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On drift estimation for non-ergodic fractional Ornstein-Uhlenbeck process with discrete observations

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Abstract. We consider parameter estimation problems for the non-ergodic fractional Ornstein-Uhlenbeck process defined as $dX_t = \theta X_t dt + dB_t^H$, $t \geq 0$, with an unknown parameter $\theta > 0$, where B^H is a fractional Brownian motion of Hurst index $H \in (\frac{1}{2}, 1)$. We assume that the process $\{X_t, t \geq 0\}$ is observed at discrete time instants $t_1 = \Delta_n, \ldots, t_n = n\Delta_n$. We construct two estimators $\hat{\theta}_n$ and $\check{\theta}_n$ of θ which are strongly consistent, namely, $\hat{\theta}_n$ and $\check{\theta}_n$ converge to θ almost surely as $n \to \infty$. We also prove that $\sqrt{n\Delta_n}(\hat{\theta}_n - \theta)$ and $\sqrt{n\Delta_n}(\check{\theta}_n - \theta)$ are tight.

Résumé. Dans ce travail, nous étudions des problèmes d'estimation paramétriques relatifs au processus d'Ornstein-Uhlenbeck fractionaire non-ergodique défini par $dX_t = \theta X_t dt + dB_t^H, t \geq 0$, où $\theta > 0$ est un paramètre et B^H est un mouvement Brownien fractionaire d'indice de Hurst $H \in]1/2, 1[$. Le processus $\{X_t, t \geq 0\}$ a été observé (de façon régulière) aux instants $t_1 = \Delta_n, \ldots, t_n = n\Delta_n$, c'est-à-dire pour tout $i \in \{0, \cdots, n\}$, $t_i = i\Delta_n$. Nous avons construit deux estimateurs $\hat{\theta}_n$ et $\check{\theta}_n$ de θ fortement consistants, c'est-à-dire, $\hat{\theta}_n$ et $\check{\theta}_n$ convergent presque surement vers θ quand $n \to \infty$. Nous avons aussi prouvé que $\sqrt{n\Delta_n}(\hat{\theta}_n - \theta)$ et $\sqrt{n\Delta_n}(\check{\theta}_n - \theta)$ sont tendus.

Key words: Drift estimation; Discrete observations; Ornstein-Uhlenbeck process; Non-ergodicity.
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1. Introduction

Consider the Ornstein-Uhlenbeck process $X = \{X_t, t \geq 0\}$ defined as

$$X_0 = 0, \quad \text{and} \quad dX_t = \theta X_t dt + dB_t^H, \, t \geq 0, \tag{1}$$

where $B^H = \{B_t^H, t \geq 0\}$ is a fractional Brownian motion of Hurst index $H > \frac{1}{2}$ and $\theta \in (-\infty, \infty)$ is an unknown parameter. An interesting problem is to estimate the parameter θ when one observes the whole trajectory of X.constant

In the continuous case, recently, by using the least squares estimator (LSE) $\tilde{\theta}_t$ of θ given by

$$\tilde{\theta}_t = \frac{\int_0^t X_s dX_s}{\int_0^t X_s^2 ds}, \quad t \geq 0,$$

Hu and Nualart (2010) and Belfadli *et al.* (2011) have studied the consistency and the asymptotic distributions of $\tilde{\theta}_t$ based on the observation $\{X_t, \ t \in [0, T]\}$ as $T \to \infty$.

The LSE $\tilde{\theta}_t$ is obtained by the least squares technique, that is, $\tilde{\theta}_t$ (formally) minimizes

