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To Professor Włodzimierz Staś on his 75th birthday

AVERAGE VALUES OF QUADRATIC TWISTS OF MODULAR L-FUNCTIONS

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Abstract: This paper studies non-vanishing of quadratic twists of automorphic forms f on $GL(2)$ over $\mathbf{Q}$ at various points inside the critical strip. Given any point w_0 inside the critical strip, and $\epsilon > 0$, we show that at least $Y^{12/17-\varepsilon}$ of the quadratic twists $L(f, \chi_d, s)$ with $|d| \leq Y$ do not vanish inside the disc $|w - w_0| < (\log Y)^{-1-\varepsilon}$. (Here $d \equiv 1 \bmod 4$ is a fundamental discriminant and χ_d denotes the Kronecker symbol.) If we assume the Ramanujan conjecture about the Fourier coefficients of f (in particular, if f is holomorphic) then $\frac{12}{17}$ above can be replaced with 1.

This should be compared with a result of Ono and Skinner [10] which states that if f is a holomorphic newform of even weight and trivial character, then at least $\gg Y/\log Y$ of the quadratic twists $L(f, \chi_d, s)$ are nonzero at the central critical point. A slightly weaker result had been proved earlier by Perelli and Pomykala [11]. By contrast, we make no restriction on the holomorphy of f and the result holds even if f has non-trivial central character. Moreover, we prove non-vanishing in a disc about any point in the critical strip. As in [11], our tools are the method of Iwaniec [4] and a mean value estimate of Heath-Brown [3].

1. Introduction

Let f be a cusp form which is a normalized eigenform for the Hecke operators, of level N, character ω and weight k (k is a positive integer and $k = 1$ if f is real-analytic due to our normalization). We have an expansion

$$f(z) = \begin{cases} \sum_{n \geq 1} a(n) e(nz) & \text{if } f \text{ is holomorphic} \\ \sum_{n \neq 0} a(n) 2\sqrt{y} K_\nu(2\pi |n| y) e(nx) & \text{if } f \text{ is real analytic.} \end{cases}$$

Here $e(z) = \exp(2\pi i z)$, $z = x + iy$ and K_ν denotes the Bessel function of degree ν. It is known that

$$|a(n)| \leq \mathbf{d}(n) n^{(k-1)/2+\alpha} \tag{1.1}$$

$$\sum_{|n| \leq x} |a(n)| \ll x^{(k+1)/2} \tag{1.2}$$

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where $\mathbf{d}(n)$ denotes the number of positive divisors of n. If f is holomorphic, the Ramanujan-Petersson conjecture is known and we may take $\alpha = 0$. By a recent result of Kim and Shahidi [6], we have $\alpha \leq \frac{5}{34}$ if f is real analytic.

Let χ_d denote the quadratic character $(d/\cdot)$. Then the Dirichlet series

$$L(f, \chi_d, s) = \sum_{n \geq 1} a(n)\chi_d(n)n^{-s} = \prod_p (1 - \alpha(p)\chi_d(p)p^{-s})^{-1}(1 - \beta(p)\chi_d(p)p^{-s})^{-1}$$

converges absolutely for $\Re(s) > \frac{1}{2}(k+1)$ and has an analytic continuation as an entire function of s. If d is a fundamental discriminant (i.e. d is squarefree and $\equiv 1 \pmod 4$ or $d = 4d_0$, d_0 squarefree $\equiv 2, 3 \pmod 4$) and $(d, N) = 1$, we have the functional equation

$$A_d^s \tilde{\Gamma}(s) L(f, \chi_d, s) = \omega_d A_d^{k-s} \tilde{\Gamma}(k-s) L(\bar{f}, \chi_d, k-s)$$

where

$$A_d = \begin{cases} d\sqrt{N}/2\pi & \text{if } f \text{ is holomorphic} \\ d\sqrt{N}/\pi & \text{if } f \text{ is real analytic,} \end{cases}$$

$$\tilde{\Gamma}(s) = \begin{cases} \Gamma(s) & \text{if } f \text{ is holomorphic} \\ \Gamma(\frac{s+\nu}{2})\Gamma(\frac{s-\nu}{2}) & \text{if } f \text{ is real analytic} \end{cases}$$

and

$$\omega_d = \omega_1 \chi_d(-N)\omega(d), \qquad \omega_1 \in \mathbf{C}, \ |\omega_1| = 1.$$

We are interested in the average value of the L-function $L(f, \chi_d, s)$ in the critical strip. In [9], Chapter 6, it was shown that if f is holomorphic and $k = 2$, then

$$\sum_{d \equiv a \pmod{4N}, |d| \leq Y} L(f, \chi_d, 1)\left(1 - \frac{|d|}{Y}\right) = cY + \mathbf{O}(Y(\log Y)^{-\beta})$$

for some $c \neq 0$ and $\beta > 0$ where the sum ranges over all d (i.e. not only over fundamental discriminants). It follows that there are infinitely many fundamental discriminants d such that $L(f, \chi_d, 1) \neq 0$ and this was the first such result for forms f with non-trivial Nebentypus character ω. The methods of [9] were a refinement of those of [8]. In [12], Stefanicki showed that the method of Iwaniec [4] could be used to prove a similar asymptotic formula ranging over fundamental discriminants and with a sharper error term. An analogous result was established by Friedberg and Hoffstein [2] for automorphic forms on $GL(2)$ over number fields using metaplectic Eisenstein series.

In this paper we use the method of Iwaniec [4] to prove the following estimate. Let $a \equiv 1 \pmod 4$, $(a, 4N) = 1$. Set

$$D_a^{\pm} = \{n \in \mathbf{N} : \operatorname{sgn}(n) = \pm, \ n \equiv a \pmod{4N}\}$$

and

$$D_a = D_a^+ \cup D_a^-.$$

Let F be a smooth compactly supported function in $\mathbf{R}^+$ with positive mean value $\int_0^\infty F(t)\,dt$ and let μ denote the Möbius function.

Theorem 1.1. *Let $\varepsilon > 0$. Let $w_0 \in \mathbf{C}$ satisfy $\Re w_0 \in [k/2, (k+1)/2)$ and for each $d \in D_a^{\pm}$, $|d| \ll Y$ choose $w_d \in \mathbf{C}$ in the disc $|w - w_0| \le \lambda \overset{\mathrm{def}}{=} 1/(\log Y)^{1+\varepsilon}$. Then*

$$\sum_{d \in D_a} \mu^2(|d|) L(f, \chi_d, w_d) F\left(\frac{|d|}{Y}\right) = cY + \mathbf{O}(|\tilde{\Gamma}(w_0)|^{-1} \lambda Y^{1+k/2-\Re w_0} \log Y \log\log Y)$$

where $c = c(f, F, w_0, a) \ne 0$.

