Nielsen numbers of selfmaps of flat 3-manifolds*

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Abstract

We compute the Nielsen number $N(f)$ of a self homeomorphism f of a three dimensional flat manifold. Furthermore, we determine the possible values of $N(f)$ when f is an arbitrary self-map.

1 Introduction

In the 1920s, J. Nielsen conjectured that for any homeomorphism $f : M \to M$ of a closed surface M there exists a map g, isotopic to f, so that g has exactly $N(f) = N(g)$ fixed points. Here, $N(f)$ is now known as the Nielsen number of f. This homotopy invariant is often a sharp lower bound for the minimal number of fixed points in the homotopy class of f (see e.g. [1, 12]). This conjecture was proven by Jiang [13], Ivanov [11] (for self-homotopy equivalences), and Jiang-Guo [14] using the Nielsen-Thurston classification of surface homeomorphisms. The Nielsen conjecture has been proven for homeomorphisms of manifolds of dimension greater than or equal to 5 [17], and for a large class of 3-manifolds including (after Thurston's geometrization theorem) all irreducible 3-manifolds [16]. Meanwhile, Nielsen numbers of surface maps have been studied using Fox Calculus and other methods of combinatorial group theory. In

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particular, M. Kelly [18] outlined a method of calculating $N(f)$ for surface homeomorphisms using the work of M. Bestvina and M. Handel based on the theory of train tracks. He also gave algorithms for $N(f)$ for homeomorphisms of certain geometric 3-manifolds [19], including the Seifert manifolds.

The purpose of this work is to make explicit calculation of the Nielsen number of a self homeomorphism of a flat 3-manifold. In particular, for a flat 3-manifold X, we compute

$$\mathrm{NSH}(X) = \{N(h) \mid h \in Home(X)\}.$$

Using appropriate group presentations for the fundamental groups of the ten flat 3-manifolds, we further analyze the possible values of $N(f)$ when f is an arbitrary selfmap. In section 2, we recall the ten 3-dimensional flat manifolds by listing their fundamental groups and their presentations. In section 3, we compute the Nielsen number of a self homeomorphism of the first five flat manifolds making use of the automorphisms of the 2-dimensional crystallographic group on which the fundamental group of the flat manifold projects. In section 4, we turn our attention to the remaining cases. In sections 5 and 6, we compute $N(f)$ for arbitrary selfmaps f. For cases 2 - 5, 9,10, we use a particular fully invariant subgroup corresponding to the fundamental group of a torus or a Klein bottle that allows us to compute $N(f)$ using fiberwise techniques. We complete the computation of $N(f)$ for the remaining cases using different techniques. In the last section, we determine the flat manifolds for which the Jiang-type condition holds.

2 Flat 3-manifolds and Nielsen numbers

Every isometry of the Euclidean space $\mathbb{R}^n$ is a rotation followed by a translation. More precisely, the group of isometries $\mathrm{Isom}(\mathbb{R}^n)$ is given by the semi-direct product $\mathbb{R}^n \rtimes O(n)$. A subgroup $\pi \subset \mathrm{Isom}(\mathbb{R}^n)$ is a *crystallographic* group on $\mathbb{R}^n$ if π is a discrete uniform subgroup. Moreover, π is called a *Bieberbach* group if in addition it is torsion free. Given a Bieberbach group π, the resulting quotient manifold $\mathbb{R}^n/\pi$ is called a *flat* n-manifold. The group π has a normal maximal abelian subgroup Γ of finite index and Γ has rank n. The quotient $\Phi = \pi/\Gamma$ is called the *holonomy* group. For more details on flat manifolds, see [3] or [22, Ch.3].

There are a total of ten flat 3-manifolds whose fundamental groups are listed below, where the first six are orientable and the remaining four are non-orientable. The following presentations can be found in [22, pp.117-121].

1. $\langle \alpha_1, \alpha_2, \alpha_3 | \ \alpha_i\alpha_j = \alpha_j\alpha_i, 1 \le i, j \le 3\rangle$ with holonomy $\Phi = \{1\}$.

2. $\langle \alpha_1, \alpha_2, \alpha_3, t | \ \alpha_1 = t^2, t\alpha_2 t^{-1} = \alpha_2^{-1}, t\alpha_3 t^{-1} = \alpha_3^{-1}, \alpha_i\alpha_j = \alpha_j\alpha_i, 1 \le i, j \le 3\rangle$ with holonomy $\Phi = \mathbb{Z}_2$.

3. $\langle \alpha_1, \alpha_2, \alpha_3, t | \ \alpha_1 = t^3, t\alpha_2 t^{-1} = \alpha_3, t\alpha_3 t^{-1} = \alpha_2^{-1}\alpha_3^{-1}, \alpha_i\alpha_j = \alpha_j\alpha_i, 1 \le i, j \le 3\rangle$ with holonomy $\Phi = \mathbb{Z}_3$.

4. $\langle \alpha_1, \alpha_2, \alpha_3, t | \ \alpha_1 = t^4, t\alpha_2 t^{-1} = \alpha_3, t\alpha_3 t^{-1} = \alpha_2^{-1}, \alpha_i\alpha_j = \alpha_j\alpha_i, 1 \le i, j \le 3\rangle$ with holonomy $\Phi = \mathbb{Z}_4$.

5. $\langle \alpha_1, \alpha_2, \alpha_3, t | \ \alpha_1 = t^6, t\alpha_2 t^{-1} = \alpha_3, t\alpha_3 t^{-1} = \alpha_2^{-1}\alpha_3, \alpha_i\alpha_j = \alpha_j\alpha_i, 1 \le i,j \le 3\rangle$ with holonomy $\Phi = \mathbb{Z}_6$.

6. $\langle \alpha_1, \alpha_2, \alpha_3, t_1, t_2, t_3 | \ \alpha_1\alpha_3 = t_3t_2t_1, \alpha_i = t_i^2, t_i\alpha_j t_i^{-1} = \alpha_j^{-1}$ for $i \neq j, \alpha_i\alpha_j = \alpha_j\alpha_i, 1 \le i,j \le 3\rangle$ with holonomy $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$.

7′. $\langle t_1, \alpha_1, \alpha_2, \alpha_3 | \ t_1^2 = \alpha_1, t_1\alpha_2 t_1^{-1} = \alpha_2, t_1\alpha_3 t_1^{-1} = \alpha_3^{-1}, \alpha_i\alpha_j = \alpha_j\alpha_i, 1 \le i,j \le 3\rangle$ with holonomy $\Phi = \mathbb{Z}_2$. The isomorphism $t_1 \mapsto \beta, \alpha_2 \mapsto t, \alpha_3 \mapsto \alpha$ gives the following alternate presentation

7. $\pi_1(K) \times \mathbb{Z} = \langle \alpha, \beta | \beta\alpha\beta^{-1} = \alpha^{-1}\rangle \times \langle t\rangle$ where K is the Klein bottle, with holonomy $\Phi = \mathbb{Z}_2$.

8′. $\langle t_1, \alpha_1, \alpha_2, \alpha_3 | \ t_1^2 = \alpha_1, t_1\alpha_2 t_1^{-1} = \alpha_2, t_1\alpha_3 t_1^{-1} = \alpha_1\alpha_2\alpha_3^{-1}, \alpha_i\alpha_j = \alpha_j\alpha_i, 1 \le i,j \le 3\rangle$ with holonomy $\Phi = \mathbb{Z}_2$. The isomorphism $\alpha_2 \mapsto (\alpha\beta)^2, \alpha_3 \mapsto (\alpha\beta)^2 t, t_1 \mapsto (\alpha\beta)t$ gives the following alternate presentation

8. $\langle \alpha, \beta, t | \beta\alpha\beta^{-1} = \alpha^{-1}, t\alpha t^{-1} = \alpha, t\beta t^{-1} = \alpha\beta\rangle$ with holonomy $\Phi = \mathbb{Z}_2$.

9′. $\langle t_1, t_2, \alpha_1, \alpha_2, \alpha_3 | \ t_1^2 = \alpha_1, t_2^2 = \alpha_2, t_2t_1t_2^{-1} = \alpha_2 t_1, t_1\alpha_2 t_1^{-1} = \alpha_2^{-1}, t_1\alpha_3 t_1^{-1} = \alpha_3^{-1}, t_2\alpha_1 t_2^{-1} = \alpha_1, t_2\alpha_3 t_2^{-1} = \alpha_3^{-1}, \alpha_i\alpha_j = \alpha_j\alpha_i, 1 \le i,j \le 3\rangle$ with holonomy $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$. The isomorphism $\alpha_3 \mapsto \alpha, t_2 \mapsto \beta, t_1t_2 \mapsto t$ gives the following alternate presentation

9. $\langle \alpha, \beta, t | \beta\alpha\beta^{-1} = \alpha^{-1}, t\alpha t^{-1} = \alpha, t\beta t^{-1} = \beta^{-1}\rangle$ with holonomy $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$.

10′. $\langle t_1, t_2, \alpha_1, \alpha_2, \alpha_3 | \ t_1^2 = \alpha_1, t_2^2 = \alpha_2, t_2t_1t_2^{-1} = \alpha_2\alpha_3 t_1, t_1\alpha_2 t_1^{-1} = \alpha_2^{-1}, t_1\alpha_3 t_1^{-1} = \alpha_3^{-1}, t_2\alpha_1 t_2^{-1} = \alpha_1, t_2\alpha_3 t_2^{-1} = \alpha_3^{-1}, \alpha_i\alpha_j = \alpha_j\alpha_i, 1 \le i,j \le 3\rangle$ with holonomy $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$. The isomorphism $\alpha_3 \mapsto \alpha^{-1}, t_2 \mapsto \beta, t_1t_2 \mapsto t$ gives the following alternate presentation

10. $\langle \alpha, \beta, t | \beta\alpha\beta^{-1} = \alpha^{-1}, t\alpha t^{-1} = \alpha, t\beta t^{-1} = \alpha\beta^{-1}\rangle$ with holonomy $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$.

All of these 10 Bieberbach groups possess natural projections onto some 2-dimensional crystallographic groups. Cases 1 and 7 are straightforward as they project onto $G_1 = \mathbb{Z} \times \mathbb{Z}$ (torus) and onto $G_1^3 = \pi_1(K)$ (Klein bottle) respectively.

We shall use the notation of the 2-dimensional crystallographic groups as given by R. Lyndon in [21].

Case 2: $p : G \to G_2$ where

$$G_2 = \langle \alpha, \beta, \tau | \alpha\beta = \beta\alpha, \alpha^\tau = \alpha^{-1}, \beta^\tau = \beta^{-1}, \tau^2 = 1\rangle.$$

and p is given by $\alpha_1 \mapsto 1, \alpha_2 \mapsto \alpha, \alpha_3 \mapsto \beta, t \mapsto \tau$.

Case 3: $p : G \to G_3$ where

$$G_3 = \langle \alpha, \beta, \tau | \alpha\beta = \beta\alpha, \alpha^\tau = \alpha^{-1}\beta, \beta^\tau = \alpha^{-1}, \tau^3 = 1\rangle.$$

and p is given by $\alpha_1 \mapsto 1, \alpha_2 \mapsto \beta^{-1}, \alpha_3 \mapsto \alpha, t \mapsto \tau$.

Case 4: $p : G \to G_4$ where

$$G_4 = \langle \alpha, \beta, \tau | \alpha\beta = \beta\alpha, \alpha^\tau = \beta, \beta^\tau = \alpha^{-1}, \tau^4 = 1 \rangle.$$

and p is given by $\alpha_1 \mapsto 1, \alpha_2 \mapsto \alpha, \alpha_3 \mapsto \beta, t \mapsto \tau$.

Case 5: $p : G \to G_6$ where

$$G_6 = \langle \alpha, \beta, \tau | \alpha\beta = \beta\alpha, \alpha^\tau = \beta, \beta^\tau = \alpha^{-1}\beta, \tau^6 = 1 \rangle.$$

and p is given by $\alpha_1 \mapsto 1, \alpha_2 \mapsto \alpha, \alpha_3 \mapsto \beta, t \mapsto \tau$.

Case 6: $p : G \to G_2^4$ where

$$G_2^4 = \langle \alpha, \beta, \tau | \beta\alpha\beta^{-1} = \alpha^{-1}, \alpha^\tau = \alpha^{-1}, \beta^\tau = \alpha\beta^{-1}, \tau^2 = 1 \rangle$$

and p is given by $t_1 \mapsto \beta^{-1}, t_2 \mapsto \tau, t_3 \mapsto \tau\beta, \alpha_1 \mapsto \beta^{-2}, \alpha_2 \mapsto 1, \alpha_3 \mapsto \alpha$.