$$\theta \longmapsto \int_0^t \left| \dot{X}_s - \theta X_s \right|^2 ds.$$

To obtain the consistency of the LSE $\tilde{\theta}_t$, in the recurrent case corresponding to $\theta < 0$, Hu and Nualart (2010) are forced to consider $\int_0^t X_s dX_s$ as a Skorohod integral rather than an integral in a path-wise sense. Assuming $\int_0^t X_s dX_s$ is a Skorohod integral and $\theta < 0$, they proved the strong consistence of $\tilde{\theta}_t$ if $H \geq \frac{1}{2}$, and that the LSE $\tilde{\theta}_t$ is asymptotically normal if $H \in [\frac{1}{2}, \frac{3}{4})$. In the non-recurrent case corresponding to $\theta > 0$, Belfadli *et al.* (2011) established, when $H > \frac{1}{2}$, that the LSE $\tilde{\theta}_t$ of θ is strongly consistent and asymptotically Cauchy, where in their case, the integral $\int_0^t X_s dX_s$ is interpreted as an integral in a path-wise sense. The almost sure central limit theorem (ASCLT) for the estimator $\tilde{\theta}_t$, in the case when $\theta < 0$, is also studied by Cénac and Es-Sebaiy (2012). They proved that, when $H \in (1/2, 3/4)$, the sequence $\{\sqrt{n}(\theta - \tilde{\theta}_n)\}_{n \geq 1}$ satisfies the ASCLT.

From a practical point of view, in parametric inference, it is more realistic and interesting to consider asymptotic estimation for X based on discrete observations.

Assume that the process X is observed equidistantly in time with the step size Δ_n: $t_i = i\Delta_n, i = 0, \ldots, n$, and $T_n = n\Delta_n$ denotes the length of the 'observation window'. The purpose of this paper, when $\theta > 0$ corresponding to the non-recurrent case, is to construct two estimators for θ converging at rate $\sqrt{n\Delta_n}$ based on the sampling data $X_{t_i}, i = 0, \ldots, n$.

Suppose that the integral $\int_0^t X_s dX_s$ is interpreted in the Young sense (path-wise sense). Then we can write

$$\tilde{\theta}_{T_n} = \frac{\int_0^{T_n} X_s dX_s}{\int_0^{T_n} X_s^2 ds} = \frac{X_{T_n}^2}{2 \int_0^{T_n} X_s^2 ds}. \tag{2}$$

Now, let us construct two discrete versions of $\tilde{\theta}_{T_n}$. If, in (2), dX_s is replaced by $(X_{t_i} - X_{t_{i-1}})$, and $\int_0^{T_n} X_s^2 ds$ by $\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2$, we obtain the following estimators of θ,

$$\hat{\theta}_n = \frac{\sum_{i=1}^{n} X_{t_{i-1}}(X_{t_i} - X_{t_{i-1}})}{\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2}, \tag{3}$$

and

$$\check{\theta}_n = \frac{X_{t_n}^2}{2\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2}. \tag{4}$$

For non-ergodic diffusion processes driven by Brownian motion based on discrete observations, parametric estimation problems have been studied for instance by Jacod (2006), Dietz and Kutoyants (2003) and Shimizu (2009).

The rest of our paper is organized as follows. In Section 2 we introduce the needed material for our study. In section 3 we prove the strong consistency of $\hat{\theta}_n$ and $\check{\theta}_n$. Finally, section 4 is devoted to establish that the sequences $\sqrt{n\Delta_n}\left(\hat{\theta}_n - \theta\right)$ and $\sqrt{n\Delta_n}\left(\check{\theta}_n - \theta\right)$ are tight.

2. Basic notions for fractional Brownian motion

In this section, we briefly recall some basic facts concerning stochastic calculus with respect to a fractional Brownian motion; we refer to Nualart (2006) for further details. Let $B^H = \{B_t^H\}_{t\in[0,T]}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$, defined on some probability space $(\Omega, \mathcal{F}, P)$. (Here, and everywhere else, we do assume that $\mathcal{F}$ is the sigma-field generated by B^H.) This means that B^H is a centered Gaussian process with the covariance function $E[B_s^H B_t^H] = R_H(s,t)$, where

$$R_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{5}$$

If $H = \frac{1}{2}$, then $B^{\frac{1}{2}}$ is a Brownian motion.

