ON APPROXIMATELY CONTINUOUS FUNCTIONS

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In a very interesting paper *Sur l'équation fonctionnelle* $g(x) = f\phi(x)$, S. Braun* established a series of theorems on the functional equation

$$g(x) = f[\phi(x)],$$

where $g(x)$ and $f(y)$ are given functions, and $\phi(x)$ is a function sought for. In this note, we consider the case for which $\phi(x)$ is an approximately continuous function.† A function $f(x)$ is said to be approximately continuous at x_0 if the density at x_0 of the set $E[f(x_0), \epsilon]$ of all points x such that $|f(x) - f(x_0)| < \epsilon$ is equal to 1, no matter what ϵ is.

Let $f(x)$ be a finite function of class 1 in $[0, 1] = [0 \leqq x \leqq 1]$, and let $\{y_n\}$ be the sequence of all rational numbers y_n such that there are two points x_n' and x_n'' belonging to $[0, 1]$ and satisfying the condition $f(x_n') < y_n < f(x_n'')$. Let E_{y_n} (E^{y_n}), $(n = 1, 2, 3, \cdots)$, denote the set of all points x such that $f(x) < y_n$ $(f(x) > y_n)$. If z is an irrational number, let E_z (E^z) denote the sum of all the sets E_{y_n} (E^{y_n}) such that $y_n < z$ $(y_n > z)$. We now prove the following theorem:

THEOREM 1. *A necessary and sufficient condition that a finite function* $\phi(x)$ *be approximately continuous in* $[0, 1]$ *is that there exist a system of perfect sets*

$$(\mathfrak{P}): \quad \mathfrak{P}^n_{y_r}, \quad \mathfrak{P}^{y_r}_n, \qquad r = 1, 2, 3, \cdots, n;\ n = 1, 2, 3, \cdots,$$

such that

(i) $E_{y_r} = \lim_{n=\infty} \mathfrak{P}^n_{y_r}$, $E^{y_r} = \lim_{n=\infty} \mathfrak{P}^{y_r}_n$, $\mathfrak{P}^n_{y_r} \subset \mathfrak{P}^{n+1}_{y_r} \subset E_{y_r}$, $\mathfrak{P}^{y_r}_n \subset \mathfrak{P}^{y_r}_{n+1} \subset E^{y_r}$;

(ii) *if* $y_r < y_s$ *and* M *is the greater of the integers* r, s, *every point of the set* $\mathfrak{P}^n_{y_r}$ $(\mathfrak{P}^{y_s}_n)$ *is a density point of the set* $\mathfrak{P}^n_{y_s}$ $(\mathfrak{P}^{y_r}_n)$ *for all* $n \geqq M$.

A point x of a set E will be called a density point in E if

$$\lim_{h=0} \left[\frac{1}{2h} \text{meas } [(x - h, x + h)E]\right] = 1.$$

PROOF. Let x_0 be an arbitrary point in $[0, 1]$, and let $f(x_0) = y_0$.

* Fundamenta Mathematicae, vol. 28 (1937), pp. 294–302.

† A. Denjoy, *Sur les fonctions dérivées sommables*, Bulletin de la Société Mathématique de France, vol. 43 (1915), pp. 161–247, especially p. 165.

We suppose that y_0 is not the least upper bound (greatest lower bound) of f in $[0, 1]$. Then, if η is an arbitrary, positive, sufficiently small number, we have integers k_1, k, s, s_1 such that $y_0-\eta<y_{k_1}<y_k<y_0<y_s<y_{s_1}<y_0+\eta$.

The point x_0 belongs to the set $E_{y_s}\cdot E_{y_{s_1}}$ as well as to the set $E^{y_k}\cdot E^{y_{k_1}}$: Consequently, x_0 belongs to the set $\mathfrak{P}^n_{y_s}\cdot\mathfrak{P}^n_{y_{s_1}}$ as well as to the set $\mathfrak{P}^{y_k}_n\cdot\mathfrak{P}^{y_{k_1}}_n$ for sufficiently large n. Thence x_0 is a density point in $\mathfrak{P}^n_{y_{s_1}}$ as well as in $\mathfrak{P}^{y_{k_1}}_n$; therefore x_0 is a density point in $\mathfrak{P}^n_{y_{s_1}}\cdot\mathfrak{P}^{y_{k_1}}_n$. But if $E[f(x_0), \eta]$ is the set of all points x satisfying the condition $|f(x)-f(x_0)|\leqq\eta$, we have

$$\mathfrak{P}^n_{y_{s_1}}\cdot\mathfrak{P}^{y_{k_1}}_n\subset E_{y_{s_1}}\cdot E^{y_{k_1}}\subset E[f(x_0), \eta];$$

so that x_0 is a density point in $E[f(x_0), \eta]$.

