ON AFFINE TRANSFORMATIONS OF A RIEMANNIAN MANIFOLD*

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In this paper we establish some theorems about the group of affine transformations on a Riemannian manifold. First we prove a decomposition theorem (Theorem 1) of the largest connected group of affine transformations on a simply connected complete Riemannian manifold, which corresponds to the decomposition theorem of de Rham [4][1] for the manifold. In the case of the largest group of isometries, a theorem of the same type is found in de Rham's paper [4] in a weaker form. Using Theorem 1 we obtain a sufficient condition for an infinitesimal affine transformation to be a Killing vector field (Theorem 2). This result includes K. Yano's theorem [13] which states that on a compact Riemannian manifold an infinitesimal affine transformation is always a Killing vector field. His proof of the theorem depends on an integral formula which is valid only for a compact manifold. Our method is quite different and is based on a result [11] of K. Nomizu.

The author expresses his deep thanks to Dr. Nomizu who suggested these problems to him.

I. Preliminaries

1. Let M be a differentiable manifold of class C^∞.[2] The set $\mathfrak{X}$ of all tangent vector fields defined on M is a module over the ring $\mathfrak{F}$ of all differentiable functions on M.

An affine connection is defined by a homomorphism over $\mathfrak{F}$: $X \rightarrow \nabla_X$ from $\mathfrak{X}$ into the module of linear mappings (over the field of all real numbers) of T, which satisfies the following condition

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1) Numbers in brackets refer to Bibliography at the end of this paper.

2) As we only consider manifolds, tangent vector fields, tensor fields and mappings which are "differentiable of class C^∞," we always omit this adjective. We deal only with connected manifolds. For the terminology concering manifolds, we follow C. Chevalley [3].

$$\nabla_X(f\cdot Y)=f\cdot\nabla_X\cdot Y+(X\cdot f)\cdot Y,$$

where X and Y are in T and f in F.[3] ∇_X is the so-called covariant differentiation along X.

The torsion tensor field T, of type (1, 2), and the curvature tensor field R, of type (1, 3), are expressed as follows:

$$T(X,\ Y)=\nabla_X\cdot Y-\nabla_Y\cdot X-[X,\ Y]$$
$$R(X,\ Y)\cdot Z=\nabla_X\cdot\nabla_Y\cdot Z-\nabla_Y\cdot\nabla_X\cdot Z-\nabla_{[X,\ Y]}Z,$$

for any X, Y and Z in T [10].

When M has a Riemannian metric defined by a positive definite symmetric quadratic tensor field G, there is one and only one affine connection such that its torsion tensor field T is zero and the covariant differentiation of the fundamental tensor field G along any X in $\mathfrak{X}$ is zero. This connection is called the Riemannian connection associated to a Riemannian metric G.

Let M_i $(i=0,1,\ldots,r)$ be a manifold with an affine connection and let M be the direct product of M_i $(i=0,1,\ldots,r)$. For any point $p=(p_0,p_1,\ldots,p_r)$, where $p_i\in M_i$, the mapping $\iota_i(p):p_i'\to(p_0,\ldots,p_{i-1},p_i',p_{i+1},\ldots,p_r)$ from M_i into M defines the submanifold $M_i(p)$.[4] The tangent space $T(p)$ at p is the direct sum of $T_i(p)$ $(i=0,1,\ldots,r)$, where $T_i(p)$ is the tangent space at p of the submanifold $M_i(p)$. Any X in $\mathfrak{X}$ can be written uniquely as a sum of X_i, where $X_i(p)\in T_i(p)$. We call X_i the i-component of X. Moreover from a tangent vector field X_i^* on M_i we can define a tangent vector field $X_i:p\to d\iota_i(p)\cdot X_i^*$ on M. Let $\mathfrak{X}_i$ be the set of all extended tangent vector fields which are so obtained from vector fields on M_i. In the submanifold $M_i(p)$ through p, we can define a connection which is isomorphic with the given connection of M_i by the mapping $\iota_i(p)$.

We can now define the product connection on M as follows. For any X and Y in $\mathfrak{X}$

$$\nabla_X\cdot Y=\sum_{i=0}^{r}(\nabla_{iX_i}\cdot Y_i+\sum_{i\neq j}[X_i,\ Y_j]_j),\tag{1}$$

where the lower index i (resp. j) of a vector field means the i (resp. j)-component and $(\nabla_{iX_i}\cdot Y_i)(p)$ is calculated on the submanifold $M_i(p)$ for each i.

