leqq \int_G |e(y)| \, \|f(xy) - f(zxy)\| \, dy$$
$$\leqq d^{1/2} \|\varphi\|^2 \left(\int_G \|f(y) - f(zy)\|^2 dy\right)^{1/2} \longrightarrow 0 \quad \text{as} \quad z \longrightarrow 1 \, .$$

Now let h be any function with compact support in $L^2(G, \lambda)$. Then

$$\begin{aligned}
&\int_G (g(x) \mid h(x)) dx \\
&= \int_G \left(\int_G e(y) f(xy) dy \mid h(x)\right) dx = \int_G dx \int_G e(y)(f(xy) \mid h(x)) dy \\
&= \int_G dy \int_G e(y)(f(xy) \mid h(x)) dx = \int_G e(y)(T(y)f \mid h) dy \\
&= \int_G e(y)(US(y)\varphi \mid h) dy && \text{(by (1.3))} \\
&= d\int_G (S(y)\varphi \mid U^*h)\overline{(S(y)\varphi \mid \varphi)} dy \\
&= (\varphi \mid \varphi)\overline{(U^*h \mid \varphi)} && \text{(by (1.1))} \\
&= (\|\varphi\|^2 f \mid h) \, .
\end{aligned}$$

Since this holds for all such h, it follows that

$$g(x) = \|\varphi\|^2 f(x) \, , \quad \text{a.e.} \, . \tag{2.2}$$

Because the complement of a set of Haar measure 0 is dense, it follows that each function class $U\varphi$ contains a unique continuous function; from now on that function will be denoted by $U\varphi$.

Suppose $\varphi \neq 0$. Then from (2.1), (2.2) and the computations above, we have

$$(U\varphi)(x) = \frac{d}{\|\varphi\|^2} \int_G (U\varphi)(xy)\overline{(S(y)\varphi \mid \varphi)} dy \tag{2.3}$$

and

$$\|U\varphi(x)\| \leqq d^{1/2} \|U\varphi\|_2 \tag{2.4}$$

for every x in G. Therefore, point evaluations are continuous.

Now suppose $U(\mathscr{H}) \neq 0$. Then, since the maps

$$E_x : f \longrightarrow f(x) \, , \quad f \in U(\mathscr{H}) \, , \quad x \in G$$

are all continuous, $U(\mathscr{H})$ is completely determined by the positive definite kernel

$$Q(x, y) = E_x E_y^* \tag{2.5}$$

in the simple fashion described in [1]. On the other hand, it is easy to see directly that $Q(x, y) = P(xy^{-1})$ where

(2.6) $$P(x) = E_1 T(x) E_1^* , \qquad x \in G$$

and that following result is valid.

THEOREM 2. *The operator-valued function P is continuous and square-integrable on G. It has formal properties*

(1) $P(x)^* = P(x^{-1})$

(2) $P(xy^{-1}) = E_x E_y^*$

(3) $P(k_1 x k_2) = \lambda(k_1) P(x) \lambda(k_2)$

(4) $P(x) = \int_G P(xy^{-1}) P(y) dy$

which are valid for all k_1, k_2 in K and x, y in G. Moreover, left convolution by P is the orthogonal projection of $L^2(G, \lambda)$ on $U(\mathscr{H})$; in particular

(5) $f(x) = \int_G P(xy^{-1}) f(y) dy$

for all f in $U(\mathscr{H})$ and x in G.

Proof. Equation (1) follows from (2.6); (2) and (3) are consequences of the relations $E_x = E_1 T(x) (x \in G)$ and $E_1 T(k) = \lambda(k) E_1 (k \in K)$. If α and β are vectors in $\mathscr{K}$, then

$$(P(x)\alpha \mid \beta) = (T(x) E_1^* \alpha \mid E_1^* \beta)$$

and since T is equivalent to S in $U(\mathscr{H})$, it follows that $x \to (P(x)\alpha \mid \beta)$ is not only continuous but square-integrable on G. We also have