The proof is essentially the same as in [4]. However, it is necessary to keep track of the appearance of α and for this reason, we write out the details.

Theorem 1.2. *With the same notation and hypotheses as above,*

$$\sum_{d \in D_a^{\pm}, |d| \ll Y} \mu^2(|d|) |L(f, \chi_d, w_d)|^2 \ll |\tilde{\Gamma}(w_0)|^{-2} Y^{1+\varepsilon+2\alpha}.$$

These mean-value estimates have the following consequence for zeros of $L(f, \chi_d, s)$.

Theorem 1.3. *With notation as in Theorem 1.1, there are $\gg_{|w_0|} Y^{1-2\alpha-\varepsilon}$ fundamental discriminants $|d| \ll Y$ such that $L(f, \chi_d, s)$ has no zero in the disc $|s - w_0| \le \lambda$.*

Thus, using $\alpha \le 5/34$, we get $\gg Y^{12/17-\varepsilon}$ non-vanishing quadratic twists. If we assume the Ramanujan conjecture, we get $\gg Y^{1-\varepsilon}$ such twists. Theorem 1.3 follows from Theorem 1.1 and 1.2 by the Cauchy-Schwartz inequality.

Remarks

1. It is often possible to obtain an asymptotic formula in Theorem 1 when we restrict summation to D_a^+ or D_a^-. Indeed, it is always possible if $\Re w_0 \ne k/2$. If $\Re w_0 = k/2$, then either D_a^+ or D_a^- will yield an asymptotic formula. The general formula is given in the final section.

2. For a general L-function which can be represented by an Euler product let us write $L_{(a)}(s)$ for the Euler product with p-factors for $p|a$ removed. Then the constant in Theorem 1.1 is given by

$$c(f, F, w_0, a) = \frac{1}{2N\zeta_{(4N)}(2)} L_{(2)}(\omega^2, 4w_0 - 2k + 2)^{-1} P(2w_0) \times$$
$$\times f_{4N}(w_0) L_{(4N)}(Sym^2(f), 2w_0) \int_0^{\infty} F(t)\, dt$$

where $\zeta(s)$ is the Riemann zeta function, $L(\omega^2, s)$ is the Dirichlet L-function associated to the character ω^2, $P(s)$ is a certain function which depends on f and which is represented by an absolutely convergent Euler product for $\Re s > k - 1 + 2\alpha$ and does not vanish for $\Re s \ge k$, $f_{4N}(s)$ is a certain function which depends on

f and which does not vanish for $\Re s \geq k/2$ and $L(Sym^2(f), s)$ is the L-function attached to the symmetric square of f.

3. Several authors have shown that in some cases, a positive proportion of the twists are nonzero. For this, we refer the reader to works of James, Kohnen, Vatsal, Ono and Skinner (see [1] for the references). Also, Ono and Skinner [10] showed that for holomorphic newforms with trivial character, there are at least $\gg Y/\log Y$ quadratic twists for which the L-function does not vanish at the central critical point. These methods do not appear to work for other points or for non-holomorphic forms as they rely on the relationship of the central critical value to the Shimura lift and on the existence of Galois representations.

2. Preliminaries

Consider the integral

$$S(f, \chi_d, w, X) = \frac{1}{2\pi i} \int_{(\gamma)} \tilde{\Gamma}(w+s) L(f, \chi_d, w+s) X^s \frac{ds}{s}.$$

We have

$$S(f, \chi_d, w, X) = \sum_{n \geq 1} a(n) \chi_d(n) n^{-w} W\left(w, \frac{n}{X}\right)$$

where

$$\begin{aligned} W(w, X) &= \frac{1}{2\pi i} \int_{(\gamma)} \tilde{\Gamma}(w+s) X^{-s} \frac{ds}{s} \\ &= \begin{cases} \int_X^\infty u^{w-1} \exp(-u)\, du & \text{if } f \text{ is holomorphic} \\ \int_X^\infty u^{w-1} K_\nu(u)\, du & \text{if } f \text{ is real analytic.} \end{cases} \end{aligned}$$

For d squarefree, $\equiv 1 \pmod 4$, the functional equation implies that

$$\tilde{\Gamma}(w) L(f, \chi_d, w) = S(f, \chi_d, w, X) + \omega_d A_d^{k-2w} S(\bar{f}, \chi_d, k-w, A_d^2 X^{-1}).$$

As in Iwaniec [4], we obtain

$$\sum_{d \in D_a^{\pm}} \mu^2(|d|) L(f, \chi_d, w_d) F\left(\frac{|d|}{Y}\right) = M_1^{\pm} + M_2^{\pm} + R_{1,1}^{\pm} + R_{2,1}^{\pm}.$$

where for $i = 1, 2$,

$$M_i^{\pm} = \omega_a^* \sum_{r \leq A, (r, 4N)=1} \mu(r) \sum_{d \in D_{a\bar{r}^2}^{\pm}} \frac{1}{\tilde{\Gamma}(w_d)} S(f^*, \chi_{dr^2}, w_d^*, A_{dr^2}) F\left(\frac{|d| r^2}{Y}\right) A_{dr^2}^{w_d^* - w_d}$$

and

$$R_{i,1}^{\pm} = \omega_a^* \sum_{b \geq 1, (b, 4N)=1} \sum_{r | b, r > A} \mu(r) \sum_{d \in D_{a\bar{b}^2}^{\pm}} \mu^2(|d|) \frac{1}{\tilde{\Gamma}(w_d)} S(f^*, \chi_{db^2}, w_d^*, A_{db^2}) F\left(\frac{|d| b^2}{Y}\right) A \tag{2.1}$$

Here, A is a power of Y to be specified later, $\bar{r}$ and $\bar{b}$ denote the multiplicative inverses of r and b modulo $4N$ and

$$(f^*, w_d^*, \omega_a^*) = \begin{cases} (f, w_d, 1) & \text{if } i = 1 \\ (\bar{f}, k - w_d, \operatorname{sgn}(d)\omega_1(\frac{a}{-N})\omega(a)) & \text{if } i = 2. \end{cases}$$

Every integer can be written uniquely as a product $n = k_1 l^2 m$ where $p|k_1 \Rightarrow p|4N$, $(lm, 4N) = 1$ and m squarefree. Then

$$\chi_d(ml^2) = \begin{cases} \chi_d(m) & \text{if } (d,l) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

To ensure that the condition $(d,l) = 1$ holds we introduce the sum $\sum_{q|(d,l)} \mu(q)$. Also, we use the expansion

$$\chi_d(m) = \bar{\varepsilon}_m m^{-\frac{1}{2}} \sum_{2|\rho|<m} \chi_{N\rho}(m) e\left(\frac{\bar{4}\bar{N}\rho d}{m}\right)$$

where

$$\varepsilon_m = \begin{cases} 1 & \text{if } m \equiv 1 \pmod 4 \\ i & \text{if } m \equiv 3 \pmod 4 \end{cases}$$

and $\bar{4}\bar{N}$ is the multiplicative inverse of $4N$ modulo m. The introduction of this expansion is a key factor of Iwaniec's argument in [4].