We now suppose that y_0 is the least upper bound (greatest lower bound) of f in $[0, 1]$. Then, if η is an arbitrary, positive, sufficiently small number, we have integers k_1, k (s, s_1) such that

$$y_0-\eta<y_{k_1}<y_k<y_0\quad(y_0<y_s<y_{s_1}<y_0+\eta).$$

The point x_0 belongs to the set

$$E^{y_k}\cdot E^{y_{k_1}}\quad(E_{y_s}\cdot E_{y_{s_1}}).$$

Consequently, x_0 belongs to the set

$$\mathfrak{P}^{y_k}_n\cdot\mathfrak{P}^{y_{k_1}}_n\quad(\mathfrak{P}^n_{y_s}\cdot\mathfrak{P}^n_{y_{s_1}})$$

for sufficiently large n. Thence x_0 is a density point in $\mathfrak{P}^{y_{k_1}}_n$ ($\mathfrak{P}^n_{y_{s_1}}$), and x_0 is therefore a density point in

$$E^{y_{k_1}}\subset E[f(x_0), \eta]\quad(E_{y_{s_1}}\subset E[f(x_0), \eta]),$$

so that x_0 is a density point in $E[f(x_0), \eta]$, and the sufficiency of the condition is established. To prove that the condition is necessary, we shall assume that $f(x)$ is an approximately continuous function. In virtue of a Theorem of A. Denjoy,* in this case $f(x)$ is a function of class 1. It follows that each of the sets E_{y_r}, E^{y_r}, $(r=1, 2, 3, \cdots)$, is the sum of an enumerable infinity of perfect sets and of an enumerable set N of points $x_1, x_2, x_3, \cdots$ of $[0, 1]$. Let E denote an arbitrary one of the sets E_{y_n}, E^{y_n}, $(n=1, 2, 3, \cdots)$. Since $f(x)$ is approximately continuous, we can find for every point x_n a perfect set $\mathfrak{P}(x_n)$ such that

* Loc. cit., p. 181.

(i) x_n is a density point of $\mathfrak{P}(x_n)$;

(ii) $f(x)$ is continuous at the point x_0 over $\mathfrak{P}(x_n)$ relative to $\mathfrak{P}(x_n)$;

(iii) $\mathfrak{P}(x_n)$ is contained in E.

In this case $\mathfrak{P}(x_n)$ will be called the perfect and dense *road* at the point x_n. It is clear that E is the sum of perfect sets and of sets $\mathfrak{P}(x_n)$, $(n=1, 2, 3, 4, \cdots)$. Since $\mathfrak{P}(x_n)$ is perfect, E is the enumerable sum of perfect sets, and we may also write

$$E = \lim_{n=\infty} P_n,$$

where P_n is a perfect set such that $P_n \subset E$.

We have also

$$E_{y_r} = \lim_{n=\infty} P_{y_r}^n, \quad E_{y_s} = \lim_{n=\infty} P_{y_s}^n, \quad E^{y_r} = \lim_{n=\infty} P_n^{y_r}, \quad E^{y_s} = \lim_{n=\infty} P_n^{y_s},$$

where $P_{y_r}^n$, $P_{y_s}^n$, $P_n^{y_r}$, $P_n^{y_s}$, are perfect sets such that

$$P_{y_r}^n \subset E_{y_r}, \quad P_{y_s}^n \subset E_{y_s}, \quad P_n^{y_r} \subset E^{y_r}, \quad P_n^{y_s} \subset E^{y_s}.$$

Hence on putting $\mathfrak{P}_{y_s}^n = P_{y_s}^n + P_{y_r}^n$, $\mathfrak{P}_n^{y_r} = P_n^{y_r} + P_n^{y_s}$, we have

$$E_{y_s} = \lim_{n=\infty} \mathfrak{P}_{y_s}^n, \qquad E^{y_r} = \lim_{n=\infty} \mathfrak{P}_n^{y_r}.$$

It is easily seen that

$$\mathfrak{P}_{y_r}^n \subset \mathfrak{P}_{y_s}^n, \qquad \mathfrak{P}_n^{y_s} \subset \mathfrak{P}_n^{y_r}.$$

Let (a, b) be any contiguous interval of the set $\mathfrak{P}_{y_r}^n$ $(\mathfrak{P}_n^{y_s})$. We shall construct a perfect set Q_{ab} (R_{ab}) such that

(i) $Q_{ab} \subset E_{y_r}$, $R_{ab} \subset E^{y_s}$;