We denote by $\mathcal{E}$ the set of step $\mathbb{R}-$valued functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(t,s).$$

We denote by $|\cdot|_{\mathcal{H}}$ the associate norm. The mapping $\mathbf{1}_{[0,t]} \mapsto B_t^H$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space associated with B^H. We denote this isometry by

$$\varphi \mapsto B^H(\varphi) = \int_0^T \varphi(s) dB_s^H. \tag{6}$$

When $H \in (\frac{1}{2}, 1)$, it follows from Pipiras and Taqqu (2000) that the elements of $\mathcal{H}$ may not be functions but distributions of negative order. It will be more convenient to work with a subspace of $\mathcal{H}$ which contains only functions. Such a space is the set $|\mathcal{H}|$ of all measurable functions φ on $[0,T]$ such that

$$|\varphi|^2_{|\mathcal{H}|} := H(2H-1)\int_0^T\int_0^T |\varphi(u)||\varphi(v)||u-v|^{2H-2}dudv < \infty.$$

If $\varphi, \psi \in |\mathcal{H}|$ then

$$E\big[B^H(\varphi)B^H(\psi)\big] = H(2H-1)\int_0^T\int_0^T \varphi(u)\psi(v)|u-v|^{2H-2}dudv. \tag{7}$$

We know that $(|\mathcal{H}|, \langle\cdot,\cdot\rangle_{|\mathcal{H}|})$ is a Banach space, but that $(|\mathcal{H}|, \langle\cdot,\cdot\rangle_{\mathcal{H}})$ is not complete (see e.g. Pipiras and Taqqu, 2000). However, we have the dense inclusions $L^2([0,T]) \subset L^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}| \subset \mathcal{H}$.For every $q \geq 1$, let $\mathcal{H}_q$ be the qth Wiener chaos of X, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(X(h)), h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$, where H_q is the qth Hermite polynomial defined as $H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}(e^{-\frac{x^2}{2}})$. The mapping $I_q(h^{\otimes q}) = H_q(X(h))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}^{\odot q}$ (equipped with the modified norm $\|\cdot\|_{\mathcal{H}^{\odot q}} = \sqrt{q!}\|\cdot\|_{\mathcal{H}^{\otimes q}}$) and $\mathcal{H}_q$. Specifically, for all $f, g \in \mathcal{H}^{\odot q}$ and $q \geq 1$, one has

$$E\big[I_q(f)I_q(g)\big] = q!\langle f, g\rangle_{\mathcal{H}^{\otimes q}}.$$

The multiple stochastic integral $I_q(f)$ satisfies hypercontractivity property:

$$\big(E\big[|I_q(f)|^p\big]\big)^{1/p} \leqslant c_{p,q}\big(E\big[|I_q(f)|^2\big]\big)^{1/2} \quad \text{for any } p \geq 2.$$

As a consequence, for any $F \in \oplus_{l=1}^q \mathcal{H}_l$, we have

$$\big(E\big[|F|^p\big]\big)^{1/p} \leqslant c_{p,q}\big(E\big[|F|^2\big]\big)^{1/2} \quad \text{for any } p \geq 2. \tag{8}$$

3. Construction and strong consistency of the estimators

From the explicit solution of (1) which is given by

$$X_t = e^{\theta t}\int_0^t e^{-\theta s}dB^H_s. \tag{9}$$

Let us introduce the following processes related to X_t:

$$\xi_t := \int_0^t e^{-\theta s}dB^H_s$$

and

$$S_n := \Delta_n \sum_{i=1}^n X^2_{t_{i-1}}.$$

So, we can write

$$\hat{\theta}_n = \frac{e^{\theta\Delta_n} - 1}{\Delta_n} + \frac{G_n}{S_n} \tag{10}$$

where

$$G_n := \sum_{i=1}^{n} e^{\theta t_i} \left(\xi_{t_i} - \xi_{t_{i-1}}\right) X_{t_{i-1}}.$$

We first recall some results of Belfadli *et al.* (2011) needed throughout the paper:

$$\lim_{t\to\infty} \xi_t = \xi_\infty := \int_0^\infty e^{-\theta s} dB_s^H \tag{11}$$

almost surely as $t \to \infty$. Moreover

$$\sup_{t\geq 0} E(\xi_t^2) \leqslant E(\xi_\infty^2) = H\Gamma(2H)\theta^{-2H} < \infty. \tag{12}$$

On the other hand

$$e^{-2\theta T_n} \int_0^{T_n} X_t^2 dt \longrightarrow \frac{\xi_\infty^2}{2\theta} \tag{13}$$

almost surely as $n \to \infty$.