Case 8: $p : G \to G_1 = \mathbb{Z} \times \mathbb{Z} = \langle \tau \rangle \times \langle b \rangle$ where p is given by $\alpha \mapsto 1$, $\beta \mapsto b, t \mapsto \tau$.

Case 9: $p : G \to G_2^2$ where

$$G_2^2 = \langle \alpha, \beta, \tau | \beta\alpha\beta^{-1} = \alpha^{-1}, \alpha^\tau = \alpha, \beta^\tau = \beta^{-1}, \tau^2 = 1 \rangle$$

and p is given by $\alpha \mapsto \alpha, \beta \mapsto \beta, t \mapsto \tau$.

Case 10: First, the isomorphism $\alpha \mapsto \alpha, \beta \mapsto \beta, t \mapsto t\beta$ gives the group the following presentation

$$G = \langle \alpha, \beta, t | \beta\alpha\beta^{-1} = \alpha^{-1}, t\alpha t^{-1} = \alpha^{-1}, t\beta t^{-1} = \alpha\beta^{-1} \rangle.$$

$p : G \to G_2^4$ where

$$G_2^4 = \langle \alpha, \beta, \tau | \beta\alpha\beta^{-1} = \alpha^{-1}, \alpha^\tau = \alpha^{-1}, \beta^\tau = \alpha\beta^{-1}, \tau^2 = 1 \rangle$$

and p is given by $\alpha \mapsto \alpha, \beta \mapsto \beta, t \mapsto \tau$.

Let M^n be a flat manifold with fundamental group π. Then there exists a maximal abelian normal subgroup Γ such that $\pi/\Gamma = \Phi$ (the holonomy) is finite. Given a selfmap $f : M \to M$, there exist lifts $D_* f$ on the $|\Phi|$-fold cover T^n whose fundamental group is Γ, for each $D \in \Phi$. There is an averaging formula for the Nielsen number [20] given by

$$N(f) = \frac{1}{|\Phi|} \sum_{D \in \Phi} |\det(1 - (D_* f)_\sharp)|. \tag{2.1}$$

There is an alternate way of computing $N(f)$ when M is fibered over S^1. Consider the fibration $N \hookrightarrow M \xrightarrow{p} S^1$ where N is a closed surface. Given a fiber-preserving map $f : M \to M$ inducing $\bar{f} : S^1 \to S^1$, we can compute $N(f)$ as follows. Let $\gamma = \deg \bar{f}$. If $\gamma = 1$, then $\bar{f} \sim 1_{S^1}$ so that $\bar{f}$ is homotopic to a fixed point free map. It follows that f is deformable to be fixed point free and thus $N(f) = 0$. If $\gamma \neq 1$, then $N(\bar{f}) = |1 - \gamma|$. Without loss of generality, we may assume that $\bar{f}$ has exactly $|1 - \gamma|$ fixed points each of which is its own fixed point class. The fixed point classes of $f|_{p^{-1}(\bar{x})} : p^{-1}(\bar{x}) \to p^{-1}(\bar{x})$ inject into the fixed point classes of f for each $\bar{x} \in Fix\bar{f}$. In fact, we have

$$N(f) = \sum_{\bar{x} \in Fix\bar{f}} N(f|_{p^{-1}(\bar{x})}). \tag{2.2}$$

This fiberwise technique and in particular the formula (2.2) will be useful in section 5 when we compute $N(f)$ for arbitrary selfmaps in most cases.

3 Nielsen numbers of self homeomorphisms: Cases 1 - 5

3.1 Case 1.

This flat manifold is the 3-torus T^3. Every homeomorphism $f : T^3 \to T^3$ induces on the fundamental group a linear map $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^3$ and the Nielsen number is $N(f) = |\det(1-\varphi)|$. It is easy to see that $\mathrm{NSH}(M) = \mathbb{N} \cup \{0\}$.

Next, we use the formula (2.1) to determine the values of the Nielsen numbers of homeomorphisms for Cases 2,3,4, and 5.

It is well known that the center $\mathcal{Z}(G)$ of a crystallographic group G coincides with the fixed point group $(\mathbb{Z}^n)^\Phi$ where $\mathbb{Z}^n$ is the translation subgroup and Φ is the holonomy group. In Case 2, the holonomy $\mathbb{Z}_2$ is generated by t so that $(\mathbb{Z}^3)^\Phi$ is the subgroup of the elements fixed by the automorphism induced by t. From the presentation of G for Case 2, the automorphism induced by t is given by $\alpha_1 \mapsto \alpha_1$, $\alpha_2 \mapsto \alpha_2^{-1}$, $\alpha_3 \mapsto \alpha_3^{-1}$. In other words, the automorphism is given by a diagonal integral matrix which has 1 as eigenvalue with one dimensional eigenspace. We now conclude that $\mathcal{Z}(G) = \langle \alpha_1 \rangle$.

For each of the Cases 3,4, and 5, a similar argument shows that $Kerp = \langle \alpha_1 \rangle = \mathcal{Z}(G)$. Thus, for every $\varphi \in \mathrm{Aut}(G)$ for each G in Cases 2-5, φ is represented by an array of the form

$$\varphi = \begin{bmatrix} \kappa & * \\ 0 & A \end{bmatrix}.$$

where $\kappa = \pm 1$ and A is a 3×3 array representing the induced automorphism $\overline{\varphi} : G/\mathcal{Z}(G) \to G/\mathcal{Z}(G)$.

Write φ to be the array

$$\varphi = \begin{bmatrix} \kappa & x & y & z \\ 0 & a & c & r \\ 0 & b & d & s \\ 0 & \epsilon & \delta & \gamma \end{bmatrix}.$$

where

$$A = \begin{bmatrix} a & c & r \\ b & d & s \\ \epsilon & \delta & \gamma \end{bmatrix}.$$

Here the columns are the exponents of the generators $\alpha_1, \alpha_2, \alpha_3, t$ of their images under φ since every word can be written in the normal form $\alpha_1^{n_1}\alpha_2^{n_2}\alpha_3^{n_3}t^n$. Furthermore, $\alpha_1, \alpha_2, \alpha_3$ generate a maximal abelian normal subgroup Γ in G so that the lift (or restriction to Γ) φ' is represented by the array

$$\varphi' = \begin{bmatrix} \kappa & x & y \\ 0 & a & c \\ 0 & b & d \end{bmatrix}.$$

If a homeomorphism f has an induced automorphism φ on the fundamental group, the averaging formula (2.1) yields

$$N(f) = \frac{1}{|\Phi|} \sum_{0 \le i < |\Phi|} |\det(1 - \theta(t^i)\varphi')| \tag{3.1}$$

where $\theta(t)$ denotes the action of t. In the Cases 2-5, t acts trivially on α_1 so that $\theta(t^i)\varphi'$ is also represented by an array of the form

$$\theta(t^i)\varphi' = \begin{bmatrix} \kappa & * \\ 0 & \overline{A_i} \end{bmatrix}.$$

for some 2×2 array $\overline{A_i}$. Thus, when $\kappa = 1$, $|\det(1 - \theta(t^i)\varphi')| = 0$ for all i, $0 \le i < |\Phi|$. For such homeomorphisms f, we have $N(f) = 0$. For the rest of this section, we consider automorphisms where $\kappa = -1$.

3.2 Case 2.

This group projects onto G_2. It follows from [6, 7] that φ can be represented by an array of the form

$$\varphi = \begin{bmatrix} -1 & x & y & z \\ 0 & a & c & r \\ 0 & b & d & s \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

with $ad - bc = \pm 1$. Now the lifts of φ are of the form (in fact, matrices)

$$\varphi' = \begin{bmatrix} -1 & x & y \\ 0 & a & c \\ 0 & b & d \end{bmatrix} \qquad \text{and} \qquad \theta(t)\varphi' = \begin{bmatrix} -1 & x & y \\ 0 & -a & -c \\ 0 & -b & -d \end{bmatrix}.$$

Let

$$\overline{A} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

A straightforward calculation using the averaging formula (3.1) shows that if $\det \overline{A} = -1$ then the Nielsen number $N(f) = 2|\mathrm{Tr}\overline{A}|$, where $\mathrm{Tr}X$ denotes the trace of a matrix X. If $\det \overline{A} = 1$ then $N(f) = 2|\mathrm{Tr}\overline{A}|$ if $|\mathrm{Tr}\overline{A}| \ge 2$ or else $N(f) = 4$. Thus, for any homeomorphism f, we have $\mathrm{NSH}(M) = 2\mathbb{N} \cup \{0\}$.

3.3 Case 3.

This group projects onto G_3. It follows from [6, 7] that the automorphism A has one of the following two forms:

$$(i) \quad A = \begin{bmatrix} a & -b & r \\ b & a+b & s \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{or} \qquad (ii) \quad A = \begin{bmatrix} a & b+a & r \\ b & -a & s \\ 0 & 0 & 2 \end{bmatrix}.$$

The maximal abelian subgroup Γ is generated by $\alpha_1, \alpha_2, \alpha_3$ with quotient the holonomy $\Phi = \mathbb{Z}_3$. Moreover, the restriction of φ on Γ is given by the *matrix*

$$\varphi' = \begin{bmatrix} -1 & x & y \\ 0 & a & -b \\ 0 & b & a+b \end{bmatrix} \qquad \text{or} \qquad \varphi' = \begin{bmatrix} -1 & x & y \\ 0 & a & b+a \\ 0 & b & -a \end{bmatrix}.$$

Note that $\varphi(\alpha_1) = \alpha_1^{-1}$. Since $\alpha_1 = t^3$, it follows that $\varphi(t) = \alpha_1^z \alpha_2^r \alpha_3^s t^{-1} = \alpha_1^z \alpha_2^r \alpha_3^s t^2$ so that

$$\varphi' = \begin{bmatrix} -1 & x & y \\ 0 & a & b+a \\ 0 & b & -a \end{bmatrix}.$$

Now, from [7], we have

$$\begin{bmatrix} a & b+a \\ b & -a \end{bmatrix} \in \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \right\}.$$

Now a straightforward calculation shows that $\det(1 - \varphi') = \det(1 - \theta(t)\varphi') = \det(1 - \theta(t^2)\varphi') = 0$. Hence such automorphisms also yield $N(f) = 0$. We conclude that $\mathrm{NSH}(M) = \{0\}$. Hence, by [16], every homeomorphism of this flat manifold is isotopic to a fixed point free homeomorphism.

3.4 Case 4.

This groups projects onto G_4. It follows from [6, 7] that the automorphism A has one of the following two forms:

$$(i) \quad A = \begin{bmatrix} a & -b & r \\ b & a & s \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{or} \qquad (ii) \quad A = \begin{bmatrix} a & b & r \\ b & -a & s \\ 0 & 0 & 3 \end{bmatrix}.$$

Note that $\varphi(\alpha_1) = \alpha_1^{-1}$. Since $\alpha_1 = t^4$, it follows that $\varphi(t) = \alpha_1^z \alpha_2^r \alpha_3^s t^3$ so that only (ii) can occur. Furthermore, we have

$$(ii) \quad \overline{A} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}.$$

Here,

$$\theta(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \theta(t^2) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \theta(t^3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now a straightforward calculation using the averaging formula shows that $N(f) = 0$. Thus we conclude that for any homeomorphism f, we have $N(f) = 0$ or $\mathrm{NSH}(M) = \{0\}$.

3.5 Case 5.

This group projects onto G_6. It follows from [6, 7] that the automorphism A has one of the following two forms:

$$(i) \quad A = \begin{bmatrix} a & -b & r \\ b & a+b & s \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{or} \qquad (ii) \quad A = \begin{bmatrix} a & a+b & r \\ b & -a & s \\ 0 & 0 & 5 \end{bmatrix}.$$

Note that $\varphi(\alpha_1) = \alpha_1^{-1}$. Since $\alpha_1 = t^6$, it follows that $\varphi(t) = \alpha_1^z \alpha_2^r \alpha_3^s t^5$ so that only (ii) can occur. Furthermore, we have

$$(ii) \quad \overline{A} = \begin{bmatrix} a & a+b \\ b & -a \end{bmatrix} \in \left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

Here,

$$\theta(t) = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \theta(t^2) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \quad \theta(t^3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\theta(t^4) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad \theta(t^5) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now a straightforward calculation using the averaging formula shows that $N(f) = 0$. Thus we conclude that for any homeomorphism f, we have $N(f) = 0$ or $\mathrm{NSH}(M) = \{0\}$.