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3) This definition of an affine connection is due to Koszul [10].

4) Clearly two mappings $\iota_i(p)$ and $\iota_i(p')$ coincide if $M_i(p)=M_i(p')$.

When each connection on M_i is a Riemannian connection associated to a fundamental tensor field G_i, the product connection on M is also a Riemannian connection which is associated to the product Riemannian metric G.

For any $p \in M$, $M_i(p)$ $(i=0, 1, \ldots, r)$ is a totally geodesic submanifold, that is, any geodesic curve tangent to $M_i(p)$ at a point is contained in $M_i(p)$.

Let T_i and T be the torsion tensor fields of M_i and M respectively, and R_i and R the curvature tensor fields of M_i and M respectively. Then it is easily seen that

$$T(X, Y) = \sum_{i=0}^{r} T_i(X_i, Y_i) \tag{2}$$

$$R(X, Y)Z = \sum_{i=0}^{r} R_i(X_i, Y_i)Z_i, \tag{3}$$

where $(T_i(X_i, Y_i))(p)$ and $(R_i(X_i, Y_i)Z_i)(p)$ are calculated in $M_i(p)$. Indeed, T and R being tensors, the above equalities have only to be verified for $X_i \in \mathfrak{X}_i$, $Y_j \in \mathfrak{X}_j$ and $Z_k \in \mathfrak{X}_k$.

2. Any tangent vector field V on a manifold M generates a local one-parameter group of local transformations $\varphi_t(-\varepsilon < t < \varepsilon)$[5] such that

$$\lim_{t\to 0} \frac{1}{t}(\varphi_t \circ f - f) = V \cdot f,$$

for any f in $\mathfrak{F}$. A vector field V on M with an affine connection is called an infinitesimal affine transformation if it generates a local one-parameter group of local affine transformations, that is,

$$d\varphi_t(\nabla_X \cdot Y) = \Delta_{d\varphi_t(X)} \cdot (d\varphi_t(Y)) \tag{4}$$

for any X and Y in $\mathfrak{X}$, and for any $t(-\varepsilon < t < \varepsilon)$. When M has a Riemannian metric, we say that V is a Killing vector field if it generates a local one-parameter group of local isometries, that is,

$$G(d\varphi_t(X), d\varphi_t(Y)) = G(X, Y) \tag{5}$$

for any X and Y in $\mathfrak{X}$, and for any $t(-\varepsilon < t < \varepsilon)$.

A necessary and sufficient condition for a vector field V to be an infinitesimal affine transformation is that we have

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[5] More precisely, for each point p, there exist a neighbourhood $U(p)$ and a positive number ε such that φ_t $(-\varepsilon < t < \varepsilon)$ defined on $U(p)$.

$$(6)\quad \nabla_X\cdot\nabla_Y\cdot(V)-\nabla_{\nabla_X\cdot Y}\cdot(V)+\nabla_X\cdot(T(V,\ Y))-T(V,\ \nabla_X\cdot Y)+R(V,\ X)Y=0$$

for any X and Y in $\mathfrak{X}$. Using the covariant differentiation and the contraction, we can write the above equality as follows:

$$<\nabla\nabla V,\ Y\otimes X>+(\nabla_X\cdot T)(V,\ Y)+T(\nabla_X\cdot V,\ Y)+R(V,\ X)Y=0.$$

A vector field V is a Killing vector field if and only if it satisfies

$$(7)\qquad G(\nabla_X\cdot V,\ Y)+G(X,\ \nabla_Y\cdot V)=0$$

for any X and Y in $\mathfrak{X}$. The equalites (6) and (7) can be obtained by computing the Lie derivatives along V of the both sides of (4) and (5) respectively.

LEMMA 1. *Let $M=M_0\times\ldots\times M_r$ be a product manifold with a product affine connection, and let V be an infinitesimal affine transformation on M with the i-component V_i. Then V_i is an infinitesimal affine transformation on M and the restriction of V_i on $M_i(p)$ is an infinitesimal affine transformation on $M_i(p)$ for each i.*

Proof. From (1), for a vector Z such that $Z(p)\in T_i(p)$ at each point, we have

$$(\nabla_X\cdot Z_i)(p)\in T_i(p),$$

for any $X\in\mathfrak{X}$. From (2) and (3) we have

$$(T(V,\ X))_i=T(V_i,\ X)\quad\text{and}\quad(R(V,\ X)Y)_i=R(V_i,\ X)Y,$$

for any X and Y in $\mathfrak{X}$. The i-component of the left hand side of (6) is equal to the left hand side of (6) in which V is replaced by V_i. This shows that V_i is an infinitesimal affine transformation on M.

