$(\mathbf{M}, cr^\gamma, \delta)$-minimizing curve regularity

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Abstract

This is a new proof that $(\mathbf{M}, Cr^\gamma, \delta)$-minimizing sets S are pieces of $\mathcal{C}^{1,\gamma/2}$ curves, $0 < \gamma \leqslant 1$. To obtain this result, the almost monotonicity property is established for balls centered on S or not. Furthermore it is proved that almost minimizing sets fulfill the epiperimetric inequality.

1 Introduction

The ξ-almost minimizers with respect to a finite boundary set B, $\xi(r)$ being a nondecreasing function tending to 0 as $r \to 0$, are compact connected 1-rectifiable sets S such that

$$\mathcal{H}^1(S \cap B(x,r)) \leqslant (1 + \xi(r))\mathcal{H}^1(C \cap B(x,r))$$

whenever

(a) $B(x,r) \cap B = \emptyset$,

(b) C is a compact connected 1-rectifiable set with $S \setminus B(x,r) = C \setminus B(x,r)$.

We assume $r < \delta$ and notation $B(x,r)$ indicates an open ball.

This definition slightly differs from that given by Almgren [1] in what we do not require comparison sets to be Lipschitz images of the original set.

There is an interesting class of such ξ-almost minimizers. Consider a function $\alpha : \mathbb{R}^n \to \mathbb{R}$ uniformly continuous and bounded below by some $\alpha_0 > 0$. For each Borel set $S \subset \mathbb{R}^n$ we put

$$E_\alpha(S) := \int_S \alpha d\mathcal{H}^1 .$$

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If $B \subset \mathbb{R}^n$ is finite we claim that there exists a compact connected 1-rectifiable set $S^* \subset \mathbb{R}^n$ containing B and such that

$$E_\alpha(S^*) = \inf\{E_\alpha(S) : B \subset S \subset \mathbb{R}^n,\ B \text{ is compact, connected and 1-rectifiable}\}\,.$$

This follows from the Blaschke selection principle [4, Theorem 3.16] together with a restricted lower semicontinuity property of E_α analogous to [4, Theorem 3.18]. It is then easy to check that S^* is ξ-almost minimizing with respect to B where

$$\xi(r) = \frac{\omega_\alpha(r)}{\alpha_0}\,,$$

$\omega_\alpha(r) := \sup\{\operatorname{diam}(\alpha(B(x,r))) : x \in \mathbb{R}^n\}$ being the modulus of continuity of α.

Another large class of ξ-almost minimizers consists of the simple $\mathcal{C}^{1,\gamma}$ curves themselves ($0 < \gamma \leqslant 1$). They can be described by an arclength parametrization $\lambda : [a,b] \to \mathbb{R}^n$ such that

$$\left|\lambda'(t_1) - \lambda'(t_2)\right|_2 \leqslant C\,|t_1 - t_2|^\gamma\,.$$

Then one can check (see [2]) that $\lambda([a,b])$ is ξ-almost minimizing with respect to $B = \{\lambda(a), \lambda(b)\}$, where $\xi(r) = C' r^{2\gamma}$.

Moreover, solutions of other variational problems, like networks of bubbles in the plane (see [1]), meet also the requirements of ξ-almost minimizing sets.

We give a new look at the regularity of ξ-almost minimizers. The main result is the following.

Theorem 1. *Let $S \subset \mathbb{R}^n$ be compact connected 1-rectifiable. Assume that $y \in S$. Let $0 < \gamma \leqslant 1$ and $C > 0$. Then the following conditions are equivalent:*

(A) $\Theta^1(S,y) = 1$ and, in a neighborhood of y, S is ξ-almost minimizing with $\xi(r) \leqslant Cr^\gamma$,

(B) in a neighborhood of y, S is a simple $\mathcal{C}^{1,\frac{\gamma}{2}}$ curve.

The present paper provides a new method for proving $(A) \Rightarrow (B)$, the first one being due to Morgan [6], and $(B) \Rightarrow (A)$ can be found in [2] for example.

2 Sketch of proof

First we show that Cr^γ-almost minimizers S are almost monotonic near y. This means that there exists $R > 0$ such that for $x \in B(y,R)$ and $0 < r \leqslant R$ the function

$$e^{Cr^\gamma}\frac{\mathcal{H}^1(S \cap B(x,r))}{2r} \tag{1}$$

is nondecreasing. Through this paper the constant C is allowed to increase from one estimate to another but only depends on γ.

Notice that the monotonicity formula of J. Taylor in [8] is actually obtained for balls centered on S. In this way this result is significantly different.