This brings $M_i^{\pm}$ to the form

$$M_i^{\pm} = \omega_a^* \sum_{r \le A, (r,4N)=1} \mu(r) \sum_{n = k_1 l^2 m, (n,r)=1} a^*(n) \left(\frac{a}{k_1}\right) \sum_{q|l} \mu(q) \sum_{d, dr^2 q \in D_a^{\pm}} n^{-w_d^*}$$
$$\sum_{2|\rho|<m} \frac{1}{\bar{\Gamma}(w_d)} \bar{\varepsilon}_m m^{-\frac{1}{2}} \chi_{N\rho q}(m) e\left(\frac{\bar{4}\bar{N}\rho d}{m}\right) W\left(w_d^*, \frac{n}{A_{dr^2 q}}\right) F\left(\frac{|d| r^2 q}{Y}\right) A_{dr^2 q}^{w_d^* - w_d} \tag{2.2}$$

where $a^*(n) = a(n)$ or $\bar{a}(n)$ depending on whether $i = 1$ or 2. Let us set

$$\Delta = \min\left(\frac{1}{2}, r^2 q Y^{\varepsilon - 1}\right)$$

Then we can write

$$M_i^{\pm} = MT_i^{\pm} + R_{i,2}^{\pm} + R_{i,3}^{\pm}$$

where in $MT_i^{\pm}$, $\rho = 0$, in $R_{i,2}^{\pm}$, $\Delta m \ge |\rho| > 0$, and in $R_{i,3}^{\pm}$, $\Delta m < |\rho| < m/2$. The following lemma is another key feature of [4] and it is very useful in estimating the above sums.

Lemma 2.1. *Suppose that ψ is a periodic function of period r and $|\psi| \leq 1$. Suppose $\alpha \in \mathbf{R}$ and $a \in \mathbf{Z}$. Then*

$$\sum_{|n| \leq x} a(n)e(\alpha n) \ll x^{k/2} \log x$$

$$\sum_{|n| \leq x, (n,a)=1} \mu^2(n)\psi(n)a(n)e(\alpha n) \ll \mathbf{d}(a)r^{1/2}x^{k/2}(\log x)^3.$$

We will also need the following standard bounds for the kernel function W and its derivatives

$$W^{(i)}(w^*, X) \ll \begin{cases} X^{\Re(w^*-\nu)-i} & \text{if } X \ll 1 \\ X^{\Re(w^*-\frac{3}{2})}\exp(-X) & \text{as } X \to \infty,\ f \text{ real-analytic} \\ X^{\Re(w^*-1)}\exp(-X) & \text{as } X \to \infty,\ f \text{ holomorphic} \end{cases} \tag{2.3}$$
$$\ll_{i,c} X^{\Re(w^*-\nu)-i}\exp(-cX)$$

where c is a positive constant.

3. The second moment

We have for d squarefree, $\equiv 1 \pmod 4$, the functional equation

$$\tilde{\Gamma}(w)L(f, \chi_d, w) = S(f, \chi_d, w, X) + \omega_d A_d^{k-2w} S(\bar{f}, \chi_d, k-w, A_d^2 X^{-1}).$$

Using the exponential decay of $W(w, n/X)$ we see that

$$\sum_{|d| \leq Y, d \in D_a^{\pm}} \left| \sum_{n} a(n)\chi_d(n)n^{-w}W\left(w, \frac{n}{X}\right)\right|^2$$
$$\ll \sum_{|d| \leq Y, d \in D_a^{\pm}} \left| \sum_{n \ll X} a(n)\chi_d(n)n^{-w}W\left(w, \frac{n}{X}\right)\right|^2$$

and this is

$$\ll (\log X)^2 \max_{M \ll X} \sum_{|d| \leq Y, d \in D_a^{\pm}} \left| \sum_{M \leq n \leq 2M} a(n)\chi_d(n)n^{-w}W\left(w, \frac{n}{X}\right)\right|^2.$$

Now by [3], Corollary 3 this is

$$\ll (\log X)^2 \max_{M \ll X} Y^{\epsilon}M^{1+\epsilon}(Y+M)\max_{M \leq n \leq 2M}|d(n)n^{(k-1)/2+\alpha-\Re w}|^2.$$

Simplifying, this is

$$\ll Y^{\epsilon}(X+Y)X^{2\epsilon+k+2\alpha-2\Re w}.$$

Now,

$$S(f,\chi_d,w_d,X) = \frac{1}{2\pi i}\int_{|w-w_0|=2\lambda} \frac{S(f,\chi_d,w,X)}{w-w_d}\,dw,$$

so

$$\begin{aligned}\sum_{|d|\le Y, d\in D_a^{\pm}} & \mu^2(|d|)|S(f,\chi_d,w_d,X)|^2 \\ & \ll \lambda^{-1}\int_0^{2\pi} \sum_{|d|\le Y, d\in D_a^{\pm}} \mu^2(|d|)|S(f,\chi_d,w_0+2\lambda e^{i\theta},X)|^2\,d\theta \\ & \ll Y^{\epsilon}(X+Y)X^{k+2\epsilon-2\Re w_0+4\lambda+2\alpha}\end{aligned}$$

uniformly for w_d as above. Now using partial summation we deduce that

$$\begin{aligned}\sum_{|d|\le Y, d\in D_a^{\pm}} & \mu^2(|d|)|S(f,\chi_d,w_d,X)A_d^{2w_d-k}|^2 \\ & \ll Y^{2(2\Re w_0-k)+\epsilon}(X+Y)X^{2\varepsilon+k+2\alpha+4\lambda-2\Re(w_0)}.\end{aligned}$$

Similarly

$$\sum_{|d|\le Y, d\in D_a^{\pm}} \mu^2(|d|)|S(\bar f,\chi_d,k-w_d,X)|^2 \ll Y^{\epsilon}(X+Y)X^{2\epsilon-k+2\alpha+2\Re(w_0)+6\lambda}.$$

Now, from the functional equation

$$\begin{aligned}|\tilde\Gamma(w_d)L(f,\chi_d,w_d)&A_d^{2w_d-k}|^2 \\ & \ll |S(f,\chi_d,w_d,X)A_d^{2w_d-k}|^2 + |S(\bar f,\chi_d,k-w_d,A_d^2X^{-1})|^2.\end{aligned}$$