$$\begin{aligned}
(P(x)\alpha \mid \beta) &= (E_1^* \alpha \mid T(x^{-1}) E_1^* \beta) \\
&= \int_G (E_1^* \alpha(y) \mid T(x^{-1}) E_1^* \beta(y)) dy \\
&= \int_G (E_y E_1^* \alpha \mid E_y T(x^{-1}) E_1^* \beta) dy \\
&= \int_G (P(y)\alpha \mid P(yx^{-1})\beta) dy && \text{(by (2))} \\
&= \int_G (P(xy^{-1}) P(y)\alpha \mid \beta) dy && \text{(by (1))}
\end{aligned}$$

for all α, β in $\mathscr{K}$; hence, (4) is true.

Now suppose $f \in L^2(G, \lambda)$. Then for any x in G and α in $\mathscr{K}$

$$\begin{aligned}
(f \mid T(x^{-1}) E_1^* \alpha) &= \int_G (f(y) \mid (T(x^{-1}) E_1^* \alpha)(y)) dy \\
&= \int_G (f(y) \mid E_y T(x^{-1}) E_1^* \alpha) dy \\
&= \int_G (f(y) \mid P(yx^{-1})\alpha) dy = \int_G (P(xy^{-1}) f(y) \mid \alpha) dy .
\end{aligned}$$

If f is orthogonal to $U(\mathscr{H})$, then $(f \mid T(x^{-1})E_1^*\alpha) = 0$ for all x and α; hence

$$\int_G (P(xy^{-1})f(y) \mid \alpha)dy = 0$$

for all x and α. Therefore

$$\int_G P(xy^{-1})f(y)dy = 0 \;, \qquad f \in U(\mathscr{H})^\perp \;. \tag{2.7}$$

On the other hand, if $f \in U(\mathscr{H})$, then

$$(f \mid T(x^{-1})E_1^*\alpha) = (E_1T(x)f \mid \alpha)$$

so that

$$(f(x) \mid \alpha) = \int_G (P(xy^{-1})f(y) \mid \alpha)dy$$

for all x and α; hence, (5) is valid. To complete the proof it is enough to observe that (5) and (2.7) imply that for any f in $L^2(G, \lambda)$, the function

$$g(x) = \int_G P(xy^{-1})f(y)dy \;, \qquad x \in G$$

is the orthogonal projection of f on $U(\mathscr{H})$.

3. The reciprocity theorem. In the statement of the next result, which is our version of the Frobenius reciprocity theorem for square-integrable representations, we retain the assumptions and notation used in §§ 1 and 2.

THEOREM 3. *The intertwining spaces $\mathscr{I}(S_K, \lambda)$ and $\mathscr{I}(S, T)$ are canonically isomorphic via an isometric linear map*

$$A \to U_A \;, \qquad A \in \mathscr{I}(S_K, \lambda)$$

that is defined by the equation

$$(U_A\varphi)(x) = cAS(x)\varphi \;, \quad \varphi \in \mathscr{H}, \quad x \in G \tag{3.1}$$

in which $c = (d/\dim(\mathscr{K}))^{1/2}$.

Proof. Let $A \in \mathscr{I}(S_K, \lambda)$, $\varphi \in \mathscr{H}$, and define f on G by

$$f(x) = AS(x)\varphi \;, \qquad x \in G \;.$$

Then f is continuous, and

$$f(kx) = AS(k)S(x)\varphi = \lambda(k)AS(x)\varphi = \lambda(k)f(x)$$

for all (k, x) in $K \times G$. If $\alpha \in \mathscr{K}$ then

$$(f(x) \mid \alpha) = (AS(x)\varphi \mid \alpha) = (S(x)\varphi \mid A^*\alpha) .$$

Since S is square-integrable, it follows that $x \longrightarrow (f(x) \mid \alpha)$ is square-integrable for each α in $\mathscr{K}$. Hence, since $\mathscr{K}$ is necessarily finite dimensional

$$\int_G \| f(x) \|^2 \, dx < \infty .$$

It follows that (3.1) defines an element $U_A\varphi$ in $L^2(G, \lambda)$, and $\varphi \longrightarrow U_A\varphi(\varphi \in \mathscr{H})$ is a linear map U_A of $\mathscr{H}$ into $L^2(G, \lambda)$.