4 Nielsen numbers: remaining cases 6 - 10

In this section, we compute the Nielsen numbers of self-homeomorphisms of flat manifolds in the remaining 5 cases, 6 - 10.

4.1 Case 6.

Lemma 4.1. *Each element in G can be written as the form $t_1^{p_1} \alpha_2^{p_2} t_3^{p_3}$.*

Proof. By definition of holonomy, the subgroup of G generated by $\alpha_1, \alpha_2, \alpha_3$ has index 4 in G. Thus, each element of G must be in one of the forms: $\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$, $t_1\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$, $t_3\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$ and $t_1t_3\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3}$. Clearly, $\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3} = t_1^{2p_1}\alpha_2^{p_2}t_3^{2p_3}$, and $t_1\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3} = t_1^{2p_1+1}\alpha_2^{p_2}t_3^{2p_3}$. By using the relation: $t_3\alpha_j t_3^{-1} = \alpha_j^{-1}$, $j = 1, 2$. We obtain:

$$t_3\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3} = \alpha_1^{-p_1}\alpha_2^{-p_2}t_3\alpha_3^{p_3} = t_1^{-2p_1}\alpha_2^{-p_2}t_3^{2p_3+1},$$

and

$$t_1t_3\alpha_1^{p_1}\alpha_2^{p_2}\alpha_3^{p_3} = t_1\alpha_1^{-p_1}\alpha_2^{-p_2}t_3\alpha_3^{p_3} = t_1^{1-2p_1}\alpha_2^{-p_2}t_3^{2p_3+1}.$$ ■

Note that $t_2 = t_3t_1$. Lemma 4.1 says that every group element has such normal form. In particular, a straightforward calculation yields

$$t_3^t t_1^a = \begin{cases} t_1^a t_3^t, & \text{if } a \text{ is even and } t \text{ is even} \\ t_1^{-a} t_3^t, & \text{if } a \text{ is even and } t \text{ is odd} \\ t_1^a t_3^{-t}, & \text{if } a \text{ is odd and } t \text{ is even} \\ t_1^{-a}\alpha_2^{-1}t_3^{-t}, & \text{if } a \text{ is odd and } t \text{ is odd.} \end{cases}$$

Now for any $\varphi \in \mathrm{Aut}(G)$, using the generators t_1, α_2 and t_3, we can represent φ by a 3×3 array of the form

$$\varphi = \begin{bmatrix} a & c & r \\ b & d & s \\ \epsilon & \delta & t \end{bmatrix}.$$

We now compute $\varphi(t_1^2)$ under all possible cases for the parities of the pair (a, ϵ).

Type	a	ϵ	$\varphi(t_1^2) = \varphi(\alpha_1)$
(I)	even	even	$t_1^{2a}\alpha_2^{2b}t_3^{2\epsilon}$
(II)	even	odd	$t_3^{2\epsilon}$
(III)	odd	even	t_1^{2a}
(IV)	odd	odd	α_2^{-2b-1}

Similarly, we compute $\varphi(t_3^2)$ under all possible cases for the parities of the pair (r, t).

Type	r	t	$\varphi(t_3^2) = \varphi(\alpha_3)$
(I′)	even	even	$t_1^{2r}\alpha_2^{2s}t_3^{2t}$
(II′)	even	odd	t_3^{2t}
(III′)	odd	even	t_1^{2r}
(IV′)	odd	odd	α_2^{-2s-1}

If Type (I) occurs, we consider the relation $\varphi(t_1 t_3^2 t_1^{-1}) = \varphi(t_3^{-2})$. With a and ϵ both even, $\varphi(t_1) = t_1^a \alpha_2^b t_3^\epsilon$ lies in the maximal abelian subgroup generated by $\alpha_1, \alpha_2, \alpha_3$ so that $\varphi(t_1)$ commutes with $\varphi(t_3^2) = \varphi(\alpha_3)$. It follows that $\varphi(t_3^2) = \varphi(t_3^{-2})$ and so $\varphi(t_3) = 1$, a contradiction to the fact that φ is an automorphism and t_3 is a generator. Likewise, if Type (I′) occurs then the relation $\varphi(t_3 t_1^2 t_1^{-1}) = \varphi(t_1^{-2})$ leads to $\varphi(t_1) = 1$, a contradiction.

Next, we consider the case Type (II) and Type (II′). Then the relation $\varphi(t_1 t_3^2 t_1^{-1}) = \varphi(t_3^{-2})$ becomes

$$\begin{aligned} t_1^a \alpha_2^b t_3^\epsilon t_3^{2t} t_3^{-\epsilon} \alpha_2^{-b} t_1^{-a} &= t_3^{-2t} \\ \Rightarrow t_3^{2t} &= t_3^{-2t} \quad \Rightarrow t = 0. \end{aligned}$$

This is a contradiction to the assumption that t is odd.

Consider the case Type (III) and Type (III′). Then the relation $\varphi(t_3 t_1^2 t_3^{-1}) = \varphi(t_1^{-2})$ becomes

$$\begin{aligned} t_1^r \alpha_2^s t_3^t t_1^{2a} t_3^{-t} \alpha_2^{-s} t_1^{-r} &= t_1^{-2a} \\ \Rightarrow t_1^{2a} &= t_1^{-2a} \quad \Rightarrow a = 0. \end{aligned}$$

This is a contradiction to the assumption that a is odd.

Consider the case Type (IV) and Type (IV′). Then the relation $\varphi(t_1 t_3^2 t_1^{-1}) = \varphi(t_3^{-2})$ becomes

$$\begin{aligned} t_1^a \alpha_2^b t_3^\epsilon \alpha_2^{-2s-1} t_3^{-\epsilon} \alpha_2^{-b} t_1^{-a} &= \alpha_2^{2s+1} \\ \Rightarrow \alpha_2^{-2s-1} &= \alpha_2^{2s+1} \quad \Rightarrow 2s + 1 = 0. \end{aligned}$$

This is not possible since s is an integer.

Thus, we only need to consider six possible cases below which we compute $\varphi(\alpha_2) = \varphi(t_2^2) = \varphi(t_3t_1)^2$.

Type	$\varphi(t_3t_1)^2 = (t_1^r\alpha_2^s t_3^t t_1^a \alpha_2^b t_3^\epsilon)^2$
(II) and (III′)	$\alpha_2^{-2s-2b-1}$
(II) and (IV′)	$t_1^{2(r-a)}$
(III) and (II′)	$\alpha_2^{2(s+b)+1}$
(III) and (IV′)	$t_3^{2(\epsilon-t)}$
(IV) and (II′)	$t_1^{2(r-a)}$
(IV) and (III′)	$t_3^{2(\epsilon-t)}$

If we denote by φ' the restriction of φ on the maximal subgroup generated by α_1, α_2 and α_3, then we have the following

Automorphism Types (4.1)

Type	(II) and (III′)	(II) and (IV′)	(III) and (II′)	(III) and (IV′)	(IV) and (II′)	(IV) and (III′)
φ'	$\begin{bmatrix} 0 & 0 & r \\ 0 & -2s-2b-1 & 0 \\ \epsilon & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & r-a & 0 \\ 0 & 0 & -2s-1 \\ \epsilon & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} a & 0 & 0 \\ 0 & 2(s+b)+1 & 0 \\ 0 & 0 & t \end{bmatrix}$	$\begin{bmatrix} a & 0 & 0 \\ 0 & 0 & -2s-1 \\ 0 & \epsilon-t & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & r-a & 0 \\ -2b-1 & 0 & 0 \\ 0 & 0 & t \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & r \\ -2b-1 & 0 & 0 \\ 0 & \epsilon-t & 0 \end{bmatrix}$

The holonomy $\Phi = \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by the images of t_1 and t_3. Their actions of α_i are given by the following matrices:

$$\theta_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \theta_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \theta_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Now a straightforward calculation together with the average formula for $N(f)$, we conclude that in all six cases we have $N(f) = 0$ or 2 so that $\mathrm{NSH}(M) = \{0, 2\}$ for any homeomorphism.

4.2 Case 7.

The group is $\pi_1(K) \times \mathbb{Z}$. Moreover, we have the following presentation

$$G = \langle \alpha, \beta, t \mid \beta\alpha\beta^{-1} = \alpha^{-1}, t\alpha t^{-1} = \alpha, t\beta t^{-1} = \beta \rangle.$$

The center of G is $\mathcal{Z}(G) = \langle \beta^2 \rangle \times \langle t \rangle$. Let $\varphi \in \mathrm{Aut}(G)$. Using the generators α, β, t, we can represent φ by a 3×3 array of the form

$$\varphi = \begin{bmatrix} a & c & r \\ b & d & s \\ \epsilon & \delta & \gamma \end{bmatrix}.$$

Since $t \in \mathcal{Z}(G)$, $\varphi(t) \in \mathcal{Z}(G)$. It follows that $r = 0$ and s is even. Now

$$\varphi(\beta^2) = \alpha^c \beta^d t^\delta \alpha^c \beta^d t^\delta = \alpha^c \alpha^{(-1)^d c} \beta^{2d} t^{2\delta} \qquad \in \mathcal{Z}(G)$$

and thus $c + (-1)^d c = 0$ and so d must be odd. Thus,

$$\varphi = \begin{bmatrix} a & c & 0 \\ b & 2q+1 & 2k \\ \epsilon & \delta & \gamma \end{bmatrix}.$$

Next, we have $\varphi(\beta\alpha\beta^{-1}) = \varphi(\alpha^{-1})$. It follows that

$$\begin{aligned} \alpha^c\beta^{2q+1}t^\delta\alpha^a\beta^b t^\epsilon t^{-\delta}\beta^{-2q-1}\alpha^{-c} &= t^{-\epsilon}\beta^{-b}\alpha^{-a} \\ \alpha^c\beta^{2q+1}\alpha^a\beta^b\beta^{-2q-1}\alpha^{-c}t^\epsilon &= \beta^{-b}\alpha^{-a}t^{-\epsilon} \qquad \text{hence } \epsilon = 0 \\ \alpha^c\alpha^{-a}\beta^b\alpha^{-c} &= \beta^{-b}\alpha^{-a} \\ \beta^b\alpha^{c-a}\beta^b &= \alpha^{c-a} \quad \Rightarrow \quad \alpha^{(-1)^b(c-a)}\beta^{2b} = \alpha^{c-a} \end{aligned}$$

It follows that $b = 0$. In other words, $\varphi(\alpha) = \alpha^a$. Since φ is an automorphism, we have $a = \pm 1$. Now, we have

$$\varphi = \begin{bmatrix} \pm 1 & c & 0 \\ 0 & 2q+1 & 2k \\ 0 & \delta & \gamma \end{bmatrix}.$$

From the calculation above, we have $\varphi(\beta^2) = \beta^{4q+2}t^{2\delta}$. Now the subgroup generated by α, β^2, t is a maximal abelian subgroup Γ and the quotient G/Γ is the holonomy group $\Phi = \langle \overline{\beta} | \overline{\beta}^2 = 1 \rangle \cong \mathbb{Z}_2$. The restriction of φ on Γ is given by the *matrix*

$$\varphi' = \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & 2q+1 & 2k \\ 0 & 2\delta & \gamma \end{bmatrix}.$$

Since φ' is an automorphism, we have $\det \varphi' = (2q+1)\gamma - 4\delta k = \pm 1$. It follows that γ must be odd. The action of Φ on Γ sends α to α^{-1} and is trivial on β^2 and t. Thus, it induces another lift $D_*\varphi'$ given by

$$D_*\varphi' = \begin{bmatrix} \mp 1 & 0 & 0 \\ 0 & 2q+1 & 2k \\ 0 & 2\delta & \gamma \end{bmatrix}.$$

A straightforward calculation shows that

$$N(f) = \frac{1}{2}(0 + 2|2q(\gamma-1) - 4\delta k|) = |2q(\gamma-1) - 4\delta k| = |\pm 1 - \gamma - 2q|$$

where

$$f_\sharp = \varphi = \begin{bmatrix} \pm 1 & c & 0 \\ 0 & 2q+1 & 2k \\ 0 & \delta & \gamma \end{bmatrix}$$

with γ an odd integer. In particular, $N(f)$ must be even. In fact, we have $\mathrm{NSH}(M) = 2\mathbb{N} \cup \{0\}$.