For the strong consistency, let us state the following direct consequence of the Borel-Cantelli Lemma (see e.g. Kloeden and Neuenkirch, 2007), which allows us to turn convergence rates in the p-th mean into pathwise convergence rates.

Lemma 1. *Let $\gamma > 0$ and $p_0 \in \mathbb{N}$. Moreover let $(Z_n)_{n\in\mathbb{N}}$ be a sequence of random variables. If for every $p \geq p_0$ there exists a constant $c_p > 0$ such that for all $n \in \mathbb{N}$,*

$$(\mathbb{E}|Z_n|^p)^{1/p} \leqslant c_p \cdot n^{-\gamma},$$

then for all $\varepsilon > 0$ there exists a random variable η_ε such that

$$|Z_n| \leqslant \eta_\varepsilon \cdot n^{-\gamma+\varepsilon} \quad \textit{almost surely}$$

for all $n \in \mathbb{N}$. Moreover, $\mathbb{E}|\eta_\varepsilon|^p < \infty$ for all $p \geq 1$.

We will need the following Lemma.

Lemma 2. *Let $H \in (\frac{1}{2}, 1)$. Assume that $\theta > 0$, $\Delta_n \to 0$ and $T_n \to \infty$ as $n \to \infty$. Then for any $\beta > 0$*

$$e^{-2\theta T_n} S_n = \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \xi_{t_{n-1}}^2 + o(n^\beta \Delta_n^{H-1} e^{-\theta T_n}) \quad \textit{almost surely.} \tag{14}$$

In addition, if we assume that $n\Delta_n^{1+\alpha} \to 0$ for some $\alpha > 0$,

$$e^{-2\theta T_n} S_n = \frac{\Delta_n}{e^{2\theta\Delta_n} - 1} \xi_{t_{n-1}}^2 + o(1) \quad \textit{almost surely,} \tag{15}$$

and hence, as $n \to \infty$

$$e^{-2\theta T_n} S_n \longrightarrow \frac{\xi_\infty^2}{2\theta} \quad \textit{almost surely.} \tag{16}$$

Proof. Let us start by noting that

$$\begin{aligned}
e^{-2\theta T_n}S_n &= \frac{\Delta_n}{e^{2\theta\Delta_n}-1}\sum_{i=1}^{n} e^{-2\theta(n-i)\Delta_n}\left(\frac{e^{2\theta\Delta_n}-1}{e^{2\theta\Delta_n}}\right)\xi_{t_{i-1}}^2 \\
&= \frac{\Delta_n}{e^{2\theta\Delta_n}-1}\sum_{i=1}^{n} e^{-2\theta(n-i)\Delta_n}\left(1-\frac{1}{e^{2\theta\Delta_n}}\right)\xi_{t_{i-1}}^2 \\
&= \frac{\Delta_n}{e^{2\theta\Delta_n}-1}\sum_{i=1}^{n} \left(e^{-2\theta(n-i)\Delta_n}-e^{-2\theta(n-i+1)\Delta_n}\right)\xi_{t_{i-1}}^2 \\
&= \frac{\Delta_n}{e^{2\theta\Delta_n}-1}\left[\xi_{t_{n-1}}^2-\sum_{i=2}^{n}(\xi_{t_{i-1}}^2-\xi_{t_{i-2}}^2)e^{-2\theta(n-i+1)\Delta_n}\right].
\end{aligned}$$