Multiplying both sides by dX/X and integrating over X in the range $(\frac12 A_d, A_d)$, we find

$$\begin{aligned}|\tilde\Gamma(w_d)L(f,\chi_d,w_d)&A_d^{2w_d-k}|^2 \\ & \ll \int_{\frac12 A_d}^{A_d} |S(f,\chi_d,w_d,X)A_d^{2w_d-k}|^2\,\frac{dX}{X} \\ & \quad + \int_{\frac12 A_d}^{A_d} \left|S\left(\bar f,\chi_d,k-w_d,\frac{A_d^2}{X}\right)\right|^2 \frac{dX}{X}.\end{aligned}$$

In the second integral we change the variable to $u = A_d^2/X$. Then we extend the range of integration in both integrals to obtain

$$\begin{aligned}|\tilde\Gamma(w_d)L(f,\chi_d,w_d)&A_d^{2w_d-k}|^2 \\ & \ll \int_1^{cNY} (|S(f,\chi_d,w_d,X)A_d^{2w_d-k}|^2 + |S(\bar f,\chi_d,k-w_d,X)|^2)\frac{dX}{X}.\end{aligned}$$

Now summing over d, we deduce that

$$\sum_{|d|\leq Y, d\in D_a^{\pm}} \mu^2(|d|)|\tilde{\Gamma}(w_d)L(f,\chi_d,w_d)A_d^{2w_d-k}|^2 \ll Y^{1+\varepsilon+2\alpha+6\lambda+2\Re(w_0)-k}.$$

Using partial summation we obtain

$$\sum_{|d|\leq Y, d\in D_a^{\pm}} \mu^2(|d|)|L(f,\chi_d,w_d)|^2 \ll |\tilde{\Gamma}(w_0)|^{-2}Y^{1+\varepsilon+2\alpha}.$$

4. Estimation of errors

Estimation of $R_{i,1}^{\pm}$ IN (2.1)

To estimate $R_{i.1}^{\pm}$ we observe that

$$S(f^*,\chi_{db^2},w^*,A_{db^2}) = \sum_{l_1,l_2|b} \frac{\alpha^*(l_1)\beta^*(l_2)}{(l_1l_2)^{w^*}}\chi_d(l_1l_2)\mu(l_1)\mu(l_2)S\left(f^*,\chi_d,w^*,\frac{A_{db^2}}{l_1l_2}\right).$$

Here $\alpha^*(n)=\alpha(n)$ or $\bar{\alpha}(n)$ depending on whether $f^*=f$ or $\bar{f}$ and similarly for $\beta^*(n)$. We also assume that $|w-w_0|=2\lambda$. Since d is square-free in $R_{i.1}^{\pm}$ we may move the integration in the integral representation of

$$S\left(f^*,\chi_d,w^*,\frac{A_{db^2}}{l_1l_2}\right)$$

to the left of zero, picking up the residue at $s=0$, and apply functional equation to obtain

$$\{\text{residue at } s=0\} - \omega_d A_d^{k-2w^*} S\left(\bar{f}^*,\chi_d,k-w^*,\frac{A_d^2l_1l_2}{A_{db^2}}\right).$$

We first estimate the non-residual contribution. Now,

$$S\left(\bar{f}^*,\chi_d,k-w^*,\frac{A_d^2l_1l_2}{A_{db^2}}\right) = \sum_{n\geq 1}\bar{a}^*(n)n^{-k+w^*}\chi_d(n)W\left(k-w^*,\frac{nA_{db^2}}{A_d^2l_1l_2}\right).$$

We split the sum according to whether $n\leq A_d^2l_1l_2/A_{db^2}$ or not and use partial summation with (1.2) and (2.3). We obtain

$$\mathbf{O}((|d|b^{-2}l_1l_2)^{(1-k)/2+\Re w^*}).$$

We sum over l_1 and l_2 to see that the contribution to $S(f^*,\chi_{db^2},w^*,A_{db^2})$ is

$$\ll A_d^{k-2\Re w^*}\sum_{l_1,l_2|b}\left|\frac{\alpha^*(l_1)\beta^*(l_2)}{(l_1l_2)^{w^*}}\right|\left(\frac{|d|l_1l_2}{b^2}\right)^{(1-k)/2+\Re w^*}$$
$$\ll |d|^{(k+1)/2-\Re w^*}b^{k-1-2\Re w^*+\alpha}\mathbf{d}^2(b)$$

using (1.1) and the fact that if f is real analytic, one of $\alpha(\cdot)$ or $\beta(\cdot)$ is bounded. Multiplying it by $A_{db^2}^{w^*-w}$, dividing by $w-w_d$ and summing over $|d| \ll Y/b^2$ gives

$$\ll Y^{(k+3)/2-\Re w} b^{\alpha-4} \mathbf{d}^2(b)\lambda^{-1}.$$

Summing it over $r|b$ and $b > A$ gives

$$\ll A^{\alpha-3} Y^{(k+3)/2-\Re w+\varepsilon}.$$

It remains to estimate the contribution from the residue

$$A_{db^2}^{w^*-w} L(f^*, \chi_d, w^*)\tilde{\Gamma}(w^*) \prod_{p|b} (1-\alpha^*(p)\chi_d(p)p^{-w^*})(1-\beta^*(p)\chi_d(p)p^{-w^*})$$

at $s=0$. Firstly we note that the b-contribution is

$$b^{2\Re(w^*-w)} \prod_{p|b} (\cdot)(\cdot) \ll \mathbf{d}^2(b) b^{2\lambda}.$$

Hence, the contribution from the residue to $R_{i,1}^{\pm}$ is

$$\begin{aligned} &\sum_{b>A} \mathbf{d}^3(b) b^{2\lambda} \sum_{|d|\ll Y/b^2} \mu^2(|d|) \frac{|L(f^*, \chi_d, w^*)|}{|w-w_d|} |d|^{\Re(w^*-w)} \\ &\qquad \ll \sum_{b>A} \mathbf{d}^3(b) b^{2\lambda} \Bigg(\sum_{|d|\ll Y/b^2} \mu^2(|d|) |L(f^*, \chi_d, w^*)|^2 |d|^{2\Re(w^*-w)} \Bigg)^{\frac{1}{2}} \left(\frac{Y}{b^2}\right)^{\frac{1}{2}} \lambda^{-1} \\ &\qquad \ll |\tilde{\Gamma}(w_0)|^{-1} A^{-1-2\alpha-2\Re(w^*-w)} Y^{1+\alpha+\varepsilon+\Re(w^*-w)} \end{aligned}$$

by Theorem 1.2. To summarize, we have proved that

$$\begin{aligned} &\sum_{b\geq 1, (b,4N)=1} \sum_{r|b, r>A} \mu(r) \sum_{d\in D_{a\bar{b}^2}^{\pm}} \frac{\mu^2(|d|)}{(w-w_d)\tilde{\Gamma}(w)} S(f^*, \chi_{db^2}, w^*, A_{db^2}) F\left(\frac{|d|b^2}{Y}\right) A_{db^2}^{w^*-w} \\ &\qquad \ll A^{-3+\alpha} Y^{(k+3)/2-\Re w+\varepsilon} \\ &\qquad\quad + |\tilde{\Gamma}(w_0)|^{-1} A^{-1-2\alpha-2\Re(w^*-w)} Y^{1+\alpha+\varepsilon+\Re(w^*-w)}. \end{aligned}$$