Now suppose A and B lie in $\mathscr{I}(S_K, \lambda)$, let $\varepsilon_1, \cdots, \varepsilon_n$ be an orthonormal base for $\mathscr{K}$, and let φ and ψ be vectors in $\mathscr{H}$. Then

$$(U_A\varphi \mid U_B\psi) = d^{-1}(\varphi \mid \psi) \sum_{i=1}^{n} (B^*\varepsilon_i \mid A^*\varepsilon_i) . \tag{3.2}$$

In fact

$$\begin{aligned} \int_G (AS(x)\varphi \mid BS(x)\psi)dx &= \sum_i \int_G (S(x)\varphi \mid A^*\varepsilon_i)\overline{(S(x)\psi \mid B^*\varepsilon_i)}dx \\ &= d^{-1}(\varphi \mid \psi) \sum_i (B^*\varepsilon_i \mid A^*\varepsilon_i) \qquad \text{(by (1.1))} . \end{aligned}$$

Because S and λ are unitary representations and $AS(k) = \lambda(k)A$, it follows that

$$AA^*\lambda(k) = \lambda(k)AA^*$$

for all $k \in K$. Since λ is irreducible this implies $AA^* = \| A \|^2 I$. Hence

$$(A^*\alpha \mid A^*B) = \| A \|^2 (\alpha \mid \beta)$$

for all α, β in $\mathscr{K}$. Using this and setting $B = A$ in (3.2), we find that

$$(U_A\varphi \mid U_A\psi) = \| A \|^2 (\varphi \mid \psi) \tag{3.3}$$

for all φ, ψ in $\mathscr{H}$. Therefore, U_A is a continuous linear map of $\mathscr{H}$ into $L^2(G, \lambda)$, and $\| U_A \| = \| A \|$.

Next note that for φ in $\mathscr{H}$ and x, y in G

$$(T(y)U_A\varphi)(x) = (U_A\varphi)(xy) = cAS(x)S(y)\varphi = (U_AS(y)\varphi)(x) .$$

Hence, $T(y)U_A = U_AS(y)$ for all y in G. Therefore, $U_A \in \mathscr{I}(S, T)$. Since

$$U_{cA+B} = cU_A + U_B$$

it follows that $A \longrightarrow U_A$ is an isometric linear map of $\mathscr{I}(S_K, T)$ into $\mathscr{I}(S, T)$.

Now suppose $U \in \mathscr{I}(S, T)$. Then by Theorem 1, we can define a continuous linear map A of $\mathscr{H}$ into $\mathscr{K}$ by setting

$$(3.4) \qquad A\varphi = c^{-1}(U\varphi)(1)\,, \qquad \varphi \in \mathscr{H}\,.$$

Then for k in K and φ in $\mathscr{H}$

$$AS(k)\varphi = c^{-1}(US(k)\varphi)(1) = c^{-1}(T(k)U\varphi)(1) = c^{-1}(U\varphi)(k) = \lambda(k)A\varphi\,.$$

Thus $A \in \mathscr{I}(S_K, \lambda)$, and

$$U_A\varphi(x) = cAS(x)\varphi = (US(x)\varphi)(1) = (T(x)U\varphi)(1) = (U\varphi)(x)$$

for φ in $\mathscr{H}$ and x in G. Hence, $U = U_A$ and $A \rightarrow U_A$ $(A \in \mathscr{I}(S_K, \lambda))$ is an isometric linear map of $\mathscr{I}(S_K, \lambda)$ onto $\mathscr{I}(S, T)$.

COROLLARY. *The multiplicity of* S *in* T *is exactly the same as the multiplicity of* λ *in* S_K.

Proof. These multiplicities are just dim $\mathscr{I}(S, T)$ and dim $\mathscr{I}(S_K, \lambda)$, respectively.

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