Next, from (1) we can see easily

$$\nabla_{X_i}\cdot Y_i=\nabla_{iX_i}\cdot Y$$

and this vector field is contained in $\mathfrak{X}_i$ for X_i and Y_i in $\mathfrak{X}_i$. Moreover for a vector field Z such that $Z(p)\in T_i(p)$ at each point p, we have

$$\nabla_{X_i}\cdot Z=\nabla_{iX_i}\cdot Z$$

for $X_i\in\mathfrak{X}_i$. From (2) and (3) it follows that

$$(T(X,\ Y))_i=T_i(X_i,\ Y_i)$$
$$(R(X,\ Y)\cdot Z)_i=R_i(X_i,\ Y_i)\cdot Z_i,$$

for any X and Y in $\mathfrak{X}$. Therefore, the i-component of the left hand side of (6) for $X = X_i$ in $\mathfrak{X}_i$ and $Y = Y_i$ in $\mathfrak{X}_i$ is equal to

$$\begin{aligned} \nabla_{iX_i} \cdot \nabla_{iY_i} \cdot (V_i) + \nabla_{i\nabla_{iX_i} \cdot Y_i} \cdot (V_i) + \nabla_{iX_i} \cdot (T_i(V_i, Y_i)) \\ - T_i(V_i, \nabla_{iX_i} \cdot (Y_i)) + R_i(V_i, X_i) \cdot Y_i. \end{aligned}$$

Since each term is calculated on $M_i(p)$, this shows that the restriction of V on $M_i(p)$ is an infinitesimal affine transformation on $M_i(p)$.

When V is a Killing vector field the situation is just the same as above.

Remark. It is almost evident that the vector field V_i in $\mathfrak{X}_i$ which is the extended vector field of an infinitesimal affine transformation (resp. a Killing vector field) of M_i is also an infinitesimal affine transformation (resp. a Killing vector field) of M.

We say that a manifold with an affine connection (or a Riemannian manifold) M is complete if every geodesic curve can be extended for any large value of the canonical parameter. When the completeness is satisfied on M, any infinitesimal affine transformation (or a Killing vector field) generates a one-parameter group of affine transformations (isometries) from M onto itself [7].

3. Let M be a manifold with an affine connection. The group $A(M)$ of all affine transformation of M onto itself is a Lie group with respect to the compact-open topology [5], [6], [9]. When M has a Riemannian metric, the group $I(M)$ of all isometries of M onto itself is a closed subgroup of $A(M)$. $I(M)$ is also a Lie group [8].

The mapping from $A(M) \times M$ onto M, which gives the transformation law, is differentiable as is known from a theorem of S. Bochner and D. Montgomery [1]. Any one-parameter subgroup in $A(M)$ (resp. $I(M)$) induces an infinitesimal affine transformation (resp. a Killing vector field) on M.

II. Decomposition Theorem of $A_0(M)$ and $I_0(M)$

4. In this section we always assume that M is a simply connected complete Riemannian manifold.

The homogeneous holonomy group $\Psi(p)$ at p operates on the tangent space $T(p)$ and is completely reducible. If $T(p)$ is irreducible, we call M irreducible.

The following theorem is due to de Rham [4]:

1) A simply connected complete Riemannian manifold M is isometric to the direct product of a Euclidean space M_0 and simply connected complete irreducible Riemannian manifolds M_i with dim $M_i \geqq 2$ $(i=1, \ldots, r)$, and this decomposition is unique up to the order and isometries.

2) The tangent space $T(p)$ at each point p is the direct sum of the $\Psi(p)$-invariant subspace $T_0(p), T_1(p), \ldots, T_r(p)$, where $T_0(p)$ is the subspace of all $\Psi(p)$-invariant vectors, $T_i(p)$ $(i=1, \ldots, r)$ are irreducible and any two subspaces $T_i(p)$ and $T_j(p)$ $(i \neq j,\ i, j=0, 1, \ldots, r)$ are mutually orthogonal. This decomposition is unique.