Hence

$$\begin{aligned}
e^{-2\theta T_n}S_n-\frac{\Delta_n}{e^{2\theta\Delta_n}-1}\xi_{t_{n-1}}^2 &= \frac{\Delta_n}{e^{2\theta\Delta_n}-1}\left[-\sum_{i=2}^{n}(\xi_{t_{i-1}}^2-\xi_{t_{i-2}}^2)e^{-2\theta(n-i+1)\Delta_n}\right] \\
:&= \frac{-\Delta_n}{e^{2\theta\Delta_n}-1}R_n.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{-\Delta_n}{e^{2\theta\Delta_n}-1} &= \frac{-\Delta_n}{2\theta\Delta_n+o(\Delta_n^2)} \\
&= \frac{-1}{2\theta+o(\Delta_n)} \\
&= \frac{-1}{2\theta}+o(\Delta_n),
\end{aligned}$$

we have

$$e^{-2\theta T_n}S_n-\frac{\Delta_n}{e^{2\theta\Delta_n}-1}\xi_{t_{n-1}}^2 = \left(\frac{-1}{2\theta}+o(\Delta_n)\right)R_n. \tag{17}$$

From the equality

$$\sqrt{\Delta_n}e^{\theta T_n}R_n=\sqrt{\Delta_n}\sum_{i=1}^{n-1}e^{\theta i\Delta_n}e^{-\theta\Delta_n(n-i)}(\xi_{t_i}^2-\xi_{t_{i-1}}^2),$$

we can write by using Minkowski and Cauchy Schwartz inequalities and (12)

$$\begin{aligned}
\left(E\left|\sqrt{\Delta_n}e^{\theta T_n}R_n\right|^2\right)^{1/2} &\leqslant \sqrt{\Delta_n}\sum_{i=1}^{n-1}e^{\theta i\Delta_n}e^{-\theta\Delta_n(n-i)}[E(\xi_{t_i}^2-\xi_{t_{i-1}}^2)^2]^{1/2} \\
&\leqslant 2\sqrt{\Delta_n}[E(\xi_\infty)^2]^{1/2}\sum_{i=1}^{n-1}e^{\theta i\Delta_n}e^{-\theta\Delta_n(n-i)}[E(\xi_{t_i}-\xi_{t_{i-1}})^4]^{1/4} \\
&= 2\sqrt{\Delta_n}[E(\xi_\infty)^2]^{1/2}\sum_{i=1}^{n-1}e^{\theta i\Delta_n}e^{-\theta\Delta_n(n-i)}[E(\xi_{t_i}-\xi_{t_{i-1}})^2]^{1/2}.
\end{aligned}$$

We now calculate

$$E\left[(e^{\theta i\Delta_n}(\xi_{t_i}-\xi_{t_{i-1}}))^2\right] = H(2H-1)e^{2\theta i\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n} e^{-\theta s}e^{-\theta r}|s-r|^{2H-2}dsdr.$$

Making the change of variables $u=\frac{s}{\Delta_n}-i+1$ and $v=\frac{r}{\Delta_n}-i+1$ yield

$$\begin{aligned} E\left[(e^{\theta i\Delta_n}(\xi_{t_i}-\xi_{t_{i-1}}))^2\right] &= H(2H-1)\Delta_n^{2H}e^{2\theta\Delta_n}\int_0^1\int_0^1 e^{-\theta u\Delta_n}e^{-\theta v\Delta_n}|u-v|^{2H-2}dudv \\ &\leqslant H(2H-1)\Delta_n^{2H}e^{2\theta\Delta_n}\int_0^1\int_0^1 |u-v|^{2H-2}dudv \\ &= \Delta_n^{2H}e^{2\theta\Delta_n}. \end{aligned} \tag{18}$$

Therefore

$$\begin{aligned} \left(E\left|\sqrt{\Delta_n}e^{\theta T_n}R_n\right|^2\right)^{1/2} &\leqslant 2\sqrt{\Delta_n}\Delta_n^H e^{\theta\Delta_n}[E(\xi_\infty)^2]^{1/2}\sum_{i=1}^{n-1}e^{-\theta\Delta_n(n-i)} \\ &= 2\sqrt{\Delta_n}\Delta_n^H[E(\xi_\infty)^2]^{1/2}\left(\sum_{i=0}^{n-2}e^{-\theta i\Delta_n}\right) \\ &= 2\sqrt{\Delta_n}\Delta_n^H[E(\xi_\infty)^2]^{1/2}\left(\frac{1-e^{-\theta(n-1)\Delta_n}}{1-e^{-\theta\Delta_n}}\right) \\ &\leqslant 2\sqrt{\Delta_n}\Delta_n^H[E(\xi_\infty)^2]^{1/2}\left(\frac{1}{1-e^{-\theta\Delta_n}}\right) \\ &= 2\Delta_n^{H-1/2}[E(\xi_\infty)^2]^{1/2}\left(\frac{\Delta_n}{1-e^{-\theta\Delta_n}}\right) \\ &\leqslant c(H,\theta)\Delta_n^{H-1/2} \end{aligned} \tag{19}$$