Now, integrating over the circle $|w-w_0| = 2\lambda$ gives

$$R_{i,1}^{\pm} \ll A^{-3+\alpha} Y^{(k+3)/2-\Re w_0+\varepsilon} + |\tilde{\Gamma}(w_0)|^{-1} A^{-1-2\alpha-2\Re(w_0^*-w_0)} Y^{1+\varepsilon+\alpha+\Re(w_0^*-w_0)}.$$

Estimation of $R_{1,2}^{(2)}$ in (2.2)

To estimate $R_{i,2}^{\pm}$ we will sum in (2.2) over m first. Let us write

$$n = k_1 l^2 l_0 m$$

where k_1 and l are as before and $(m,l)=1$, $p|l_0 \Rightarrow p|l$, $\mu^2(l_0)=1$. We rewrite $R^{\pm}_{i.2}$ as

$$\frac{1}{2\pi i}\int_{|w-w_0|=2\lambda}\frac{\omega_a^*}{\tilde\Gamma(w)}\sum_{r\le A,(r,4N)=1}\mu(r)\sum_{k_1,l}\Big(\sum_{q|l}\mu(q)\Big)a^*(k_1)k_1^{-w^*}\Big(\frac{a}{k_1}\Big)$$
$$\times\sum_{d,dr^2q\in D_a^{\pm}}F\Big(\frac{|d|r^2q}{Y}\Big)A_{dr^2q}^{w^*-w}\sum_{|\rho|\ge 1}\sum_{l_0}a^*(l^2l_0)(l^2l_0)^{-w^*}\bar\varepsilon_{l_0}l_0^{-\frac12}\chi_{N\rho q}(l_0)$$
$$\times\sum_{m\ge\frac{|\rho|}{\Delta l_0}}a^*(m)m^{-w^*-\frac12}\mu^2(m)\bar\varepsilon_m\chi_{N\rho q}(m)e\Big(\frac{\bar4\bar N\rho d}{ml_0}\Big)W\Big(w^*,\frac{k_1l^2l_0m}{A_{dr^2q}}\Big)\frac{dw}{w-w_d}.$$

For $T\gg|\rho|/\Delta l_0$ set

$$A(T)=\sum_{m\ll T,(m,4Nl)=1}\mu^2(m)a^*(m)\bar\varepsilon_m\chi_{N\rho q}(m)e\Big(-\frac{\bar m\bar l_0\rho d}{4N}\Big)$$

where $\bar m$ and $\bar l_0$ are the multiplicative inverses of m and l_0 modulo $4N$. By Lemma 1

$$A(T)\ \ll\ \mathbf{d}(l)(|\rho|q)^{\frac12}T^{k/2+\varepsilon}.$$

By partial summation

$$A_1(T)\overset{\text{def}}{=}\sum_{|\rho|/\Delta l_0\ll m\ll T,(m,4Nl)=1}\mu^2(m)a^*(m)\bar\varepsilon_m\chi_{N\rho q}(m)e\Big(-\frac{\bar m\bar l_0\rho d}{4N}\Big)e\Big(\frac{\rho d}{4Nl_0m}\Big)$$
$$\ll\mathbf{d}(l)(|\rho|q)^{\frac12}T^{k/2+\varepsilon}(1+|d|\Delta)\ \ll\ BT^{k/2+\varepsilon}$$

where $B=\mathbf{d}(l)(|\rho|q)^{\frac12}Y^{\varepsilon}$. Here we used $|d|\ll\frac{Y}{r^2q}$ and $\Delta\le r^2qY^{\varepsilon-1}$. Let us set $c=k_1l^2l_0/A_{dr^2q}$. Then

$$\sum_{m\ge\frac{|\rho|}{\Delta l_0},(m,4Nl)=1}\mu^2(m)a^*(m)\chi_{N\rho q}(m)\bar\varepsilon_m e\Big(\frac{\bar4\bar N\rho d}{ml_0}\Big)m^{-w^*-\frac12}W(w^*,cm)$$
$$=-A_1\Big(\frac{|\rho|}{\Delta l_0}\Big)\Big(\frac{|\rho|}{\Delta l_0}\Big)^{-w^*-\frac12}W\Big(w^*,\frac{c|\rho|}{\Delta l_0}\Big)$$
$$-\int_{|\rho|/\Delta l_0}^{\infty}A_1(t)t^{-w^*-\frac12}W'(w^*,ct)\,d(tc)\ +\ g(t)W(w^*,ct)]_{t=|\rho|/\Delta l_0}^{\infty}$$
$$-\int_{|\rho|/\Delta l_0}^{\infty}g(t)W'(w^*,ct)\,d(tc)$$

by partial summation and integration by parts where $g(t)=\int_t^{\infty}A_1(u)u^{-w^*-\frac32}(-w^*-\frac12)du$. Notice that the integral defining $g(t)$ converges and is bounded by $Bt^{k/2+\varepsilon-1/2-\Re}$

We see that in order to estimate the sum over m we need to estimate

a) $$B\left(\frac{|\rho|}{\Delta l_0}\right)^{k/2+\varepsilon-\Re w^*-1/2}\left|W\left(w^*,\frac{c|\rho|}{\Delta l_0}\right)\right|$$

and

b) $$B\int_{|\rho|/\Delta l_0}^{\infty} t^{(k-1)/2+\varepsilon-\Re w^*}|W'(w^*,ct)|c\,dt.$$

We estimate the contribution from (b) - the contribution from (a) is exactly the same. We notice that by (2.3) it is enough to estimate

$$Bc^{\Re w^*-(k-1)/2-\varepsilon}\left(\frac{c|\rho|}{\Delta l_0}\right)^{(k-3)/2+\varepsilon-\Re\nu}\exp\left(-\frac{c|\rho|}{2\Delta l_0}\right).$$