4.3 Case 8.

The group is $\pi_1(K) \rtimes \mathbb{Z}$. Moreover, we have the following presentation

$$G = \langle \alpha, \beta, t | \beta\alpha\beta^{-1} = \alpha^{-1}, t\alpha t^{-1} = \alpha, t\beta t^{-1} = \alpha\beta \rangle.$$

Note that α, β^2, t generate an index 2 abelian subgroup in G and hence is the maximal abelian subgroup whose quotient group $\mathbb{Z}_2$ is the holonomy. Let $\varphi \in \mathrm{Aut}(G)$. Using the generators α, β, t, we can represent φ by a 3×3 array of the form

$$\varphi = \begin{bmatrix} a & c & r \\ b & d & s \\ \epsilon & \delta & \gamma \end{bmatrix}.$$

Since $\varphi(t\beta t^{-1}) = \varphi(\alpha\beta)$, we have

$$\alpha^r \beta^s t^\gamma \alpha^c \beta^d t^\delta t^{-\gamma} \beta^{-s} \alpha^{-r} = \alpha^a \beta^b t^\epsilon \alpha^c \beta^d t^\delta. \tag{4.2}$$

Using the group relations, (4.2) can be rewritten as

$$w_1 t^\delta = w_2 t^{\epsilon+\delta}$$

where w_1, w_2 are words in α and β. It follows that $\epsilon = 0$.

Note that $t^x\beta = \alpha^x\beta t^x$ so that $t^x\beta^y t^{-x} = (\alpha^x\beta)^y$. Moreover, $\alpha^x\beta\alpha^x\beta = \beta^2$.

Since $\varphi(\beta\alpha\beta^{-1}) = \varphi(\alpha^{-1})$, we have

$$\begin{aligned} \alpha^c \beta^d t^\delta \alpha^a \beta^b t^{-\delta} \beta^{-d} \alpha^{-c} &= \beta^{-b} \alpha^{-a} \\ \Rightarrow \alpha^c \beta^d \alpha^a (\alpha^\delta \beta)^b \beta^{-d} \alpha^{-c+a} &= \beta^{-b}. \end{aligned} \tag{4.3}$$

Case (i): b even

In this case, (4.3) yields

$$\begin{aligned} \alpha^c \beta^d \alpha^a \beta^{b-d} \alpha^{-c+a} &= \beta^{-b} \\ \Rightarrow \alpha^c \alpha^{(-1)^d a} \beta^b \alpha^{-c+a} &= \beta^{-b} \\ \Rightarrow \alpha^{c+(-1)^d a + (-1)^b(-c+a)} \beta^b &= \beta^{-b} \quad \Rightarrow b = 0. \end{aligned}$$

Case (ii): b odd

In this case, (4.3) yields

$$\begin{aligned} \alpha^c \beta^d \alpha^a \beta^{b-1} \alpha^\delta \beta \beta^{-d} \alpha^{-c+a} &= \beta^{-b} \\ \Rightarrow \alpha^c \alpha^{(-1)^d a} \beta^{b+d-1} a f^\delta \beta^{1-d} \alpha^{-c+a} &= \beta^{-b} \\ \Rightarrow \alpha^{c+(-1)^d a + (-1)^{b+d-1}\delta} \beta^b \alpha^{-c+a} &= \beta^{-b} \\ \Rightarrow \alpha^w \beta^b &= \beta^{-b} \end{aligned}$$

for some $w \Rightarrow b = 0$ a contradiction since b is odd.

Thus, we conclude that $b = 0$. Now, φ is an automorphism and $\varphi(\alpha) = \alpha^a$. It follows that $a = \pm 1$.

Since $\varphi(t\alpha t^{-1}) = \alpha$, we have

$$\begin{aligned} \alpha^r\beta^s t^\gamma \alpha^a t^{-\gamma}\beta^{-s}\alpha^{-r} &= \alpha^a \\ \Rightarrow \alpha^r\beta^s\alpha^a\beta^{-s}\alpha^{-r} &= \alpha^a \\ \Rightarrow \alpha^r\alpha^{(-1)^s a}\alpha^{-r} &= \alpha^a \quad \Rightarrow (-1)^s a = a \quad \Rightarrow s \text{ is even.} \end{aligned} \tag{4.4}$$

Suppose $a = -1$ so that

$$\varphi = \begin{bmatrix} -1 & c & r \\ 0 & d & s \\ 0 & \delta & \gamma \end{bmatrix}$$

where s is even. Now (4.3) yields $c - (-1)^d - c - 1 = 0$ so that d must be odd. (Note that d is also odd when $a = 1$.)

The equality (4.2) becomes

$$\begin{aligned} \alpha^r\beta^s t^\gamma\alpha^c\beta^d t^\delta t^{-\gamma}\beta^{-s}\alpha^{-r} &= \alpha^{-1}\alpha^c\beta^d t^\delta \\ \Rightarrow \alpha^r\beta^s\alpha^c t^\gamma\beta^d t^{-\gamma}t^\delta\beta^{-s}\alpha^{-r} &= \alpha^{c-1}\beta^d t^\delta \\ \Rightarrow \alpha^r\beta^s\alpha^c(\alpha^\gamma\beta)^d t^\delta\beta^{-s}\alpha^{-r} &= \alpha^{c-1}\beta^d t^\delta \\ \Rightarrow \alpha^r\alpha^{(-1)^s c}\beta^s\beta^{d-1}\alpha^\gamma\beta t^\delta\beta^{-s}\alpha^{-r} &= \alpha^{c-1}\beta^d t^\delta \quad \text{since } d \text{ is odd} \\ \Rightarrow \alpha^{r+c}\beta^{s+d-1}\alpha^\gamma\beta(\alpha^\delta\beta)^{-s}\alpha^{-r} &= \alpha^{c-1}\beta^d \quad \text{since } s \text{ is even} \\ \Rightarrow \alpha^{r+c}\beta^{s+d-1}\alpha^\gamma\beta\beta^{-s}\alpha^{-r} &= \alpha^{c-1}\beta^d \quad \text{since } s \text{ is even} \\ \Rightarrow \alpha^{r+c}\alpha^{(-1)^{s+d-1}\gamma}\beta^d\alpha^{-r} &= \alpha^{c-1}\beta^d \\ \Rightarrow \alpha^{r+c}\alpha^\gamma\alpha^{(-1)^d(-r)}\beta^d &= \alpha^{c-1}\beta^d \\ \Rightarrow r + c + \gamma - (-1)^d r = c - 1 \quad &\Rightarrow \gamma = -1 - 2r \quad \text{since } d \text{ is odd.} \end{aligned}$$

It follows that γ must be odd.

Let φ' denote the restriction of φ on the maximal abelian subgroup generated by α, β^2 and t. A straightforward calculation show that

$$\varphi' = \begin{bmatrix} \pm 1 & -\delta & r \\ 0 & d & s/2 \\ 0 & 2\delta & \gamma \end{bmatrix}.$$

The other lift $D_*\varphi'$ induced by the holonomy action is given by

$$D_*\varphi'(w) = \beta\varphi'(w)\beta^{-1}.$$

Again, a straightforward calculation yields

$$D_*\varphi' = \begin{bmatrix} \mp 1 & 3\delta & -r - (-1)^{s/2}\gamma \\ 0 & d & s/2 \\ 0 & 2\delta & \gamma \end{bmatrix}.$$

Thus, if $a = 1$ then $\det(1 - \varphi') = 0$ while $\det(1 - D_*\varphi') = 2[(1-d)(1-\gamma) - \delta s]$. The averaging formula shows that $N(f) = |1 - (d+\gamma) + (\pm 1)|$ is even. Similarly, if $a = -1$ then $\det(1 - \varphi') = 2[(1-d)(1-\gamma) - \delta s]$ while $\det(1 - D_*\varphi') = 0$. Again using the averaging formula yields that $N(f)$ is even. In fact, all even non negative integers can occur as $N(f)$ and hence $\mathrm{NSH}(M) = 2\mathbb{N} \cup \{0\}$

4.4 Case 9.

The isomorphism $\alpha \mapsto \alpha, \beta \mapsto \beta, t \mapsto t\beta$ gives the group G the following presentation

$$G = \langle \alpha, \beta, t | \beta\alpha\beta^{-1} = \alpha^{-1}, t\alpha t^{-1} = \alpha^{-1}, t\beta t^{-1} = \beta^{-1} \rangle. \tag{4.5}$$

This group is the mapping torus $\pi_1(K) \rtimes_\varphi \mathbb{Z}$ where $\varphi(\alpha) = \alpha^{-1}$ and $\varphi(\beta) = \beta^{-1}$. Here K denotes the Klein bottle. Using the calculation in [7] and the fact that this group projects onto the group G_2^2, the normal subgroup $\pi_1(K)$ is characteristic. In fact, the corresponding flat manifold M is a Klein bottle bundle over the unit circle S^1. Given a homeomorphism f, it induces the following commutative diagram at the fundamental group level.

$$\begin{array}{ccccc}
\pi_1(K) & \longrightarrow & \pi_1(M) & \longrightarrow & \pi_1(S^1) \\
{\scriptstyle \varphi'}\big\downarrow & & {\scriptstyle \varphi}\big\downarrow & & {\scriptstyle \overline{\varphi}}\big\downarrow \\
\pi_1(K) & \longrightarrow & \pi_1(M) & \longrightarrow & \pi_1(S^1)
\end{array}$$

Choose a homeomorphism $\overline{f}$ with induced automorphism $\overline{\varphi}$. Then the following diagram is commutative, up to homotopy.

$$\begin{array}{ccc}
M & \xrightarrow{p} & S^1 \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \overline{f}} \\
M & \xrightarrow{p} & S^1
\end{array} \tag{4.6}$$

This implies that there is a homotopy $\overline{H} : M \times [0,1] \to S^1$ from $p \circ f$ to $\overline{f} \circ p$. The Covering Homotopy Property for the fibration $p : M \to S^1$ yields a homotopy $H : M \times [0,1] \to M$ covering $\overline{H}$ from f to $\hat{f}$. It follows that the diagram (4.6) gives rise to the following commutative diagram.

$$\begin{array}{ccccc}
K & \longrightarrow & M & \xrightarrow{p} & S^1 \\
{\scriptstyle f'}\big\downarrow & & {\scriptstyle \hat{f}}\big\downarrow & & {\scriptstyle \overline{f}}\big\downarrow \\
K & \longrightarrow & M & \xrightarrow{p} & S^1
\end{array}$$

Since $\overline{f}$ is a self homeomorphism of the unit circle, $N(\overline{f}) = 0$ or 2. If $N(\overline{f}) = 0$, it follows that $N(f) = 0$. Suppose $N(\overline{f}) = 2$. We may assume that $\overline{f}$ has exactly two fixed points at $z = 1$ and at $z = -1$. The corresponding fiber maps are f' and f'' respectively. It is easy to see that the fixed subgroups $Fixf'_\sharp$ and $Fixf''_\sharp$ are both trivial so that the fixed point classes of f' and of f'' inject into the set of fixed point classes of $\hat{f}$ (or f). Since there are only four isomorphism classes of automorphisms of $\pi_1(K)$, we may assume without loss of generality that the map f' induces the automorphism $\alpha \mapsto \alpha, \beta \mapsto \alpha\beta^{-1}$ or $\beta \mapsto \beta^{-1}$ while f'' induces the automorphism $\alpha \mapsto \alpha, \beta \mapsto \alpha\beta$ or $\beta \mapsto \beta$. By computing the Nielsen number of f' and f'', we see that $N(f') = 2$ while $N(f'') = 0$. Hence, we conclude that $\mathrm{NSH}(M) = \{0, 2\}$.