where, here and everywhere else, $c(H,\theta)$ is a generic positive constant depending only on H and θ.

Hence for any $\beta>0$

$$\left(E\left|n^{-\beta}\Delta_n^{1-H}e^{\theta T_n}R_n\right|^2\right)^{1/2}\leqslant c(H,\theta)n^{-\beta}.$$

Now, applying (8) and Lemma 1 there exists a random variable η_β such that

$$\left|\Delta_n^{1-H}e^{\theta T_n}R_n\right|\leqslant|\eta_\beta|n^{\beta/2}\quad\text{almost surely.} \tag{20}$$

for all $n\in\mathbb{N}$ with $\mathbb{E}|\eta_\beta|^p<\infty$ for all $p\geq 1$.

Thus, the estimation (14) is obtained. For the convergence (15), we suppose that $n\Delta_n^{1+\alpha}\to 0$ for some $\alpha>0$.

Choosing a constant $\gamma>0$ such that $\frac{\beta+1-H}{\gamma}<\alpha$,

$$n\Delta_n^{1+\frac{\beta+1-H}{\gamma}}\to 0, \tag{21}$$

and by using (14) and the fact that $T_n^{\beta+\gamma}e^{-\theta T_n}\to 0$ the estimations (15) and (16) are satisfied.

Thus we arrive at our main theorem of this section.

Theorem 1. *Let $H \in (\frac{1}{2}, 1)$. Suppose that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to 0$ as $n \to \infty$ for some $\alpha > 0$. Then, as $n \to \infty$,*

$$\hat{\theta}_n \longrightarrow \theta \quad \textit{almost surely,} \tag{22}$$

and also,

$$\check{\theta}_n \longrightarrow \theta \quad \textit{almost surely.} \tag{23}$$

Proof. We first prove (22). From (10) and (16) it suffices to show that $e^{-2\theta T_n} G_n$ converges to 0 almost surely as $n \to \infty$.

By using (17) we have

$$\begin{aligned}
\left(E\left|e^{-2\theta T_n} G_n\right|^2\right)^{1/2} &\leqslant e^{-2\theta T_n} \sum_{i=1}^{n} e^{\theta i \Delta_n} (EX_{t_{i-1}}^2)^{1/2} \left[E(\xi_{t_i} - \xi_{t_{i-1}})^2\right]^{1/2} \\
&\leqslant e^{-2\theta T_n} \Delta_n^H e^{\theta \Delta_n} \sum_{i=1}^{n} (EX_{t_{i-1}}^2)^{1/2} \\
&\leqslant e^{-2\theta T_n} \Delta_n^H e^{\theta \Delta_n} (E\xi_\infty^2)^{1/2} \sum_{i=1}^{n} e^{\theta i \Delta_n} \\
&\leqslant c(H,\theta) e^{-\theta T_n} \Delta_n^H \frac{1 - e^{-\theta T_n}}{e^{\theta \Delta_n} - 1} \\
&\leqslant c(H,\theta) e^{-\theta T_n} \Delta_n^{H-1}.
\end{aligned} \tag{24}$$

Fix $\beta > 0$. Then there exists γ a positive constant which verifies (21).

Hence (24) leads to

$$\left(E\left|e^{-2\theta T_n} G_n\right|^2\right)^{1/2} \leqslant c(H,\theta,\alpha,\beta) n^{-\beta}.$$

By applying (8) and Lemma 1 we conclude that for every $\beta > 0$ there exists a random variable η_β such that

$$\left|e^{-2\theta T_n} G_n\right| \leqslant |\eta_\beta| n^{-\beta} \quad \text{almost surely.} \tag{25}$$

for all $n \in \mathbb{N}$ with $\mathbb{E}|\eta_\beta|^p < \infty$ for all $p \geq 1$. Hence, the convergence (22) is proved.