Summation over $|\rho|$ gives

$$\sum_{|\rho|\geq 1}|\rho|^{-1-\Re\nu+k/2+\varepsilon}\exp\left(-\frac{c|\rho|}{2\Delta l_0}\right)\ll\left(\frac{\Delta l_0}{c}\right)^{-\Re\nu+k/2+\varepsilon}$$

so that after multiplying by $A_{dr^2q}^{w^*-w}/w-w_d$ we see that the contribution is

$$|A_{dr^2q}^{w^*-w}|\mathbf{d}(l)q^{\frac{1}{2}}Y^{\varepsilon}\Delta^{\frac{3}{2}}l_0^{\frac{3}{2}}c^{-\frac{k}{2}+\Re w^*-\varepsilon-1}\frac{1}{|w-w_d|}.$$

The sum over $|d|$ is

$$\ll\sum_{|d|\ll Y/r^2q}|d|^{k/2+1+\varepsilon-\Re w}\frac{1}{|w-w_d|}\ll\left(\frac{Y}{r^2q}\right)^{k/2+2+\varepsilon-\Re w}$$

so that the total contribution is

$$Y^{1/2+k/2-\Re w+\varepsilon}\sum_{r\leq A}\sum_{k_1,l,l_0}\sum_{q|l}rq\mathbf{d}(l)\frac{|a(k_1l^2l_0)|}{l_0^{1/2}(k_1l^2l_0)^{k/2+\varepsilon+1}}\quad.$$

Hence, using (1.1) and summing over q, l_0, l, k_1 and $r\leq A$ yields

$$\ll A^2Y^{1/2+k/2-\Re w+\varepsilon}.$$

Integration over the circle $|w-w_0|=2\lambda$ finally shows that

$$R_{i,2}^{\pm}\ll A^2Y^{1/2+k/2-\Re w_0+\varepsilon}.$$

Estimation of $R_{i,3}^{\pm}$ in (2.2)

We will start by summing over d in (2.2). We set $c_f = A_d|d|^{-1}$ and rewrite $R^{\pm}_{i,3}$ as

$$\frac{\omega_a^*}{2\pi i}\int_{|w-w_0|=2\lambda} \sum_{\substack{r\le A\\(r,4N)=1}} \mu(r) \sum_{\substack{n=k_1 l^2 m\\(n,r)=1}} \frac{a^*(n)}{n^{w^*}} \sum_{q|l}\mu(q) \sum_{\Delta m<|\rho|<\frac{m}{2}} \frac{\bar{\varepsilon}_m}{m^{1/2}}\chi_{N\rho q}(m)$$
$$\cdot \left(\frac{a}{k_1}\right) \sum_{d,dr^2q\in D_a^{\pm}} F\left(\frac{|d|r^2q}{Y}\right) W\left(w^*, \frac{n}{c_f|d|r^2q}\right)(r^2qc_f|d|)^{w^*-w}$$
$$e\left(\frac{\bar{4}\bar{N}\rho d}{m}\right)\frac{dw}{(w-w_d)\tilde{\Gamma}(w)}.$$

We want to estimate the sum

$$(*) \qquad \sum_{d,dr^2q\in D_a^{\pm}} h(\pm d) e\left(\frac{\bar{4}\bar{N}\rho d}{m}\right)\frac{1}{w-w_d}$$

where

$$h(x) = F\left(\frac{r^2q}{Y}x\right) W\left(w^*, \frac{n}{c_f r^2 q x}\right)(xc_f r^2 q)^{w^*-w}.$$

Observe that the presence of F restricts the range of summation to

$$c_1\frac{Y}{r^2q} < |d| < c_2\frac{Y}{r^2q}$$

if $\mathrm{Supp}(F)\subset(c_1,c_2)$. For any $T\le c_2Y/r^2q$ we want to estimate

$$(**) \qquad \sum_{d,dr^2q\in D_a^{\pm},|d|\le T} h(\pm d) e\left(\frac{\overline{4N}\rho d}{m}\right).$$

To do so it is sufficient to estimate

$$(***) \qquad \sum_{d,dr^2q\in D_a^{\pm}} h(\pm d) g(\pm d) e\left(\frac{\overline{4N}\rho d}{m}\right)$$

where g is smooth, compactly supported function in $[M,2M]$ with

$$g^{(i)}(x) \ll M^{-i}.$$

Here we take $M = c_3Y/r^2q$ for some constant c_3. By Poisson summation formula $(***)$ is equal to

$$\frac{1}{4N}\sum_u e\left(\frac{ua\bar{r}^2\bar{q}}{4N}\right)(\widehat{hg})\left(\frac{u}{4N}-\frac{\overline{4N}\rho}{m}\right)$$

where $(\widehat{hg})$ denotes the Fourier transform of $h(\pm x)g(\pm x)$. We assume for a moment that we can find two positive constants X_1 and X_2, such that

$$(hg)^{(j)}(x) \ll \frac{X_1}{(x+X_2)^j}$$

for some $j \geq 2$ (the constant in $\ll$ depending only on F, W, and j). Then integration by parts shows that

$$(\widehat{hg})(t) \ll X_1 X_2^{1-j}|t|^{-j}$$

so that writing $4N\overline{4N} = 1 + em$ for some integer e we see that

$$(\widehat{hg})\left(\frac{u}{4N} - \frac{\overline{4N}\rho}{m}\right) \ll \frac{X_1 X_2^{1-j}}{|u - \rho/m - e\rho|^j}.$$

Summation over u gives then

$$(***) \ll_j X_1 X_2^{1-j}\left(\frac{|\rho|}{m}\right)^{-j}.$$

To estimate $(hg)^{(j)}(x)$ we must estimate

$$F^{(i_1)}\left(\frac{r^2q}{Y}x\right)\left(\frac{r^2q}{Y}\right)^{i_1} g^{(i_2)}(x) W^{(i_3)}\left(w^*, \frac{n}{c_f r^2 qx}\right)\left(\frac{n}{r^2q}\right)^{i_3}$$
$$x^{-2i_3 - i_4 + \Re(w^*-w) - i_5}(r^2q)^{\Re(w^*-w)}$$
$$\ll \frac{1}{x^j}(r^2qx)^{\Re(w^*-w)}\left|W^{(i_3)}\left(w^*, \frac{n}{c_f r^2 qx}\right)\right|\left(\frac{n}{r^2q}\right)^{i_3} x^{-i_3}$$

using $\sum i_{(\cdot)} = j$ and $x \sim Y/r^2q$. By (2.3), the fact that $x \sim Y/r^2q$ and assumption about g we estimate

$$(hg)^{(j)}(x) \ll_{j,c} \exp\left(-c\frac{n}{Y}\right) Y^{\Re(w^*-w)}\left(\frac{n}{Y}\right)^{\Re(w^*-\nu)}\frac{1}{x^j} \ll \frac{X_1}{(x+X_2)^j}$$

where

$$X_1 = \left(\frac{n}{Y}\right)^{\Re(w^*-\nu)} Y^{\Re(w^*-w)}\exp\left(-c\frac{n}{Y}\right),\ X_2 = \frac{Y}{r^2q}, \qquad c - \text{positive constant.}$$