(ii) every point of the set $\mathfrak{P}_{y_r}^n$ $(\mathfrak{P}_n^{y_s})$ is the density point of the set

$$\mathfrak{Q}_{y_r}^n = \mathfrak{P}_{y_r}^n + \sum_{ab} Q_{ab}\left(\mathfrak{Q}_n^{y_s} = \mathfrak{P}_n^{y_s} + \sum_{ab} R_{ab}\right).*$$

Now we can adjoin the set $\mathfrak{Q}_{y_r}^n$ $(\mathfrak{Q}_n^{y_s})$ to the set $\mathfrak{P}_{y_s}^n$ $(\mathfrak{P}_n^{y_r})$. Thus, without loss of generality, we may assume that every point of the set $\mathfrak{P}_{y_r}^n$ $(\mathfrak{P}_n^{y_s})$ is a density point of the set $\mathfrak{P}_{y_s}^n$ $(\mathfrak{P}_n^{y_r})$ for all $n \geqq M$.

We now turn to the functional equation

$$g(x) = f[\phi(x)].$$

Consider the sequence of all rational numbers (y): $y_1, y_2, y_3, \cdots$.

* See V. Bogomoloff, *Sur une classe des fonctions asymptotiquement continues*, Recueil Mathématique, Moscow, vol. 32 (1924), pp. 152–171.

Let $\mathfrak{E}_{y_r}$ be the set of all points (x, y) such that

$$g(x) = f(y), \qquad y < y_r, 0 \leqq x \leqq 1,$$

and let $\mathfrak{E}^{y_r}$ be the set of all points (x, y) such that

$$g(x) = f(y), \qquad y > y_r, 0 \leqq x \leqq 1.$$

Denote by $\mathfrak{P}\mathfrak{E}$ the orthogonal projection of a set $\mathfrak{E}$ on the x axis.

Suppose now that M is the greater of the integers r, s, and assume $y_r < y_s$.

THEOREM 2. *A necessary and sufficient condition that there exist a finite function $\phi(x)$ of class 1 having the property of Darboux and satisfying the equation $g(x) = f[\phi(x)]$ is that there exist a sequence of perfect sets*

$$(\mathfrak{P}): \quad \mathfrak{P}_{y_r}^{n}, \quad \mathfrak{P}_{n}^{y_r}, \qquad r = 1, 2, 3, \cdots, n; n = 1, 2, 3, \cdots,$$

such that

(i) *the sets satisfy*

$$\lim_{n=\infty} \mathfrak{P}_{y_r}^{n} \subset \mathfrak{P}\mathfrak{E}_{y_r}, \ \lim_{n=\infty} \mathfrak{P}_{n}^{y_r} \subset \mathfrak{P}\mathfrak{E}^{y_r}, \ \mathfrak{P}_{y_r}^{n} \subset \mathfrak{P}_{y_r}^{n+1} \subset \mathfrak{P}\mathfrak{E}_{y_r}, \ \ \mathfrak{P}_{n}^{y_r} \subset \mathfrak{P}_{n+1}^{y_r} \subset \mathfrak{P}\mathfrak{E}^{y_r};$$

(ii) *every point of the set $\mathfrak{P}_{y_r}^{n}$ $(\mathfrak{P}_{n}^{y_s})$ is a density point of the set $\mathfrak{P}_{y_s}^{n}$ $(\mathfrak{P}_{n}^{y_r})$ for all $n \geqq M$;*

(iii) $\lim_{n=\infty} \sum_{r=1}^{r=n} \mathfrak{P}_{y_r}^{n} = \lim_{n=\infty} \sum_{r=1}^{r=n} \mathfrak{P}_{n}^{y_r} = [0 \leqq x \leqq 1]$.

PROOF. To prove that the condition is necessary, we shall assume that there exists a finite function $\phi(x)$ of class 1 having the property of Darboux and satisfying the equation $g(x) = f[\phi(x)]$. Let $\overline{\mathfrak{E}}_{y_r}$ $(\overline{\mathfrak{E}}^{y_r})$ be the set of all points $[x, \phi(x)]$, where x belongs to the set E_{y_r} (E^{y_r}). It is clear that

$$\mathfrak{P}\overline{\mathfrak{E}}_{y_r} \subset \mathfrak{P}\mathfrak{E}_{y_r}, \qquad \mathfrak{P}\overline{\mathfrak{E}}^{y_r} \subset \mathfrak{P}\mathfrak{E}^{y_r}.$$