The next step is to improve inequality (1) for points $x \in S \cap B(y,R)$. The so-called epiperimetry property is the following. For $x \in S \cap B(y,R)$ and $0 < r \leqslant R$ we have that

$$\left| \frac{\mathcal{H}^1(S \cap B(x,r))}{2r} - 1 \right| \leqslant Cr^\gamma. \tag{2}$$

Next we study Cr^γ-almost minimizers taking into account the epiperimetry property. Let S be such a set. Let $x \in S \cap B(y,R)$ and $0 < r \leqslant R$. Then there exists a line $L_{x,r}$ through the origin which is a good approximation of S in $B(x,r)$ in the sense that

$$d_{\mathcal{H}}\left(S \cap B(x,r), (x + L_{x,r}) \cap B(x,r)\right) \leqslant Cr^{1+\frac{\gamma}{2}}, \tag{3}$$

where $d_{\mathcal{H}}$ is the Hausdorff distance. It is a Reifenberg-like property (see [7]). Notice that the exponent $\frac{\gamma}{2}$ instead of γ follows the use of Pythagoras' Formula. This explains why we get $\mathcal{C}^{1,\frac{\gamma}{2}}$-regularity at the end.

Thanks to (3) we obtain that

(i) the approximation lines $L_{x,r}$ stabilize to a unique line L_x whenever $r \to 0$,

(ii) $d_{\mathcal{H}}\left(L_{x_1} \cap B(0,1), L_{x_2} \cap B(0,1)\right) \leqslant C|x_1 - x_2|^{\frac{\gamma}{2}}_{2}$.

Finally, for $r > 0$ small enough, $S \cap B(0,r)$ is the graph of a function u over L_0. Observing that L_x is also the tangent line to graph(u) in the sense of the classical derivative, the fact that u is $\mathcal{C}^{1,\frac{\gamma}{2}}$ follows from (ii).

The fact that almost monotonicity together with epiperimetry imply regularity near points of density 1 has been observed in [2]. The novelty here is the fact that the epiperimetric inequality (2) follows in dimension 1 from a comparison argument.

3 Almost minimizing sets

The first definition is just a convenient abbreviation. The second one sets the class of objects of our interest.

Definition 1. *A* gauge *is a nondecreasing function $\xi : \mathbb{R}_+^* \to \mathbb{R}_+$ such that* $\lim_{r \to 0^+} \xi(r) = 0$.

Definition 2. *Given a gauge ξ we say that a compact connected 1-rectifiable set $S \subset \mathbb{R}^n$ is* ξ-almost minimizing *with respect to a finite set $B \subset \mathbb{R}^n$ provided that the following conditions hold:*

(a) $B \subset S$,

(b) for every $r > 0$ and $x \in \mathbb{R}^n$ with $B(x,r) \cap B = \emptyset$ one has

$$\mathcal{H}^1(S \cap B(x,r)) \leqslant (1 + \xi(r))\mathcal{H}^1(C \cap B(x,r)) \tag{4}$$

whenever $C \subset \mathbb{R}^n$ is a compact connected 1-rectifiable set such that

$$S \setminus B(x,r) = C \setminus B(x,r).$$

In case ξ vanishes identically we simply say that S is minimizing *with respect to B.*

We have an immediate geometric information about these sets.

Proposition 1. *Let $S \subset \mathbb{R}^n$ be ξ-almost minimizing with respect to B, $x \in S$ and $0 < r < \operatorname{dist}(x, B)$. Then* $\operatorname{card}[S \cap \partial B(x,r)] \geqslant 2$.

Proof. The connectedness of S implies that $X = S \cap \partial B(x,r)$ is not empty. Assuming if possible that X were a singleton, the almost minimizing property (4) applied with $C := (S \setminus B(x,r)) \cup X$ would yield

$$0 < r \leqslant \mathcal{H}^1(S \cap B(x,r)) \leqslant (1+\xi(r))\mathcal{H}^1(C \cap B(x,r)) = (1+\xi(r))\mathcal{H}^1(X) = 0,$$

a contradiction. ∎

4 Almost monotonic measures

We introduce here the concept of almost monotonic measure.

Definition 3. *Given an open set $U \subset \mathbb{R}^n$, a Radon measure ϕ on U and a gauge ζ, we say that ϕ is ζ-*almost monotonic in *U if for every $x \in U$ the function*

$$]0, \operatorname{dist}(x, \mathbb{R}^n \setminus U)[\to \mathbb{R}_+ : r \mapsto e^{\zeta(r)} \frac{\phi(B(x,r))}{2r}$$

is nondecreasing. In case ζ vanishes identically we simply say that ϕ is monotonic in *U.*

The purpose of this section is to show that if S is ξ-almost minimizing with respect to B then $\mathcal{H}^1 \llcorner S$ is ζ-almost monotonic in $\mathbb{R}^n \setminus B$ with respect a gauge ζ depending on ξ (if $\xi(r) = Cr^\gamma$, $0 < \gamma \leqslant 1$, then ζ will be a multiple of ξ).