Hence

$$(***) \ll X_1 X_2^{1-j}\left(\frac{|\rho|}{m}\right)^{-j} \ll \frac{Y^{1-\varepsilon j}}{r^2q}\exp\left(-c\frac{n}{Y}\right) n^{\Re(w^*-\nu)} Y^{\Re(\nu-w)}$$

since $\Delta = r^2qY^{\varepsilon-1} < |\rho|/m$, and we obtain the same estimation for $(**)$ (multiplied only by the factor $\log Y$ say). We return to the estimation of $(*)$. Let $g_1(x)$ be a smooth function such that $g_1(|d|) = 1/w - w_d$ and $g_1'(x) \ll \lambda^{-1}$. By partial summation, using the estimation of $(**)$ we deduce that

$$(*) \ll \left(\frac{Y}{r^2q}+1\right)\lambda^{-1}\frac{Y^{1-\varepsilon j}}{r^2q}\exp\left(-c\frac{n}{Y}\right)n^{\Re(w^*-\nu)}Y^{\Re(\nu-w)}$$

for any $j \geq 2$. Summing over $r, |\rho| \ll m$, and q gives

$$\sum n^{\frac{k}{2}+\alpha}\exp\left(-c\frac{n}{Y}\right)Y^{2-\varepsilon j-\Re w+\Re\nu} \ll 1$$

by choosing j large enough. Integrating over the circle $|w-w_0| = 2\lambda$ we conclude that

$$R_{i,3}^{\pm} \ll 1.$$

5. Main term

We now consider the sums $MT_i^{\pm}$. As $\rho = 0$ in these sums, only the terms with $m = 1$ in (2.2) give a nontrivial contribution. Thus we rewrite $MT_i^{\pm}$ as

$$\begin{aligned}
\omega_a^*\sum_{k_1} a^*(k_1)\left(\frac{a}{k_1}\right)k_1^{-w_0^*} &\sum_{l\geq 1,(l,4N)=1} a^*(l^2)l^{-2w_0^*}\sum_{q|l}\mu(q)\sum_{r\leq A,(r,4Nl)=1}\mu(r)\\
&\sum_{d,dr^2q\in D_a^{\pm}}\frac{1}{\tilde\Gamma(w_0)}F\left(\frac{|d|r^2q}{Y}\right)W\left(w_0^*,\frac{k_1l^2}{c_f|d|r^2q}\right)(c_f|d|r^2q)^{w_0^*-w_0}\\
+\mathbf{O}\Bigg(&\sum_{k_1}|a(k_1)|\sum_{l\geq 1}|a(l^2)|\sum_{q|l}|\mu(q)|\sum_{r\leq A}|\mu(r)|\\
&\sum_{d,dr^2q\in D_a^{\pm}}\left|F\left(\frac{|d|r^2q}{Y}\right)\right|(k_1l^2)^{-\Re w_0^*}(c_f|d|r^2q)^{\Re(w_0^*-w_0)}\frac{1}{|\tilde\Gamma(w_0)|}\\
&\times\Bigg|\Bigg[(k_1l^2)^{-w_d^*+w_0^*}(c_f|d|r^2q)^{w_d^*-w_0^*+w_0-w_d}\frac{\tilde\Gamma(w_0)}{\tilde\Gamma(w_d)}W\left(w_d^*,\frac{k_1l^2}{c_f|d|r^2q}\right)\\
&\qquad - W\left(w_0^*,\frac{k_1l^2}{c_f|d|r^2q}\right)\Bigg]\Bigg|\Bigg).
\end{aligned}$$

We begin by estimating the above error term. The expression in the square brackets is bounded by

$$\ll \left(\frac{k_1l^2}{|d|r^2q}\right)^{\Re(w_0^*-\nu)}\exp\left(-c_1\frac{k_1l^2}{|d|r^2q}\right)\lambda(k_1l^2)^{\lambda}(|d|r^2q)^{2\lambda^*}\max\{\log Y,\log k_1l^2\}$$

by (2.3) and the fact that $|d|r^2q \sim Y$. Here $\lambda^* = 0$ if $i = 1$, $\lambda^* = \lambda$ if $i = 2$ and c_1 is some positive constant. Summation over d contributes

$$\sum_{|d| \ll Y/r^2q} |d|^{-\Re(w_0-\nu)+2\lambda^*} \ll \left(\frac{Y}{r^2q}\right)^{-\Re w_0+\Re\nu+1+2\lambda^*}$$

so that the sum over d above is

$$\ll \frac{Y^{1-\Re w_0+\Re\nu}}{r^2q}\lambda(k_1l^2)^{-\Re\nu}\max\{\log Y, \log k_1l^2\}\exp\left(-c_2\frac{k_1l^2}{Y}\right)$$

for some positive constant c_2. In order to sum over l we will use the following estimate

$$(*) \qquad \sum_{l\leq x}|a(l^2)| \ll x^k.$$

Indeed, we notice first that

$$(**) \qquad \sum_{l\leq x}|a(l^2)|^2$$

are the partial sums of the coefficients of the (not normalized) Dirichlet series attached to the Rankin-Selberg convolution (on GL_3) of $Sym^2(f) \times Sym^2(\bar{f})$. The normalized Rankin-Selberg L-function has a meromorphic continuation to the whole s-plane with simple poles at $s = 1, 0$, [5]. Hence it follows that $(**)$ is bounded by x^{2k-1}. We use Cauchy-Schwarz inequality to deduce $(*)$. Using $(*)$ and summing over r, q, l and k_1 (breaking the sum over k_1l^2 at Y) we find that the error term is

$$\ll \lambda Y^{1+\frac{k}{2}-\Re w_0}\log Y\log\log Y.$$

We return to the evaluation of the main term. Summation over d gives

$$\sum_{d,dr^2q\in D_a^\pm} = \frac{Y^{1+w_0^*-w_0}}{4Nr^2q}\frac{c_f^{w_0^*-w_0}}{\tilde{\Gamma}(w_0)}\int F(t)W\left(w_0^*, \frac{k_1l^2}{c_fYt}\right)t^{w_0^*-w_0}\,dt$$
$$+\mathbf{O}\left(Y^{\Re(w_0^*-w_0)}\int\left|\left(F(t)W\left(w_0^*, \frac{k_1l^2}{c_fYt}\right)t^{\Re(w_0^*-w_0)}\right)'\right|dt\right).$$