If we identify M with the direct product $M_0 \times M_1 \times \ldots \times M_r$, the subspace $T_i(p)$ is the tangent space at p of the submanifold $M_i(p)$ which corresponds to M_i in the manner mentioned in 1.

We call these decompositions the de Rham decomposition of the manifold, or of the tangent space $T(p)$.

5. The aim of this section is to prove the following

THEOREM 1. *Let M be a simply connected complete Riemannian manifold, and $M=M_0 \times M_1 \times \ldots \times M_r$ be the de Rham decomposition of M. Then the group $A_0(M)$ is isomorphic to the direct product $A_0(M_0) \times A_0(M_1) \times \ldots \times A_0(M_r)$, and the group $I_0(M)$ is isomorphic to the direct product $I_0(M_0) \times I_0(M_1) \times \ldots \times I_0(M_r)$, where $A_0(M)$ and $A_0(M_i)$ (resp. $I_0(M)$ and $I_0(M_i)$) are the connected components of the identity in $A(M)$ and $A(M_i)$ (resp. $I(M)$ and $I(M_i)$) respectively.*

To prove this theorem the following lemma, which is given by K. Nomizu [11], [12] is useful.

LEMMA 2. *Let $T(p)=T_0(p)+T_1(p)+\ldots+T_r(p)$ be the de Rham decomposition of the tangent space. Then*

$$d\varphi(T_i(p))=T_i(\varphi(p)),$$

for any $\varphi \in A_0(p)$ and for any $p \in M$.

Proof of Theorem 1. Let φ_i be an element in $A_0(M_i)$ $(i=0, 1, \ldots, r)$, a transformation $\varphi(p)=(\varphi_0(p_0), \ldots, \varphi_i(p_i), \ldots, \varphi_r(p_r))$, where $p=(p_0, \ldots, p_i, \ldots, p_r)$ is an element in $A_0(M)$. The mapping from $A_0(M_0) \times A_0(M_1) \times \ldots \times A_0(M_r)$ into $A_0(M)$ which maps $(\varphi_0, \varphi_1, \ldots, \varphi_r)$ to φ is an isomorphism. Then to prove the theorem, we have only to show that the induced isomorphism from the direct sum of the Lie algebras associated to $A_0(M_i)$ $(i=0, 1, \ldots, r)$ into the Lie algebras $\mathfrak{a}$ associated to $A_0(M)$ is onto. The Lie algebra $\mathfrak{a}$ is the Lie algebra of all infinitesimal affine transformations on M. The set $\mathfrak{a}_i$ of all infinitesimal affine trasformations contained in $\mathfrak{T}_i$, namely the image of the Lie algebra associated to $A_0(M_i)$ by the induced isomorphism, is an ideal for each i.

Let V be an infinitesimal affine transformation in $\mathfrak{a}$ and V_i the i-component of V. We now fix an arbitrary point $p^*=(p_0^*, \ldots, p_i^*, \ldots, p_r^*)$ and consider the mapping $\iota_i(p^*)$ from M_i onto $M_i(p^*)$ as stated in 2. From Lemma 1 V_i and the restriction of V_i on $M_i(p^*)$ are infinitesimal affine transformations on M and $M_i(p^*)$ respectively, and there is an infinitesimal affine transformation V_i' on M_i such that $d\iota_i(p^*)(V_i')$ is equal to the restriction of V_i on $M_i(p^*)$. Let V_i^* be the infinitesimal affine transformation on M which is the extension of V_i'. Clearly V_i^* is contained in $\mathfrak{T}_i$. We shall prove that $V_i^*=V_i$.