From (4) we can write

$$\check{\theta}_n = \frac{\xi_{T_n}^2}{2e^{-2\theta T_n} S_n}.$$

Thus the convergence (23) is a direct consequence of (13) and (16).

4. Rate consistency of the estimators

In this section, we will establish that $\sqrt{n\Delta_n}\left(\hat{\theta}_n - \theta\right)$ and $\sqrt{n\Delta_n}\left(\check{\theta}_n - \theta\right)$ are tight.

Theorem 2. *Let $H \in (\frac{1}{2}, 1)$. Assume that $\theta > 0$, $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$ as $n \to \infty$ for some $\alpha > 0$. Then, for any $q \geq 0$,*

$$\Delta_n^q e^{\theta T_n}(\hat{\theta}_n - \theta) \text{ is not tight (equivalently: not bounded in probability).} \tag{26}$$

In addition, we assume that $n\Delta_n^3 \to 0$ as $n \to \infty$. Then the estimator $\hat{\theta}_n$ is $\sqrt{T_n}-$consistent, in the sense that the sequence

$$\sqrt{T_n}(\hat{\theta}_n - \theta) \quad \text{is tight.} \tag{27}$$

Proof. We shall only prove the case where $q = 1$. Similarly, we can prove the case where $q > 1$, and the case where $0 \leqslant q < 1$ is a direct consequence.
From (10) we obtain

$$\Delta_n e^{\theta T_n}(\widehat{\theta}_n - \theta) = e^{\theta T_n}(e^{\theta\Delta_n} - 1 - \theta\Delta_n) + \frac{\Delta_n e^{-\theta T_n} G_n}{e^{-2\theta T_n} S_n}. \tag{28}$$

Since $n\Delta_n^{1+\alpha} \to \infty$ and $\frac{e^{\theta\Delta_n} - 1 - \theta\Delta_n}{\Delta_n^2} \longrightarrow \theta^2/2$, we deduce that

$$e^{\theta T_n}(e^{\theta\Delta_n} - 1 - \theta\Delta_n) \to \infty. \tag{29}$$

By using (24) we have

$$E|\Delta_n e^{-\theta T_n} G_n| \leqslant c(H,\theta)\Delta_n^H \to 0. \tag{30}$$

Combining (28), (29), (30) and (16) we get (26).

Let us now prove (27). We have from (10) that

$$\sqrt{T_n}(\hat{\theta}_n - \theta) = \sqrt{\frac{n}{\Delta_n}}(e^{\theta\Delta_n} - 1 - \theta\Delta_n) + \frac{\sqrt{T_n} e^{-2\theta T_n} G_n}{e^{-2\theta T_n} S_n}. \tag{31}$$

Since $n\Delta_n^3 \to 0$,

$$\begin{aligned} \sqrt{\frac{n}{\Delta_n}}(e^{\theta\Delta_n} - 1 - \theta\Delta_n) &= \sqrt{n\Delta_n^3}\frac{(e^{\theta\Delta_n} - 1 - \theta\Delta_n)}{\Delta_n^2} \\ &\to 0. \end{aligned} \tag{32}$$

On the other hand, the inequality (30) leads to

$$\begin{aligned} E|\sqrt{T_n} e^{-2\theta T_n} G_n| &\leqslant c(H,\theta)\sqrt{T_n^3}\Delta_n^{H-2} e^{-\theta T_n} \\ &\to 0. \end{aligned} \tag{33}$$

The last convergence comes from $n\Delta_n^3 \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$.

Consequently, by (31), (32), (33) and (16) we deduce (27).