We use $(*)$, (2.3) and partial summation to find that the above error term is

$$\ll AY^{k/2-\Re w_0+\varepsilon}.$$

We use

$$\sum_{r\leq A,(r,4Nl)=1}\mu(r)r^{-2} = \zeta_{(4N)}^{-1}(2)\prod_{p|l}\left(1-\frac{1}{p^2}\right)^{-1} + \mathbf{O}(A^{-1})$$

to rewrite the main term above as

$$\omega_a^* \frac{c_f^{w_0^*-w_0}}{\tilde{\Gamma}(w_0)} \frac{Y^{1+w_0^*-w_0}}{4N\zeta_{(4N)}(2)}$$
$$\times \int F(t) f_{4N}^*(w_0^*) \sum_{(l,4N)=1} \prod_{p|l} (1+\frac{1}{p})^{-1} a^*(l^2) l^{-2w_0^*} W\left(w_0^*, \frac{k_1 l^2}{c_f Y t}\right) dt$$
$$+ \mathbf{O}\left(A^{-1} Y^{1+\Re(w_0^*-w_0)} \sum_{n=k_1 l^2} |a(n)| n^{-\Re w_0^*} \sum_{q|l} \frac{|\mu(q)|}{q} \right.$$
$$\left. \int \left| F(t) W\left(w_0^*, \frac{n}{c_f Y t}\right) t^{\Re(w_0^*-w_0)} \right| dt \right)$$

where

$$f_{4N}^*(s) = \sum_{k_1, p|k_1 \Rightarrow p|4N} a^*(k_1) \left(\frac{a}{k_1}\right) k_1^{-s}.$$

As before, using $(*)$, (2.3) and partial summation we find that the error term above is

$$\ll A^{-1} Y^{1+k/2-\Re w_0+\varepsilon}.$$

Consider the functions

$$B^*(s) \stackrel{\text{def}}{=} \prod_{p \nmid 4N} \left(1 + \frac{p}{p+1}(a^*(p^2)p^{-s} + a^*(p^4)p^{-2s} + \ldots.)\right)$$

$$A_p^*(s) \stackrel{\text{def}}{=} (1-\alpha^*(p)^2 p^{-s})^{-1}(1-\beta^*(p)^2 p^{-s})^{-1}(1+\omega^*(p)p^{k-1-s}) - 1$$

$$L_{(4N)}^*(s) \stackrel{\text{def}}{=} \prod_{p \nmid 4N} (1 + A_p^*(s))$$

where $\omega^* = \omega$ or $\bar{\omega}$ depending whether $f^* = f$ or $\bar{f}$. Then

$$B^*(s) = P^*(s) L_{(4N)}^*(s)$$

where

$$P^*(s) = \prod_{p \nmid 4N} \left(1 + \frac{1}{p+1}\left(\frac{1}{1+A_p^*(s)} - 1\right)\right).$$

The function $L_{(4N)}^*(s)$ is related to the symmetric square L-function of f^* by

$$L_{(4N)}(Sym^2(f^*), s) = L_{(2)}(\omega^{*2}, 2s-2k+2) L_{(4N)}^*(s).$$

It is known that $L(Sym^2(f), s)$ is entire and satisfies an appropriate functional equation [13]. Now, we see that $P^*(s)$ converges absolutely for $\Re s > k-1+2\alpha$

and does not vanish for $\Re s > k - \frac{3}{5}$ by (1.1). The sum of $f_{4N}^*(s)$ converges absolutely for $\Re s > (k-1)/2 + \alpha$ and does not vanish there. Now replacing W by its integral, we see that the main term is

$$\omega_a^* Y^{1+w_0^*-w_0} \frac{c_f^{w_0^*-w_0}}{\tilde{\Gamma}(w_0)} \frac{1}{4N\zeta_{(4N)}(2)} \int F(t) \Bigg(\frac{1}{2\pi i} \int_\gamma f_{4N}^*(w_0^*+s) \times \frac{L_{(4N)}(Sym^2 f^*, 2w_0^*+2s)}{L_{(2)}(\omega^{*2}, 4s+4w_0^*-2k+2)} P^*(2w_0^*+2s)\tilde{\Gamma}(w_0^*+s)(c_f Y t)^s \frac{ds}{s} \Bigg) dt.$$

Here $\gamma \gg 0$. Moving the line of integration to the line $\Re s = -1/4 + k/2 - \Re w_0^*$ we get the residue from a possible simple pole at $s = 0$ (which gives the main term) and an error term

$$\ll Y^{3/4+k/2-\Re w_0}.$$

Here we used that $L(Sym^2(f^*), 2w_0^*+2s)$ has only polynomial growth for $\Re s \geq -1/4 + k/2 - \Re w_0^*$ by Phragmén-Lindelöf principle and functional equation. To summarize, we have shown that

$$\begin{aligned} &\sum_{d.d\in D_a^\pm} \mu^2(|d|) L(f, \chi_d, w_d) F\left(\frac{|d|}{Y}\right) \\ &= Y \cdot \big(\frac{1}{4N\zeta_{(4N)}(2)} P(2w_0) f_{4N}(w_0) \frac{L_{(4N)}(Sym^2 f, 2w_0)}{L_{(2)}(\omega^2, 4w_0-2k+2)} \int F(t)dt \\ &+ Y^{1+k-2w_0} \cdot \operatorname{sgn}(d)\omega_1\left(\frac{a}{-N}\right)\omega(a) \frac{1}{4N\zeta_{(4N)}(2)} \frac{\tilde{\Gamma}(k-w_0)}{\tilde{\Gamma}(w_0)} c_f^{k-2w_0} \\ &\times f_{4N}^*(k-w_0) P^*(2k-2w_0) \frac{L_{(4N)}(Sym^2 \bar{f}, 2k-2w_0)}{L_{(2)}(\bar{\omega}^2, 2k-4w_0+2)} \int F(t)dt \\ &+ \mathbf{O}(Y^{3/4+k/2-\Re w_0} + Y^\varepsilon (A^2 Y^{(k+1)/2-\Re w_0} + A^{-1} Y^{1+k/2-\Re w_0} \\ &+ A^{-1-2\alpha} Y^{1+\alpha} + A^{-3+\alpha} Y^{(3+k)/2-\Re w_0}) \\ &+ \lambda Y^{1+k/2-\Re w_0} \log Y \log\log Y) \end{aligned}$$

where the second term above is present only if $\Re w_0 < k/2 + 1/4$. Also f^* and P^* in the second term correspond to $\bar{f}$. We take $A = Y^{\frac{9}{37}}$ to write the error as

$$\mathbf{O}(Y^{\frac{731}{740}+\varepsilon} + \lambda Y^{1+k/2-\Re w_0} \log Y \log\log Y).$$

Summation over $d \in D_a$ eliminates the second term so the Theorem 1.1 follows.

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