We now consider the difference $V_i-V_i^*$. Clearly it is an infinitesimal affine transformation on M, and $(V_i-V_i^*)(p)$ is zero when p is on $M_i(p^*)$. From our assumption that M is complete, $V_i-V_i^*$ generates a one-parameter subgroup φ_t in $A_0(M)$. First, as $(V_i-V_i^*)(p)=0$ on $M_i(p^*)$, each transformation φ_t leaves every point on $M_i(p^*)$ fixed. And as $(V_i-V_i^*)(p^*)\in T_i(p)$ at any point on M, we have $\varphi_t(p)\in M_i(p)$, namely, each φ_t maps $M_i(p)$ onto itself. Next we consider the submanifold $M_{\hat{i}}(p)=\{p' ; p'\in M, p'=(p_0', \ldots, p_{i-1}', p_i, p_{i+1}', \ldots, p_r'), \text{ where } p_j'\in M_j \ (j\neq i)\}$ through $p=(p_0^*, \ldots, p_{i-1}^*, p_i, p_{i+1}^*, \ldots, p_r^*)$ which is on $M_i(p^*)$. The tangent space of $M_{\hat{i}}(p)$ at p' is $T_{\hat{i}}(p') = T_0(p')+\ldots+T_{i-1}(p')+T_{i+1}(p')+\ldots+T_r(p')$. Let q be an arbitrary point on $M_{\hat{i}}(p)$. As $M_{\hat{i}}(p)$ is a totally geodesic submanifold and is complete, p and q can be joined by a geodesic curve $\sigma(s)$ on $M_{\hat{i}}(p)$, and the tangent vector $X(p)$ of $\sigma(s)$ at p is contained in $T_{\hat{i}}(p)$. According to Lemma 2, we have $d\varphi_t\cdot X(p)\in T_{\hat{i}}(p)$ for any real number t, and $d\varphi_t\cdot X(p)$ is the tangent vector of a geodesic curve $\varphi_t\cdot\sigma(s)$ at p. Therefore the geodesic curve $\varphi_t\cdot\sigma(s)$ lies on $M_{\hat{i}}(p)$, and $\varphi_t(q)$ is in $M_{\hat{i}}(p)$. This shows $\varphi_t(M_{\hat{i}}(p))\subset M_{\hat{i}}(p)$ for any p on $M_i(p^*)$. Any point q on M is contained in one and only one $M_{\hat{i}}(p)$ through some p on $M_{\hat{i}}(p^*)$ and, on the other hand, in $M_i(q)$. Hence $\varphi_t(q)\in M_{\hat{i}}(p)\frown M_i(q)$. But $M_i(q)$ and $M_{\hat{i}}(p)$ have one and only one common point q, and hence $\varphi_t(q)=q$. Thus we conclude that φ_t is the identity transformation for any t and $V_i = V_i^*$ on M. This shows that any element V in $\mathfrak{a}$ can be written as a sum of elements in $\mathfrak{a}_i$ $(i = 0, 1, \ldots, r)$, and our isomorphism is onto.

III. An Application of the Decomposition Theorem

6. Let M be a Riemannian manifold with the fundamental tensor field G. For any φ in $A(M)$, $d\varphi \cdot G$ is also a positive definite symmetric tensor field and as $d\varphi$ is commutative with the covariant differentiation, $\nabla(d\varphi \cdot G) = 0$. Then $(d\varphi \cdot G)(p)$ is invariant by the operations of the homogeneous holonomy group $\Psi(p)$. If M is irreducible, there is a positive constant $c(\varphi)$ such that $d\varphi \cdot G = c^2(\varphi) \cdot G$, and $\varphi \to c(\varphi)$ is a continuous representation of $A(M)$ into the multiplicative group of all positive real numbers [11], [12]. This follows from the fact that, the invariant positive definite bilinear form on $T(p)$ is uniquely determined up to a positive constant, as the homogeneous holonomy group is irreducible. From this fact the following lemma is easily obtained.

LEMMA 3. *When a Riemannian manifold M is irreducible and complete, an infinitesimal affine transformation V is a Killing vector field if and only lf there is a non-trivial orbit on which the length of V is bounded.*

Proof. Let φ_t be the one-parameter group generated by V. Then $V(\varphi_t(p)) = d\varphi_t \cdot V(p)$ and $|V(\varphi_t(p))| = c(\varphi)|V(p)|$, where $|X(p)|$ denotes the length of a vector $X(p)$. If V is a Killing vector field, $c(\varphi_t) = 1$ for every t and surely the length of V is bounded on any orbit $\{\varphi_t(p)\}$. Conversely, let $\{\varphi_t(p)\}$ be a non-trivial orbit on which the length of V is bounded. Then $V(p)$ is not zero and the function $c(\varphi_t)$ of t is bounded. As already mentioned, $c(\varphi_t)$ is a continuous representation of the additive group of all real numbers. Therefore $c(\varphi_t)$ must be equal to 1 for every t, and V is a Killing vector field.