Theorem 3. *Let $H \in (\frac{1}{2}, 1)$. Suppose that $\Delta_n \to 0$ and $n\Delta_n^{1+\alpha} \to \infty$ as $n \to \infty$ for some $\alpha > 0$. Then, for any $q \geq 0$,*

$$\Delta_n^q e^{\theta T_n}(\check{\theta}_n - \theta) \text{ is not tight (equivalently: not bounded in probability).} \tag{34}$$

In addition, we assume that $n\Delta_n^3 \to 0$ as $n \to \infty$. Then the estimator $\check{\theta}_n$ is $\sqrt{T_n}-$consistent, in the sense that the sequence

$$\sqrt{T_n}(\check{\theta}_n - \theta) \quad \text{is tight.} \tag{35}$$

Proof. We shall only prove the case where $q = \frac{1}{2}$. Similarly, we can prove the case where $q > \frac{1}{2}$, and the case where $0 \leqslant q < \frac{1}{2}$ is a direct consequence.

Using the definition of $\check{\theta}_n$, we have

$$\begin{aligned}
\sqrt{\Delta_n}e^{\theta T_n}(\check{\theta}_n - \theta) &= \sqrt{\Delta_n}e^{\theta T_n}\left(\frac{X_{t_n}^2}{2\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2} - \theta\right) \\
&= \sqrt{\Delta_n}e^{\theta T_n}\left(\frac{e^{2\theta T_n}\xi_{t_n}^2}{2\Delta_n \sum_{i=1}^{n} X_{t_{i-1}}^2} - \theta\right) \\
&= \frac{\sqrt{\Delta_n}}{2} S_n^{-1} e^{3\theta T_n}(\xi_{t_n}^2 - 2\theta S_n e^{-2\theta T_n}).
\end{aligned}$$

We can write

$$\begin{aligned}
\sqrt{\Delta_n}e^{\theta T_n}(\check{\theta}_n - \theta) &= \frac{\sqrt{\Delta_n}e^{\theta T_n}}{2e^{-2\theta T_n}S_n}\left[(\xi_{t_n}^2 - \xi_{t_{n-1}}^2) + \left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n} - 1}\right)\xi_{t_{n-1}}^2\right. \\
&\quad \left. -2\theta\left(e^{-2\theta T_n}S_n - \frac{\Delta_n}{e^{2\theta\Delta_n} - 1}\xi_{t_{n-1}}^2\right)\right]. \qquad (36)
\end{aligned}$$

By (17), (18) and (19) we obtain

$$\begin{aligned}
E\left|\sqrt{\Delta_n}e^{\theta T_n}\left[(\xi_{t_n}^2 - \xi_{t_{n-1}}^2) - 2\theta\left(e^{-2\theta T_n}S_n - \frac{\Delta_n}{e^{2\theta\Delta_n} - 1}\xi_{t_{n-1}}^2\right)\right]\right| &\leqslant c(H,\theta)\Delta_n^{H-\frac{1}{2}} \\
&\rightarrow 0. \qquad (37)
\end{aligned}$$

On the other hand

$$\begin{aligned}
\sqrt{\Delta_n}e^{\theta T_n}\left(1 - \frac{2\theta\Delta_n}{e^{2\theta\Delta_n} - 1}\right) &= \Delta_n^{3/2}e^{\theta T_n}\left(\frac{e^{2\theta\Delta_n - 1 - 2\theta\Delta_n}}{\Delta_n^2}\frac{\Delta_n}{e^{2\theta\Delta_n} - 1}\right) \\
&\rightarrow \infty. \qquad (38)
\end{aligned}$$

The last convergence comes from the fact that $n\Delta_n^{1+\alpha} \rightarrow \infty$ as $n \rightarrow \infty$. Combining (36), (37) and (38) we obtain (34).

Furthermore, using $n\Delta_n^3 \rightarrow 0$ as $n \rightarrow \infty$ the result (35) is obtained.

Remark 1. Assume that $\theta > 0$. Belfadli *et al.* (2011) proved that, in the continuous case, $e^{\theta t}(\tilde{\theta}_t - \theta)$ is asymptotically Cauchy. Then one may also expect that, in the discrete case, $\hat{\theta}_n$ and $\check{\theta}_n$ are $e^{\theta T_n}$-consistent. But the answer is negative, they are $\sqrt{T_n}$−consistent (see Theorem 2 and Theorem 3).

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