4.5 Case 10.

This case is similar to Case 9. This group is the mapping torus $\pi_1(K) \rtimes_\varphi \mathbb{Z}$ where $\varphi(\alpha) = \alpha^{-1}$ and $\varphi(\beta) = \alpha\beta^{-1}$. Thus G has the following presentation

$$G = \langle \alpha, \beta, t | \beta\alpha\beta^{-1} = \alpha^{-1}, t\alpha t^{-1} = \alpha^{-1}, t\beta t^{-1} = \alpha\beta^{-1} \rangle. \tag{4.7}$$

Let $\eta \in \mathrm{Aut}(G)$ be given by the following array

$$\eta = \begin{bmatrix} a & c & r \\ b & d & s \\ \epsilon & \delta & \gamma \end{bmatrix}.$$

Since $\eta(\beta\alpha\beta^{-1}) = \eta(\alpha^{-1})$, we have

$$\alpha^c\beta^d t^\delta \alpha^a \beta^b t^\epsilon t^{-\delta} \beta^{-d} \alpha^{-c} = t^{-\epsilon}\beta^{-b}\alpha^{-a}.$$

This equality can be rewritten as $w_1 t^\epsilon = w_2 t^{-\epsilon}$ where w_i are words in α and β. Thus, $\epsilon = 0$. Similarly, $\eta(t\beta t^{-1}) = \eta(\alpha\beta^{-1})$, we have

$$\alpha^r\beta^s t^\gamma \alpha^c \beta^d t^\delta t^{-\gamma} \beta^{-s} \alpha^{-r} = \alpha^a \beta^b t^\epsilon t^{-\delta} \beta^{-d} \alpha^{-c}.$$

This equality can be rewritten as $\tilde{w}_1 t^\delta = \tilde{w}_2 t^{\epsilon-\delta}$ where $\tilde{w}_i$ are words in α and β. It follows that $\epsilon = 2\delta$ so that $\delta = 0$. Since $\epsilon = 0 = \delta$, this shows that $\pi_1(K)$ is characteristic. Now we use the same arguments as in Case 9 to conclude that $\mathrm{NSH}(M) = \{0, 2\}$ for every homeomorphism f of the flat manifold M.

5 Nielsen numbers of arbitrary selfmaps: Cases 2-5,9,10

In the previous two sections, with the exception of cases 9 and 10 for which we used fiberwise techniques to compute $N(f)$ for self homeomorphisms, we employed the average formula (3.1) in terms of the Nielsen numbers of the associated lifts to the universal cover $\mathbb{R}^3$. For arbitrary selfmaps, it is more manageable to classify these maps up to fiberwise homotopy since for all but two of the ten cases, the flat manifold M fibers over S^1 with typical fiber N corresponding to a fully invariant subgroup of $\pi_1(M)$. Thus, we can apply fiberwise techniques. For cases 2-5, $N = T^2$ is the 2-torus. For cases 9 and 10, $N = K$ is the Klein bottle.

For each of the cases 2-5, the crystallographic group G is isomorphic to a mapping torus of the form $\langle \alpha_2, \alpha_3 | \alpha_2\alpha_3 = \alpha_3\alpha_2 \rangle \rtimes_{\theta_i} \langle t \rangle$ where $i = 2, 3, 4, 5$ for each case i and

$$\theta_2(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \theta_3(t) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \quad \theta_4(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \theta_5(t) = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Moreover, the automorphisms $\theta_2, \theta_3, \theta_4, \theta_5$ have finite orders of $2, 3, 4,$ and 6 respectively. Every endomorphism of G will be given by a 3×3 array of the form

$$\varphi = \begin{bmatrix} a & c & r \\ b & d & s \\ \epsilon & \delta & \gamma \end{bmatrix}$$

where the columns represent the images of α_2, α_3, and t under φ in terms of the generators α_2, α_3, t.

The relations defining (i) $t\alpha_2 t^{-1}$ and (ii) $t\alpha_3 t^{-1}$ yield two relations of the form $wt^m = w't^n$ where w, w' are words in α_2, α_3. More precisely, we have the following:

Case 2: (i) $w_1 t^\epsilon = w_1' t^{-\epsilon}$ and (ii) $w_2 t^\delta = w_2' t^{-\delta}$. It follows that $\epsilon = 0 = \delta$.
Case 3: (i) $w_1 t^\epsilon = w_1' t^{-\delta}$ and (ii) $w_2 t^\delta = w_2' t^{-\epsilon-\delta}$. It follows that $\epsilon = 0 = \delta$.
Case 4: (i) $w_1 t^\epsilon = w_1' t^{\delta}$ and (ii) $w_2 t^\delta = w_2' t^{-\epsilon}$. It follows that $\epsilon = 0 = \delta$.
Case 5: (i) $w_1 t^\epsilon = w_1' t^{\delta}$ and (ii) $w_2 t^\delta = w_2' t^{-\epsilon+\delta}$. It follows that $\epsilon = 0 = \delta$.

Thus every endomorphism of G is of the form

$$\varphi = \begin{bmatrix} a & c & r \\ b & d & s \\ 0 & 0 & \gamma \end{bmatrix}$$

so that $N = \pi_1(T^2) = \langle \alpha_2, \alpha_3 | \alpha_2\alpha_3 = \alpha_3\alpha_2 \rangle$ is fully invariant.

For cases 9 and 10, the crystallographic group G is isomorphic to $\langle \alpha, \beta | \beta\alpha\beta^{-1} = \alpha^{-1} \rangle \rtimes_{\theta_i} \langle t \rangle$ where $i = 9, 10$ and

$$\theta_9(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \theta_{10}(t) = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}.$$

Here each of θ_9, θ_{10} is represented by a 2×2 array where the columns are the images of α, β under the action θ_i.

Case 9: Given an endomorphism φ, the relation $\varphi(t\beta t^{-1}) = \varphi(\beta^{-1})$ yields

$$\begin{aligned} &\alpha^r \beta^s t^\gamma \alpha^c \beta^d t^\delta t^{-\gamma} \beta^{-s} \alpha^{-r} = t^{-\delta} \beta^{-d} \alpha^{-c} \\ \Rightarrow &w_1 t^\delta = w_1' t^{-\delta}, \end{aligned}$$

for some words w_1, w_1' in α, β. It follows that $\delta = 0$.

Similarly the relation $\varphi(\beta\alpha\beta^{-1}) = \varphi(\alpha^{-1})$ yields

$$\begin{aligned} &\alpha^c \beta^d \alpha^a \beta^b t^\epsilon \beta^{-d} \alpha^{-c} = t^{-\epsilon} \beta^{-b} \alpha^{-a} \\ \Rightarrow &w_2 t^\epsilon = w_2' t^{-\epsilon}, \end{aligned}$$

for some words w_2, w_2' in α, β. It follows that $\epsilon = 0$.

Case 10: Given an endomorphism φ, similar to Case 9 above, the relation $\varphi(\beta\alpha\beta^{-1}) = \varphi(\alpha^{-1})$ yields $\epsilon = 0$. Now, the relation $\varphi(t\beta t^{-1}) = \varphi(\alpha\beta^{-1})$ yields

$$\begin{aligned} &\alpha^r \beta^s t^\gamma \alpha^c \beta^d t^\delta t^{-\gamma} \beta^{-s} \alpha^{-r} = \alpha^a \beta^b t^{-\delta} \beta^{-d} \alpha^{-c} \\ \Rightarrow &w_1 t^\delta = w_1' t^{-\delta}, \end{aligned}$$

for some words w_1, w_1' in α, β. It follows that $\delta = 0$.

Furthermore, for both cases 9 and 10, the relation $\varphi(\beta\alpha\beta^{-1}) = \varphi(\alpha^{-1})$ yields

$$\begin{aligned} \alpha^c \beta^d \alpha^a \beta^b \beta^{-d} \alpha^{-c} &= \beta^{-b} \alpha^{-a} \\ \Rightarrow \alpha^c (\alpha^{(-1)^d a})(\alpha^{(-1)^b(-c)}) \beta^b \alpha^a \beta^{-b} &= \beta^{-2b} \\ \Rightarrow \alpha^c (\alpha^{(-1)^d a})(\alpha^{(-1)^b(-c)})(\alpha^{(-1)^b a}) &= \beta^{-2b}. \end{aligned}$$

This implies that $b = 0$.

Thus for cases 9 and 10, every endomorphism is of the form

$$\varphi = \begin{bmatrix} a & c & r \\ 0 & d & s \\ 0 & 0 & \gamma \end{bmatrix} \tag{5.1}$$

so that $N = \pi_1(K) = \langle \alpha, \beta | \beta\alpha\beta^{-1} = \alpha^{-1} \rangle$ is fully invariant.

We are now ready to compute $N(f)$ for an arbitrary selfmap in the cases 2 - 5, 9,10.

5.1 Case 2

Using fiberwise techniques, it follows from (2.2) that the Nielsen number of a selfmap f is given by

$$N(f) = \sum_{i=0}^{|1-\gamma|-1} |\det(I - \theta^i(B))|.$$

Here, f induces on the fundamental group the endomorphism given by

$$\varphi = \begin{bmatrix} a & c & r \\ b & d & s \\ 0 & 0 & \gamma \end{bmatrix}$$

with $\deg \bar{f} = \gamma$ where $\bar{f} : S^1 \to S^1$ is the induced map on the base of the fibration $T^2 \to M \to S^1$. The matrix B is the restriction $\varphi|_{\mathbb{Z}^2} = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ and $\theta = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ is of order 2.

The relation $\varphi(t\alpha_3 t^{-1}) = \varphi(\alpha_3^{-1})$ yields

$$\begin{aligned} \alpha_2^r \alpha_3^s t^\gamma \alpha_2^c \alpha_3^d t^{-\gamma} \alpha_3^{-s} \alpha_2^{-r} &= \alpha_3^{-d} \alpha_2^{-c} \\ \Rightarrow \alpha_2^r \alpha_3^s \alpha_2^{(-1)^\gamma c} \alpha_3^{(-1)^\gamma d} \alpha_3^{-s} \alpha_2^{-r} &= \alpha_3^{-d} \alpha_2^{-c}. \end{aligned} \tag{5.2}$$

This implies that (1) γ is odd or $c = 0$ and (2) γ is odd or $d = 0$.

Similarly, the relation $\varphi(t\alpha_2 t^{-1}) = \varphi(\alpha_2^{-1})$ yields

$$\begin{aligned} \alpha_2^r \alpha_3^s t^\gamma \alpha_2^a \alpha_3^b t^{-\gamma} \alpha_3^{-s} \alpha_2^{-r} &= \alpha_3^{-b} \alpha_2^{-a} \\ \Rightarrow \alpha_2^r \alpha_3^s \alpha_2^{(-1)^\gamma a} \alpha_3^{(-1)^\gamma b} \alpha_3^{-s} \alpha_2^{-r} &= \alpha_3^{-d} \alpha_2^{-c}. \end{aligned} \tag{5.3}$$

This implies that (1) γ is odd or $a = 0$ and (2) γ is odd or $b = 0$. Thus, if γ is even then $B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and hence $N(f) = |1 - \gamma|$.

Suppose γ is odd then $B = \left[\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right]$ and $\theta B = \left[\begin{smallmatrix} -a & -c \\ -b & -d \end{smallmatrix}\right]$. It follows that $\det(I - B) = 1 + ad - bc - (a + d)$ and $\det(I - \theta B) = 1 + ad - bc + (a + d)$. When γ is odd, $|1 - \gamma|$ is even. (1) If $|1 + ad - bc| \geq |a + d|$ then we have

$$\begin{aligned} N(f) &= \frac{|1-\gamma|}{2}(|\det(I - B)| + |\det(I - \theta B)|) \\ &= |1 - \gamma| \cdot |1 + ad - bc|. \end{aligned}$$

(2) Otherwise, we have

$$N(f) = |1 - \gamma| \cdot |a + d|.$$

5.2 Case 3

In this case,

$$\theta = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \theta^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

and $\theta^3 = I$.

The relation $\varphi(t\alpha_2 t^{-1}) = \varphi(\alpha_3)$ yields

$$\alpha_2^r \alpha_3^s t^\gamma \alpha_2^a \alpha_3^b t^{-\gamma} \alpha_3^{-s} \alpha_2^{-r} = \alpha_2^c \alpha_3^d \tag{5.4}$$

and $\varphi(t\alpha_3 t^{-1}) = \varphi(\alpha_2^{-1}\alpha_3^{-1})$ yields

$$\alpha_2^r \alpha_3^s t^\gamma \alpha_2^c \alpha_3^d t^{-\gamma} \alpha_3^{-s} \alpha_2^{-r} = \alpha_2^{-a-c} \alpha_3^{-b-d}. \tag{5.5}$$

Suppose $\gamma \equiv 0 \bmod 3$. Then (5.4) implies that $a = c$ and $b = d$; (5.5) implies that $c = -a - c, d = b - d \Rightarrow a = b = c = d = 0$. Thus $B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $N(f) = |1 - \gamma|$.