Since $\mathfrak{P}\overline{\mathfrak{E}}_{y_r} = E_{y_r}$, $\mathfrak{P}\overline{\mathfrak{E}}^{y_r} = E^{y_r}$, we conclude at once that $E_{y_r} \subset \mathfrak{P}\mathfrak{E}_{y_r}$, $E^{y_r} \subset \mathfrak{P}\mathfrak{E}^{y_r}$. The sum of all sets E_{y_r} (E^{y_r}) is equal to the segment $[0 \leqq x \leqq 1]$, hence

$$\text{(A)} \qquad \sum_{r=1}^{r=\infty} \mathfrak{P}\mathfrak{E}_{y_r} = \sum_{r=1}^{r=\infty} \mathfrak{P}\mathfrak{E}^{y_r} = [0 \leqq x \leqq 1].$$

In virtue of Theorem 1 for the function $\phi(x)$, there exists a sequence of perfect sets

$$(\mathfrak{P}): \quad \mathfrak{P}_{y_r}^{n}, \quad \mathfrak{P}_{n}^{y_r}$$

satisfying the conditions of Theorem 1. It is readily seen that

$$\text{(B)} \qquad \mathfrak{P}_{y_r}^{n} \subset \mathfrak{P}\mathfrak{E}_{y_r}, \qquad \mathfrak{P}_{n}^{y_r} \subset \mathfrak{P}\mathfrak{E}^{y_r}.$$

We have just seen that the condition is necessary; let us next show that it is sufficient. For this purpose, we shall assume that there exists a system of perfect sets

$$(\mathfrak{P}): \qquad \mathfrak{P}_{y_r}^{n}, \quad \mathfrak{P}_{n}^{y_r}, \qquad r = 1, 2, 3, \cdots, n;\ n = 1, 2, 3, \cdots,$$

satisfying the following conditions:

(i) $\mathfrak{P}_{y_r}^{n} \subset \mathfrak{P}_{y_r}^{n+1} \subset \mathfrak{P}\mathfrak{E}_{y_r}$, $\mathfrak{P}_{n}^{y_r} \subset \mathfrak{P}_{n+1}^{y_r} \subset \mathfrak{P}\mathfrak{E}^{y_r}$.

(ii) Every point of the set $\mathfrak{P}_{y_r}^{n}$ ($\mathfrak{P}_{n}^{y_s}$) is a density point of the set $\mathfrak{P}_{y_s}^{n}$ ($\mathfrak{P}_{n}^{y_r}$) for all $n \geqq M$.

(iii) $\lim_{n=\infty}\sum_{r=1}^{r=n}\mathfrak{P}_{y_r}^{n} = \lim_{n=\infty}\sum_{r=1}^{r=n}\mathfrak{P}_{n}^{y_r} = [0 \leqq x \leqq 1]$.

It follows that there exists a finite function $\phi(x)$ of class 1 having the property of Darboux and satisfying the condition $g(x) = f[\phi(x)]$. In order to find the function $\phi(x)$, we shall proceed as follows. We first effect the uniformization* of the set $\mathfrak{E}_{y_1}$ relative to the y axis over the set $\mathfrak{P}_{y_1}^{1}$; thus the value of the function $\phi(x)$ is determined at each point x of $\mathfrak{P}_{y_1}^{1}$. We next effect the uniformization of the set $\mathfrak{E}^{y_1}$ over the set

$$R_1 = \mathfrak{P}_{1}^{y_1} - \mathfrak{P}_{1}^{y_1}\cdot\mathfrak{P}_{y_1}^{1}.$$

We set $U_1 = \mathfrak{P}_{1}^{y_1} + \mathfrak{P}_{y_1}^{1}$. We then effect the uniformization of the set $\mathfrak{E}_{y_1}$ over the set

$$R_2 = \mathfrak{P}_{y_1}^{2} - U_1\cdot\mathfrak{P}_{y_1}^{2}.$$

Set $U_2 = U_1 + \mathfrak{P}_{y_1}^{2}$. We now effect the uniformization of the set $\mathfrak{E}_{y_2}$ over the set

$$R_3 = \mathfrak{P}_{y_2}^{2} - U_2\cdot\mathfrak{P}_{y_2}^{2}$$

and set $U_3 = U_2 + \mathfrak{P}_{y_2}^{2}$. We next effect the uniformization of the set $\mathfrak{E}_{y_1}$ over the set

$$R_4 = \mathfrak{P}_{y_1}^{3} - U_3\cdot\mathfrak{P}_{y_1}^{3},$$

and so on.

We carry out this process until the function $\phi(x)$ is completely determined.

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* See N. Lusin, *Sur le problème de M. Jacques Hadamard d'uniformisation des ensembles*, Comptes Rendus de l'Académie, Paris, vol. 189 (1930), p. 349.