7. A Riemannian manifold is called locally flat when the homogeneous holonomy group is discrete.

If there is an absolutely parallel vector field[6] on a complete Riemannian manifold, it is a Killing vector field. We define a *translation* as an isometry which lies on the one-parameter group of isometries generated by such a Killing

[6] A vector field X is an absolutely parallel vector field when for any two points p and q, $X(p)$ and $X(q)$ are parallel along any piece-wise differentiable curve which joins p and q. This condition is equivalent to the condition $\nabla X = 0$.

vector field.

For the necessity in the next section we prepare

LEMMA 4. *If the length of an infinitesimal affine transformation V on a complete locally flat Riemannian manifold is bounded, then V is a Killing vector field and generates a one-parameter group of translations.*

Proof. The universal covering manifold $\tilde{M}$ of M has a natural Riemannian metric induced from that of M, with respect to which $\tilde{M}$ is also complete and locally flat. As is well known,[7] a complete locally flat Riemannian manifold M is isometric to a Euclidean space with the usual metric when it is simply connected, and the group of all affine transformations $A(\tilde{M})$ is the usual affine group. If we take a system of cartesian coordinates $x^1, x^2, \ldots, x^n$ on $\tilde{M}$, then an infinitesimal affine transformation $\tilde{V}$ is given by

$$\sum_{i=1}^{n}\Big(\sum_{j=1}^{n} a_{ij}\cdot x^j + b_i\Big)\frac{\partial}{\partial x^i}$$

where a_{ij} and b_i are arbitrary real numbers. It is easily seen that the length $\sqrt{\sum_{i=1}^{n}(\sum_{j=1}^{n} a_{ij}\cdot x^j + b_i)}$ of V is bounded on M if and only if each a_{ij} $(i, j = 1, 2, \ldots, n)$ is zero.

For a Killing vector field V on M whose length is bounded, there is a Killing vector field $\tilde{V}$ on $\tilde{M}$ such that the projection from $\tilde{M}$ to M maps $\tilde{V}$ on V, and its length is bounded on M. Then $\tilde{V}$ and accordingly V are absolutely parallel vector fields on $\tilde{M}$ and M respectively.

COROLLARY. *If M is a compact locally flat Riemannian manifold, $A_0(M)$ coincies with $I_0(M)$ and is abelian.*

8. THEOREM 2. *Let M be a complete Riemannian manifold. If the length of an infinitesimal affine transformation V is bounded on M, then V is a Killing vector field.*

Proof. Let $\tilde{M}$ be the universal covering manifold of M and let $\tilde{M} = \tilde{M}_0 \times \tilde{M}_1 \times \ldots \times \tilde{M}_r$ be the de Rham decomposition of M. It has a Riemannian connection which is naturally induced from that of M, and is complete with respect to it. If $\tilde{V}$ is the infinitesimal affine transformation

[7] For example, cf. [2] Chapitre III, II.

which is mapped on V by the projection from $\tilde{M}$ onto M, then the length of $\tilde{V}$ is also bounded on $\tilde{M}$. As $\tilde{M}$ is simply connected and complete, we can apply the arguments in II.

Since the length of $\tilde{V}$ is bounded on $\tilde{M}$, the length of the i-component $\tilde{V}_i$ of $\tilde{V}$ is also bounded on $\tilde{M}$. As we have seen in the proof of Theorem 1, $\tilde{V}_i$ is the extension of a certain infinitesimal affine transformation $\tilde{V}'_i$ on $\tilde{M}_i$, and the length of $\tilde{V}'_i$ is clearly bounded. We see that $\tilde{V}'_i$ is a Killing vector field from Lemma 3 when $1 \leqq i \leqq r$, and from Lemma 4 when $i = 0$. As we already remarked, the extension $\tilde{V}_i$ of a Killing vector field $\tilde{V}'_i$ on $\tilde{M}_i$ is also a Killing vector field on $\tilde{M}$ for each i. Hence $\tilde{V}$ and accordingly V are Killing vector fields on $\tilde{M}$ and M respectively.

On a compact Riemannian manifold, the length of any vector field is of course bounded. Hence we have

THEOREM 3. (K. Yano) *On a compact Riemannian manifold M, every infinitesimal affine transformation is a Killing vector field. Therefore $A_0(M)$ coincides with $I_0(M)$.*

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