Suppose $\gamma \equiv 1 \bmod 3$. Then (5.4) becomes

$$\alpha_2^r \alpha_3^s \alpha_3^a (\alpha_2^{-1}\alpha_3^{-1})^b \alpha_3^{-s} \alpha_2^{-r} = \alpha_2^c \alpha_3^d$$

This implies that $-b = c, a - b = d$ and (5.5) becomes

$$\alpha_2^r \alpha_3^s \alpha_3^c (\alpha_2^{-1}\alpha_3^{-1})^d \alpha_3^{-s} \alpha_2^{-r} = \alpha_2^{-a-c} \alpha_3^{-b-d}.$$

This implies that $-d = -a - c, c - d = -b - d$. It follows that $B = \left[\begin{smallmatrix} a & -b \\ b & a-b \end{smallmatrix}\right]$ so that $\det(I - B) = 1 + a^2 + b^2 - ab - 2a + b$. Moreover, we have $\theta B = \left[\begin{smallmatrix} -b & -a+b \\ a-b & -a \end{smallmatrix}\right]$ and $\theta^2 B = \left[\begin{smallmatrix} -a+b & a \\ -a & b \end{smallmatrix}\right]$. Now, we have $\det(I - \theta B) = 1 + a^2 + b^2 - ab + a + b$ and $\det(I - \theta^2 B) = 1 + a^2 + b^2 - ab + a - 2b$.

It is straightforward to show that $\det(I - B), \det(I - \theta B)$ and $\det(I - \theta^2 B)$ have the same sign. Thus, we conclude that

$$N(f) = (1 + a^2 + b^2 - ab) \cdot |1 - \gamma|.$$

Suppose $\gamma \equiv 2 \bmod 3$. Similar calculations show that $B = \left[\begin{smallmatrix} a & -a+b \\ b & -a \end{smallmatrix}\right]$, $\theta B = \left[\begin{smallmatrix} -b & a \\ a-b & b \end{smallmatrix}\right]$ and $\theta^2 B = \left[\begin{smallmatrix} b-a & -b \\ -a & a-b \end{smallmatrix}\right]$. It follows that $\det(I - B) = \det(I - \theta B) = \det(I - \theta^2 B) = 1 - a^2 - b^2 + ab$. Thus,

$$N(f) = |1 - a^2 - b^2 + ab| \cdot |1 - \gamma|.$$

5.3 Case 4

In this case,

$$\theta = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \theta^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \theta^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and $\theta^4 = I$.

The relation $\varphi(t\alpha_2 t^{-1}) = \varphi(\alpha_3)$ yields

$$\alpha_2^r \alpha_3^s t^\gamma \alpha_2^a \alpha_3^b t^{-\gamma} \alpha_3^{-s} \alpha_2^{-r} = \alpha_2^c \alpha_3^d \tag{5.6}$$

and $\varphi(t\alpha_3 t^{-1}) = \varphi(\alpha_2^{-1})$ yields

$$\alpha_2^r \alpha_3^s t^\gamma \alpha_2^c \alpha_3^d t^{-\gamma} \alpha_3^{-s} \alpha_2^{-r} = \alpha_2^{-a} \alpha_3^{-b}. \tag{5.7}$$

Note that $t^2\alpha_2 t^{-2} = \alpha_2^{-1}$ and $t^2\alpha_3 t^{-2} = \alpha_3$. When γ is even, we have $t^\gamma \alpha_3 t^{-\gamma} = \alpha_3$. Thus (5.6) becomes

$$\alpha_2^r \alpha_3^s \alpha_2^{(-1)^{(\gamma/2)}a} \alpha_3^b \alpha_3^{-s} \alpha_2^{-r} = \alpha_2^c \alpha_3^d$$

which then implies that $b = d$ and $(-1)^{(\gamma/2)}a = c$. Now (5.7) becomes

$$\alpha_2^r \alpha_3^s \alpha_2^{(-1)^{(\gamma/2)}c} \alpha_3^d \alpha_3^{-s} \alpha_2^{-r} = \alpha_3^{-b} \alpha_2^{-a}$$

which then implies that $d = -b$ and $(-1)^{(\gamma/2)}c = -a$. It follows that $b = d = 0$ and $c = a = 0$ so that $B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Hence, we have

$$N(f) = |1 - \gamma|.$$

When $\gamma \equiv 1 \bmod 4$, (5.6) becomes

$$\alpha_2^r \alpha_3^s \alpha_3^a \alpha_2^{-b} \alpha_3^{-s} \alpha_2^{-r} = \alpha_2^c \alpha_3^d$$

which implies that $a = d$ and $-b = c$. It follows that $B = \left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$. It is straightforward to see that $\det(I - B) = (1-a)^2 + b^2$, $\det(I - \theta B) = (1+b)^2 + a^2$, $\det(I - \theta^2 B) = (1+a)^2 + b^2$, and $\det(I - \theta^3 B) = (1-b)^2 + a^2$. Thus

$$N(f) = |1 - \gamma| \cdot (1 + a^2 + b^2).$$

When $\gamma \equiv 3 \bmod 4$, (5.6) becomes

$$\alpha_2^r \alpha_3^s \alpha_3^{-a} \alpha_2^{-b} \alpha_3^{-s} \alpha_2^{-r} = \alpha_2^c \alpha_3^d$$

which then implies that $c = -b$ and $d = -a$. It follows that $B = \left[\begin{smallmatrix} a & -b \\ b & -a \end{smallmatrix}\right]$. It is straightforward to see that $\det(I - B) = 1 - a^2 + b^2$, $\det(I - \theta B) = (1+b)^2 - a^2$, $\det(I - \theta^2 B) = 1 - a^2 + b^2$, and $\det(I - \theta^3 B) = (1-b)^2 - a^2$. It is not difficult to see that $\det(I - B), \det(I - \theta^i B)$, for $i = 1, 2, 3$, are either all non-positive or all non-negative. Thus

$$N(f) = |1 - \gamma| \cdot |1 - a^2 + b^2|.$$

5.4 Case 5

In this case,

$$\theta = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \qquad \theta^2 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \theta^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$\theta^4 = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad \theta^5 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

and $\theta^6 = I$.

The relation $\varphi(t\alpha_2 t^{-1}) = \varphi(\alpha_3)$ yields

$$\alpha_2^r \alpha_3^s t^\gamma \alpha_2^a \alpha_3^b t^{-\gamma} \alpha_3^{-s} \alpha_2^{-r} = \alpha_2^c \alpha_3^d \tag{5.8}$$

and $\varphi(t\alpha_3 t^{-1}) = \varphi(\alpha_2^{-1}\alpha_3)$ yields

$$\alpha_2^r \alpha_3^s t^\gamma \alpha_2^c \alpha_3^d t^{-\gamma} \alpha_3^{-s} \alpha_2^{-r} = \alpha_2^{c-a} \alpha_3^{d-b}. \tag{5.9}$$

Suppose $\gamma \equiv 0 \bmod 6$. Then (5.8) implies that $a = c, b = d$ and (5.9) implies that $c = c - a, d = d - b$. It follows that $a = b = c = d = 0$ and hence $B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

Suppose $\gamma \equiv 1 \bmod 6$. Then (5.8) implies that $-b = c, a + b = d$ and (5.9) implies that $-d = c - a, c + d = d - b$. It follows that $B = \left[\begin{smallmatrix} a & -b \\ b & a+b \end{smallmatrix}\right]$.

Suppose $\gamma \equiv 2 \bmod 6$. Then (5.8) implies that $-a - b = c, a = d$ and (5.9) implies that $-c - d = c - a, c = d - b$. It follows that $a = b = c = d = 0$ and hence $B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

Suppose $\gamma \equiv 3 \bmod 6$. Then (5.8) implies that $-a = c, -b = d$ and (5.9) implies that $-c = c - a, -d = d - b$. It follows that $a = b = c = d = 0$ and hence $B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

Suppose $\gamma \equiv 4 \bmod 6$. Then (5.8) implies that $b = c, -a - b = d$ and (5.9) implies that $d = c - a, -c - d = d - b$. It follows that $a = b = c = d = 0$ and hence $B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

Suppose $\gamma \equiv 5 \bmod 6$. Then (5.8) implies that $a + b = c, -a = d$ and (5.9) implies that $c + d = c - a, -c = d - b$. It follows that $B = \left[\begin{smallmatrix} a & a+b \\ b & -a \end{smallmatrix}\right]$.

Therefore, $N(f) = |\gamma - 1|$ if $\gamma \equiv 0, 2, 3, 4 \bmod 6$.

If $\gamma \equiv 1 \bmod 6$. Then we have

$$\begin{aligned}
\det(I - B) &= 1 + a^2 + b^2 - 2a - b + ab, \\
\det(I - \theta B) &= 1 + a^2 + b^2 - a + b + ab, \\
\det(I - \theta^2 B) &= 1 + a^2 + b^2 + a + 2b + ab, \\
\det(I - \theta^3 B) &= 1 + a^2 + b^2 + 2a + b + ab, \\
\det(I - \theta^4 B) &= 1 + a^2 + b^2 + a - b + ab, \\
\det(I - \theta^5 B) &= 1 + a^2 + b^2 - a - 2b + ab.
\end{aligned}$$

It is easy to see that $\det(I - B), \det(I - \theta^i B)$ for $i = 1, \ldots, 5$ are either all non-negative or all non-positive. It is straightforward to show that (3.1) yields

$$N(f) = |\gamma - 1| \cdot (1 + a^2 + b^2 + ab).$$

If $\gamma \equiv 5 \bmod 6$. Then we have

$$\det(I-B) = 1 - a^2 - b^2 - ab = \det(I-\theta B) = \det(I-\theta^2 B) = \det(I-\theta^3 B) = \det(I-\theta^4 B) = \det(I-\theta^5 B).$$

It is straightforward to show that (3.1) yields

$$N(f) = |\gamma - 1| \cdot |1 - a^2 - b^2 - ab|.$$

5.5 Case 9

Every endomorphism is of the form (5.1). Thus, the relation $\varphi(t\alpha t^{-1}) = \varphi(\alpha)$ yields

$$\begin{aligned} \alpha^r\beta^s t^\gamma \alpha^a t^{-\gamma}\beta^{-s}\alpha^{-r} &= \alpha^a \\ \Rightarrow \alpha^{(-1)^s a} &= \alpha^a \\ \Rightarrow (-1)^s a &= a. \end{aligned} \tag{5.10}$$

The relation $\varphi(t\beta t^{-1}) = \varphi(\beta^{-1})$ yields

$$\begin{aligned} \alpha^r\beta^s t^\gamma \alpha^c\beta^d t^{-\gamma}\beta^{-s}\alpha^{-r} &= \beta^{-d}\alpha^{-c} \\ \Rightarrow \alpha^{r+(-1)^s c + (-1)^{[(-1)^\gamma d]}(-r)}\beta^{(-1)^\gamma d} &= \alpha^{(-1)^d(-c)}\beta^{-d} \\ \Rightarrow r + (-1)^s c + (-1)^{[(-1)^\gamma d]}(-r) = (-1)^d(-c) \quad &\text{and} \quad (-1)^\gamma d = -d. \end{aligned} \tag{5.11}$$

When γ is even, (5.10) implies that s is even or $a = 0$. Similarly, (5.11) implies that $d = 0$ and also s is odd or $a = 0$. Now, if s is even then $c = 0$ and if s is odd then $a = 0$. Thus, these relations yield that φ has one of the following form:

$$\varphi = \begin{bmatrix} 0 & 0 & r \\ 0 & 0 & \text{even} \\ 0 & 0 & \text{even} \end{bmatrix} \quad \text{or} \quad \varphi = \begin{bmatrix} 0 & c & r \\ 0 & 0 & \text{odd} \\ 0 & 0 & \text{even} \end{bmatrix}$$

When γ is odd, similar calculations show that φ has one of the following form:

$$\varphi = \begin{bmatrix} 0 & c & r \\ 0 & \text{even} & \text{odd} \\ 0 & 0 & \text{odd} \end{bmatrix} \quad \text{or} \quad \varphi = \begin{bmatrix} 0 & 0 & r \\ 0 & \text{even} & \text{even} \\ 0 & 0 & \text{odd} \end{bmatrix}$$

when d is even and

$$\varphi = \begin{bmatrix} a & c & 0 \\ 0 & \text{odd} & \text{even} \\ 0 & 0 & \text{odd} \end{bmatrix} \quad \text{or} \quad \varphi = \begin{bmatrix} 0 & c & c \\ 0 & \text{odd} & \text{odd} \\ 0 & 0 & \text{odd} \end{bmatrix}$$

when d is odd.

Thus, if $\varphi = \begin{bmatrix} 0 & 0 & r \\ 0 & 0 & \text{even} \\ 0 & 0 & \text{even} \end{bmatrix}$ or $\varphi = \begin{bmatrix} 0 & c & r \\ 0 & 0 & \text{odd} \\ 0 & 0 & \text{even} \end{bmatrix}$ then (2.2) yields $N(f) = |\gamma - 1|$.

If $\varphi = \begin{bmatrix} 0 & c & r \\ 0 & \text{even} & \text{odd} \\ 0 & 0 & \text{odd} \end{bmatrix}$ then $|\gamma - 1|$ is even, $B = \left[\begin{smallmatrix} 0 & c \\ 0 & d \end{smallmatrix}\right]$ and $\theta B = \left[\begin{smallmatrix} 0 & c \\ 0 & -d \end{smallmatrix}\right]$. Now, (2.2) yields

$$N(f) = (|d-1| + |d+1|) \cdot \frac{|\gamma - 1|}{2} = \begin{cases} d \cdot |\gamma - 1|, & \text{if } d \geq 1; \\ |\gamma - 1|, & \text{if } d = 0; \\ (-d) \cdot |\gamma - 1|, & \text{if } d < 0. \end{cases} \tag{5.12}$$

In fact, the Nielsen number is given by (5.12) for the following types of endomorphisms:
$\varphi = \begin{bmatrix} 0 & 0 & r \\ 0 & \text{even} & \text{even} \\ 0 & 0 & \text{odd} \end{bmatrix}$ with $B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right]$ and $\theta B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & -d \end{smallmatrix}\right]$ or $\varphi = \begin{bmatrix} 0 & c & c \\ 0 & \text{odd} & \text{odd} \\ 0 & 0 & \text{odd} \end{bmatrix}$ with $B = \left[\begin{smallmatrix} 0 & c \\ 0 & d \end{smallmatrix}\right]$ and $\theta B = \left[\begin{smallmatrix} 0 & c \\ 0 & -d \end{smallmatrix}\right]$.

Finally, for the type $\varphi = \begin{bmatrix} a & c & 0 \\ 0 & \text{odd} & \text{even} \\ 0 & 0 & \text{odd} \end{bmatrix}$ with $B = \left[\begin{smallmatrix} a & c \\ 0 & d \end{smallmatrix}\right]$ and $\theta B = \left[\begin{smallmatrix} a & c \\ 0 & -d \end{smallmatrix}\right]$, (2.2) yields

$$N(f) = \begin{cases} (|a(d-1)| + |a(d+1)|) \cdot \frac{|\gamma-1|}{2}, & \text{if } a \neq 0; \\ (|d-1| + |d+1|) \cdot \frac{|\gamma-1|}{2}, & \text{if } a = 0. \end{cases}$$

5.6 Case 10

Every endomorphism is of the form (5.1). Thus, the relation $\varphi(t\alpha t^{-1}) = \varphi(\alpha)$ yields

$$\begin{aligned} \alpha^r \beta^s t^\gamma \alpha^a t^{-\gamma} \beta^{-s} \alpha^{-r} &= \alpha^a \\ \Rightarrow \alpha^{(-1)^s a} &= \alpha^a \\ \Rightarrow (-1)^s a &= a. \end{aligned} \tag{5.13}$$

This implies that s is even or $a = 0$.

The relation $\varphi(t\beta t^{-1}) = \varphi(\alpha\beta^{-1})$ yields

$$\alpha^r \beta^s t^\gamma \alpha^c \beta^d t^{-\gamma} \beta^{-s} \alpha^{-r} = \alpha^a \beta^{-d} \alpha^{-c}. \tag{5.14}$$

The relation $\varphi(\beta\alpha\beta^{-1}) = \varphi(\alpha^{-1})$ yields

$$\alpha^c (\alpha^{(-1)^d a})(\alpha^{(-c)})(\alpha^a) = 1.$$

This implies that d is odd or $a = 0$.

Straightforward calculations similar to those in Case 9 show that an endomorphism of G is of one of the following types:

$$\varphi = \begin{bmatrix} 0 & 0 & r \\ 0 & \text{even} & \text{even} \\ 0 & 0 & \text{odd} \end{bmatrix} \quad \text{or} \quad \varphi = \begin{bmatrix} 0 & 0 & r \\ 0 & 0 & \text{even} \\ 0 & 0 & \text{even} \end{bmatrix} \quad \text{or} \quad \varphi = \begin{bmatrix} 2r+\gamma & c & r \\ 0 & \text{odd} & \text{even} \\ 0 & 0 & \text{odd} \end{bmatrix}$$

when s is even or

$$\varphi = \begin{bmatrix} 0 & c & r \\ 0 & \text{even} & \text{odd} \\ 0 & 0 & \text{odd} \end{bmatrix} \quad \text{or} \quad \varphi = \begin{bmatrix} 0 & c & r \\ 0 & 0 & \text{odd} \\ 0 & 0 & \gamma \end{bmatrix}$$

when s is odd.

If $\varphi = \begin{bmatrix} 0 & 0 & r \\ 0 & \text{even} & \text{even} \\ 0 & 0 & \text{odd} \end{bmatrix}$ then d is even, $B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & d \end{smallmatrix}\right]$ and $\theta B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & -d \end{smallmatrix}\right]$. In fact, for all non-negative integer i, we have $\theta^i B = \theta^{i+2} B$. It follows from (2.2) that

$$N(f) = (|d-1| + |d+1|) \cdot \frac{|\gamma - 1|}{2} = \begin{cases} d \cdot |\gamma - 1|, & \text{if } d \geq 1; \\ |\gamma - 1|, & \text{if } d = 0; \\ (-d) \cdot |\gamma - 1|, & \text{if } d < 0. \end{cases} \tag{5.15}$$

Similarly, if $\varphi = \begin{bmatrix} 0 & c & r \\ 0 & \text{even} & \text{odd} \\ 0 & 0 & \text{odd} \end{bmatrix}$ then d is even, $B = \left[\begin{smallmatrix} 0 & c \\ 0 & d \end{smallmatrix}\right]$ and $\theta B = \left[\begin{smallmatrix} 0 & c \\ 0 & -d \end{smallmatrix}\right]$. For all non-negative integer i, we have $\theta^i B = \theta^{i+2} B$. Thus, the Nielsen number $N(f)$ is given by (5.15).

If

$$\varphi = \begin{bmatrix} 0 & 0 & r \\ 0 & 0 & \text{even} \\ 0 & 0 & \text{even} \end{bmatrix} \quad \text{or} \quad \varphi = \begin{bmatrix} 0 & c & r \\ 0 & 0 & \text{odd} \\ 0 & 0 & \gamma \end{bmatrix},$$

then $B = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ such that $N(f) = |\gamma - 1|$.

Finally, for type

$$\varphi = \begin{bmatrix} 2r+\gamma & c & r \\ 0 & \text{odd} & \text{even} \\ 0 & 0 & \text{odd} \end{bmatrix},$$

$$B = \begin{bmatrix} 2r+\gamma & c \\ 0 & d \end{bmatrix}, \quad \theta B = \begin{bmatrix} 2r+\gamma & c+1 \\ 0 & -d \end{bmatrix}, \cdots$$

such that

$$\theta^i B = \begin{bmatrix} 2r+\gamma & c+i \\ 0 & (-1)^i d \end{bmatrix}.$$

Since γ is odd, $|\gamma - 1|$ is even. Since d is odd, it follows from (2.2) that

$$N(f) = (|2r+\gamma||d-1| + |2r+\gamma|| - d - 1|) \cdot \frac{|\gamma - 1|}{2} = |(2r+\gamma)d(\gamma - 1)|.$$

6 Nielsen numbers of arbitrary selfmaps: Remaining Cases 1,7,8, and 6

In this section, we compute $N(f)$ for arbitrary selfmaps f on flat 3-manifolds in the four remaining cases. Case 1 is well-known. For case 7 and 8, the flat manifold is a S^1-bundle over the torus T^2 and every self-map is fiber-preserving since the subgroup corresponding to S^1 is fully-invariant. Moreover, the formula (2.2) is also valid in these situations and therefore can be used to compute $N(f)$. For case 6, we shall use (3.1) for the computation of the Nielsen number.

6.1 Case 1

The corresponding flat manifold is the 3-torus T^3 with fundamental group $\mathbb{Z}^3$. Given a selfmap f inducing an endomorphism φ on fundamental group, it is well-known that $N(f) = 0$ if $\det(I - \varphi) = 0$ and $N(f) = |\det(I - \varphi)|$ otherwise.

6.2 Case 7

In this case, G has the following presentation

$$G = \langle \alpha, \beta, t | \beta\alpha\beta^{-1} = \alpha^{-1}, t\alpha t^{-1} = \alpha, t\beta t^{-1} = \beta \rangle.$$

Let φ be an endomorphism given by the following 3×3 array

$$\varphi = \begin{bmatrix} a & c & r \\ b & d & s \\ \epsilon & \delta & \gamma \end{bmatrix}$$

where the columns are the images under φ of the generators α, β, t. The relation $\varphi(\beta\alpha\beta^{-1}) = \varphi(\alpha^{-1})$ yields

$$\begin{aligned}
&\alpha^c\beta^d t^\delta \alpha^a \beta^b t^\epsilon t^{-\delta} \beta^{-d} \alpha^{-c} = t^{-\epsilon}\beta^{-b}\alpha^{-a} \\
\Rightarrow &\alpha^c\beta^d\alpha^a\beta^b\beta^{-d}\alpha^{-c}t^\epsilon = \beta^{-b}\alpha^{-a}t^{-\epsilon} \quad \text{thus} \quad \epsilon = 0 \\
\Rightarrow &\alpha^c\beta^d\alpha^a\beta^b\beta^{-d}\alpha^{-c} = \beta^{-b}\alpha^{-a} \\
\Rightarrow &\alpha^c\alpha^{(-1)^d a}\beta^b\alpha^{-c} = \beta^{-b}\alpha^{-a} \\
\Rightarrow &\alpha^c\alpha^{(-1)^d a}\alpha^{(-1)^b(-c)}\beta^b = \alpha^{(-1)^b(-a)}\beta^{-b} \quad \text{thus} \quad b = 0 \\
\Rightarrow &\alpha^{c+(-1)^d a - c} = \alpha^{(-a)} \quad \Rightarrow (-1)^d a = -a.
\end{aligned} \tag{6.1}$$

This implies that d is odd or $a = 0$.

The relation $\varphi(t\beta t^{-1}) = \varphi(\beta)$ yields

$$\begin{aligned}
&\alpha^r\beta^s t^\gamma \alpha^c \beta^d t^\delta t^{-\gamma} \beta^{-s} \alpha^{-r} = \alpha^c\beta^d t^\delta \\
\Rightarrow &\alpha^r\beta^s\alpha^c\beta^d\beta^{-s}\alpha^{-r} = \alpha^c\beta^d.
\end{aligned} \tag{6.2}$$

Suppose d is even so $a = 0$. Moreover, β^d commutes with α so (6.2) becomes

$$\alpha^r\beta^s\alpha^c\beta^{-s}\alpha^{-r} = \alpha^c$$

This implies that s is even or $c = 0$. Thus, when d is even, we have

$$(i) \quad \varphi = \begin{bmatrix} 0 & c & r \\ 0 & \text{even} & \text{even} \\ 0 & \delta & \gamma \end{bmatrix} \qquad \text{or} \qquad (ii) \quad \varphi = \begin{bmatrix} 0 & 0 & r \\ 0 & \text{even} & \text{odd} \\ 0 & \delta & \gamma \end{bmatrix}.$$

Suppose d is odd. Then the relation $\varphi(\beta\alpha\beta^{-1}) = \varphi(\alpha)$ yields

$$\begin{aligned} &\alpha^r\beta^s t^\gamma \alpha^a t^{-\gamma}\beta^{-s}\alpha^{-r} = \alpha^a \\ \Rightarrow &\alpha^r\beta^s\alpha^a\beta^{-s}\alpha^{-r} = \alpha^a \\ \Rightarrow &(-1)^s a = a. \end{aligned} \tag{6.3}$$

This implies that s is even or $a = 0$. Now, (6.2) becomes

$$\begin{aligned} &\alpha^r\beta^s\alpha^c\beta^{-s}\beta^d\alpha^{-r} = \alpha^c\beta^d \\ \Rightarrow &\alpha^r\alpha^{(-1)^s c}\beta^d\alpha^{-r} = \alpha^c\beta^d \\ \Rightarrow &\alpha^r\alpha^{(-1)^s c}\alpha^{(-1)^d(-r)}\beta^d = \alpha^c\beta^d \\ \Rightarrow &r + (-1)^s c + (-1)^d(-r) = c. \end{aligned}$$

Now d is odd, so we have $2r + (-1)^s c = c$. It follows that if s is even then $r = 0$ and if s is odd then $r = c$.

Thus, when d is odd, we have

$$(iii) \quad \varphi = \begin{bmatrix} a & c & 0 \\ 0 & \text{odd} & \text{even} \\ 0 & \delta & \gamma \end{bmatrix} \qquad \text{or} \qquad (iv) \quad \varphi = \begin{bmatrix} 0 & c & c \\ 0 & \text{odd} & \text{odd} \\ 0 & \delta & \gamma \end{bmatrix}.$$

For the cases $(i), (ii), (iv)$, the Nielsen number is $N(f) = |(1-d)(1-\gamma) - \delta s|$. For case (iii), since d is odd and s is even, $|(1-d)(1-\gamma) - \delta s|$, which is the Nielsen number of the map $\bar{f}$ on the base T^2, must be even. Since the base torus has fundamental group generated by β and t whereas the fiber S^1 has fundamental group generated by α, the action of $\pi_1(T^2)$ on the fiber is induced by the relation $\beta\alpha\beta^{-1} = \alpha^{-1}$. It follows that we have

$$N(f) = (|1-a| + |1+a|) \cdot \frac{|(1-d)(1-\gamma) - \delta s|}{2}.$$

6.3 Case 8

In this case, G has the following presentation

$$G = \langle \alpha, \beta, t | \beta\alpha\beta^{-1} = \alpha^{-1}, t\alpha t^{-1} = \alpha, t\beta t^{-1} = \alpha\beta \rangle.$$

Calculations similar to those in Case 7 show that any endomorphism is of one of the following types:

When d is even, we have

$$(i) \quad \varphi = \begin{bmatrix} 0 & c & r \\ 0 & \text{even} & \text{even} \\ 0 & \delta & \gamma \end{bmatrix} \qquad \text{or} \qquad (ii) \quad \varphi = \begin{bmatrix} 0 & c & r \\ 0 & \text{even} & \text{odd} \\ 0 & -2c & \gamma \end{bmatrix}.$$

When d is odd, we have

$$(iii) \quad \varphi = \begin{bmatrix} 0 & c & \frac{1}{2}(2c - \delta - \gamma) \\ 0 & \text{odd} & \text{odd} \\ 0 & \delta & \gamma \end{bmatrix} \qquad \text{or} \qquad (iv) \quad \varphi = \begin{bmatrix} 2r+\gamma & c & r \\ 0 & \text{odd} & \text{even} \\ 0 & \delta & \gamma \end{bmatrix}.$$

For the cases $(i), (ii), (iii)$, the Nielsen number is $N(f) = |(1-d)(1-\gamma) - \delta s|$. For case (iv), similar arguments as in Case 7 show that

$$N(f) = (|1-2r-\gamma| + |1+2r+\gamma|) \cdot \frac{|(1-d)(1-\gamma)-\delta s|}{2}.$$

6.4 Case 6

In this final case, we make use of the calculations already done in subsection 4.1. For any endomorphism φ, the restriction φ' on the maximal abelian subgroup is of one of the six forms as in (4.1) or $\varphi' = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$. For this latter type of endomorphisms, $N(f) = 1$. We now compute the Nielsen number of a selfmap which induces an endomorphism φ given by

$$\varphi = \begin{bmatrix} a & c & r \\ b & d & s \\ \epsilon & \delta & t \end{bmatrix}$$

where the columns are the images under φ of the generators t_1, α_2, t_3. We will make use of the restriction φ' of φ to the maximal abelian subgroup and φ' can be represented by a 3×3 matrix where the columns are images under φ' of the generators $\alpha_1, \alpha_2, \alpha_3$.

Suppose φ' is of type (II) and (III'), that is, $\varphi' = \left[\begin{smallmatrix} 0 & 0 & r \\ 0 & -2s-2b-1 & 0 \\ \epsilon & 0 & 0 \end{smallmatrix}\right]$. It follows that

$$\begin{aligned} \det(I - \varphi') &= (2+2s+2b)(1-r\epsilon), \\ \det(I - \theta_1\varphi') &= (2s+2b)(-1-r\epsilon), \\ \det(I - \theta_2\varphi') &= (2s+2b)(-1-r\epsilon), \\ \det(I - \theta_3\varphi') &= (2+2s+2b)(1-r\epsilon). \end{aligned}$$

It follows that

$$N(f) = \frac{1}{4}(4|1+s+b||1-r\epsilon| + 4|s+b||1+r\epsilon|).$$

Suppose φ' is of type (II) and (IV'), that is, $\varphi' = \left[\begin{smallmatrix} 0 & r-a & 0 \\ 0 & 0 & -2s-1 \\ \epsilon & 0 & 0 \end{smallmatrix}\right]$. It follows that

$$\begin{aligned} \det(I - \varphi') = 1 - (a-r)\epsilon(2s+1) = \det(I - \theta_1\varphi') = \\ \det(I - \theta_2\varphi') = \det(I - \theta_3\varphi'). \end{aligned}$$

It follows that

$$N(f) = |1 - (a-r)\epsilon(2s+1)|.$$

Suppose φ' is of type (III) and (II'), that is, $\varphi' = \left[\begin{smallmatrix} a & 0 & 0 \\ 0 & 2(s+b)+1 & 0 \\ 0 & 0 & t \end{smallmatrix}\right]$. It follows that

$$\begin{aligned} \det(I - \varphi') &= (1-a)(1-t)(-2(s+b)), \\ \det(I - \theta_1\varphi') &= (1-a)(1+t)(2+2(s+b)), \\ \det(I - \theta_2\varphi') &= (1+a)(1-t)(2+2(s+b)), \\ \det(I - \theta_3\varphi') &= (1+a)(1+t)(-2(s+b)). \end{aligned}$$

It follows that

$$N(f) = \frac{1}{4}(|(1-a)(1-t)(-2(s+b))| + |(1-a)(1+t)(2+2(s+b))| + \\ |(1+a)(1-t)(2+2(s+b))| + |(1+a)(1+t)(-2(s+b))|).$$

Suppose φ' is of type (III) and (IV′), that is, $\varphi' = \left[\begin{smallmatrix} a & 0 & 0 \\ 0 & 0 & -2s-1 \\ 0 & \epsilon-t & 0 \end{smallmatrix}\right]$. It follows that

$$\begin{aligned} \det(I-\varphi') &= (1-a)(1-(2s+1)(t-\epsilon)), \\ \det(I-\theta_1\varphi') &= (1-a)(1-(2s+1)(t-\epsilon)), \\ \det(I-\theta_2\varphi') &= (1+a)(1+(2s+1)(t-\epsilon)), \\ \det(I-\theta_3\varphi') &= (1+a)(1+(2s+1)(t-\epsilon)). \end{aligned}$$

It follows that

$$N(f) = \frac{1}{4}(2|(1-a)(1-(2s+1)(t-\epsilon))| + 2|(1+a)(1+(2s+1)(t-\epsilon))|).$$

Suppose φ' is of type (IV) and (II′), that is, $\varphi' = \left[\begin{smallmatrix} 0 & r-a & 0 \\ -2b-1 & 0 & 0 \\ 0 & 0 & t \end{smallmatrix}\right]$. It follows that

$$\begin{aligned} \det(I-\varphi') &= (1-(a-r)(2b+1))(1-t), \\ \det(I-\theta_1\varphi') &= (1+(a-r)(2b+1))(1+t), \\ \det(I-\theta_2\varphi') &= (1-(a-r)(2b+1))(1-t), \\ \det(I-\theta_3\varphi') &= (1+(a-r)(2b+1))(1+t). \end{aligned}$$

It follows that

$$N(f) = \frac{1}{4}(2|(1-(a-r)(2b+1))(1-t)| + 2|(1+(a-r)(2b+1))(1+t)|).$$

Suppose φ' is of type (IV) and (III′), that is, $\varphi' = \left[\begin{smallmatrix} 0 & 0 & r \\ -2b-1 & 0 & 0 \\ 0 & \epsilon-t & 0 \end{smallmatrix}\right]$. It follows that

$$\det(I-\varphi') = 1 - r(2b+1)(t-\epsilon) = \det(I-\theta_1\varphi') = \\ \det(I-\theta_2\varphi') = \det(I-\theta_3\varphi').$$

It follows that

$$N(f) = |1 - r(2b+1)(t-\epsilon)|.$$

7 Jiang-type condition

Recall that a space M is of Jiang-type or M satisfies the Jiang-type condition, if for any selfmap $f : M \to M$, either $L(f) = 0 \Rightarrow N(f) = 0$ or $L(f) \neq 0 \Rightarrow N(f) = R(f)$. Here, $L(f), N(f), R(f)$ denote the Lefschetz, Nielsen, and Reidemeister numbers of f respectively. A group G is said to have property R_∞ if for all $\varphi \in \mathrm{Aut}(G)$, $R(\varphi) = \infty$.

In [5], flat and nilmanifolds whose fundamental groups possess property R_∞ were constructed. In particular, it was shown that for any $n \geq 5$, there is a compact nilmanifold of dimension n such that every homeomorphism is isotopic to a fixed point free homeomorphism. This is due to the fact that nilmanifolds are known to be of Jiang-type and by constructing finitely generated nilpotent groups with R_∞ property, such a nilmanifold has the property that every self homeomorphism f must have $N(f) = 0$. It is therefore natural to ask whether there exists manifold M that is not of Jiang-type but $N(f) = 0$ for every self homeomorphism f (see Remark 7.1). In this section, we determine which of the flat 3-manifolds are of Jiang-type.

For Case 1, the 3-torus, it is well-known that the Jiang type condition is satisfied.

For Case 2, the flat manifold is a torus bundle over S^1. Consider the fiberwise homeomorphism which induces on the fundamental group of the base the homomorphism given by multiplication by -1 and on the fundamental group of the fiber the automorphism given by the matrix

$$B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

The Lefschetz number of this map restricted to one fiber has value -2 but the Lefschetz number restricted to the other fiber, by routine calculation, is 6. Therefore the indices of the Nielsen classes have different values, i.e., 2 classes have index -1 and 6 classes have index $+1$. Now, consider a homeomorphism which induces on the fundamental group of the base the homomorphism given by multiplication by -1 and on the fundamental group of the fiber the automorphism given by the matrix

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The Lefschetz number of this map restricted to one fiber is 0 but the Lefschetz number restricted to the other fiber, by routine calculation, is 4. This implies that the Nielsen number is 4 but the Reidemeister number is infinite. Therefore the Jiang type condition does not hold.

For Cases 3-5, none of these manifolds is of Jiang type. For Case 3 (section 5.2), consider the map inducing $\gamma \equiv 1 \bmod 3$ with $a = 1$ and $b = 0$ so that $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. In this case, $\det(I - B) = 0$ so that $R(f) = \infty$. For Case 4 (section 5.3), consider the map inducing $\gamma \equiv 3 \bmod 4$ with $a = 0$ and $b = 1$ so that $B = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. In this case, $\det(I - \theta^3 B) = 0$ so that $R(f) = \infty$. For Case 5 (section 5.4), consider the map inducing $\gamma \equiv 1 \bmod 6$ with $a = 1$ and $b = 0$ so that $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. In this case, $\det(I - B) = 0$ so that $R(f) = \infty$. Thus, we conclude that the Jiang type condition does not hold in general in any of these three cases.

For the remaining Cases 6-10, each of these flat manifolds is not of Jiang-type. For Case 6, one can choose a self-homeomorphism (see section 4.1 and section 6.4) of type (II), (III′) with $r = \epsilon = 1, s = 0, b = -1$ so that $N(f) = 2 = |L(f)|$ but $R(f) = \infty$. For Cases 7-8 (see sections 4.2-4.3), there exist homeomorphisms f so that $N(f) = |L(f)| \neq 0$ but $R(f) = \infty$. Similarly for Cases 9-10, see sections 4.4-4.5.

For convenience, we summarize our results in the following table:

G	NSH(M)	Jiang Type
1	$\mathbb{N} \cup \{0\}$	Yes
2	$2\mathbb{N} \cup \{0\}$	No
3	$\{0\}$	No
4	$\{0\}$	No
5	$\{0\}$	No
6	$\{0, 2\}$	No
7	$2\mathbb{N} \cup \{0\}$	No
8	$2\mathbb{N} \cup \{0\}$	No
9	$\{0, 2\}$	No
10	$\{0, 2\}$	No

Remark 7.1. Based upon our calculations, the flat manifolds in Cases 3-5 have the property that they are not of Jiang-type but every self-homeomorphism has zero Nielsen number while $N(f) = |L(f)|$ (see e.g. [9, 10]) and their fundamental groups have property R_∞ (see [8]).

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