Lemma 1. *If S is ξ-almost minimizing with respect to B then for every $x \in \mathbb{R}^n \setminus B$ and every $\rho \in]0, \operatorname{dist}(x,B)[$ one has*

$$\mathcal{H}^1(S \cap B(x,\rho)) \leqslant (1+\xi(\rho))\,\rho \operatorname{card}[S \cap \partial B(x,\rho)].$$

Proof. Fix $\rho \in]0, \operatorname{dist}(x,B)[$, set $l := \operatorname{card}[S \cap \partial B(x,\rho)]$, and let $c_1, \dots, c_l$ be the members of $S \cap \partial B(x,\rho)$. If $[x, c_i]$ denotes the closed line segment joining x and c_i, then (4) applied with $C := (S \setminus B(x,\rho)) \cup \bigcup_{i=1}^{l} [x, c_i]$ yields the inequality. ∎

Lemma 2. *If S is 1-rectifiable and $\mathcal{H}^1(S) < \infty$ then for every $x \in \mathbb{R}^n$ and $\mathcal{H}^1$-almost every $\rho > 0$ one has*

$$\frac{d}{d\rho}\mathcal{H}^1(S \cap B(x,\rho)) \geqslant \operatorname{card}[S \cap \partial B(x,\rho)].$$

Proof. Fix $x \in \mathbb{R}^n$ an define $\varphi(\rho) := \mathcal{H}^1(S \cap B(x,\rho))$, $\rho > 0$. The coarea formula [5, 3.2.22] applied to the set S and the function $\delta_x(y) := |y - x|$ shows that

$$\varphi'_{ac}(\rho) = \sum_{y \in S \cap \partial B(x,\rho)} \frac{1}{J_1 \delta_x(y)} \geqslant \sum_{y \in S \cap \partial B(x,\rho)} 1 = \operatorname{card}[S \cap \partial B(x,\rho)]$$

where φ_{ac} is the absolutely continuous part of φ in its Lebesgue decomposition as nondecreasing function. Since $\varphi'(\rho) = \varphi'_{ac}(\rho)$ for $\mathcal{H}^1$-almost every $\rho > 0$ the conclusion follows. ■

We are now able to proof the expected result.

Proposition 2. *If S is ξ-almost minimizing with respect to B then $\mathcal{H}^1 \llcorner S$ is ζ-almost monotonic in $\mathbb{R}^n \setminus B$ where*

$$\zeta(r) := \int_0^r \frac{\xi(\rho)}{\rho}\, d\rho\,, \quad \text{provided that} \quad \lim_{r\to 0^+} \int_r^1 \frac{\xi(\rho)}{\rho}\, d\rho < +\infty.$$

Proof. Let $x \in \mathbb{R}^n \setminus B$. Define $\varphi(\rho) := \mathcal{H}^1(S \cap B(x,\rho))$, $0 < \rho < \operatorname{dist}(x, B)$. By Lemmas 1 and 2, we have that

$$\varphi(\rho) \leqslant (1 + \xi(\rho))\, \rho\varphi'(\rho)$$

whenever $\varphi'(\rho)$ is defined. Consequently, for such ρ,

$$\frac{d}{d\rho}\{\ln[\varphi(\rho)]\} \geqslant \frac{(1-\xi(\rho))}{\rho} = \frac{d}{d\rho}\left\{\ln\left[\rho e^{-\zeta(\rho)}\right]\right\}.$$

Thus, for $0 < r < R < \operatorname{dist}(x, B)$, we obtain

$$\int_r^R \frac{d}{d\rho}\left\{\ln\left[\rho e^{-\zeta(\rho)}\right]\right\} d\rho \leqslant \int_r^R \frac{d}{d\rho}\{\ln[\varphi(\rho)]\}\, d\rho$$

and

$$e^{\zeta(r)}\frac{\varphi(r)}{2r} \leqslant e^{\zeta(R)}\frac{\varphi(R)}{2R}.$$

This shows that the function

$$]0, \operatorname{dist}(x, B)[\; \to \mathbb{R} : \rho \mapsto e^{\zeta(\rho)}\frac{\mathcal{H}^1(S \cap B(x,\rho))}{2\rho}$$

is nondecreasing. ■

5 Epiperimetry

The regularity study of almost minimal sets is based on the epiperimetry property.

Definition 4. *Let ϕ be a gauge and let $R > 0$. We say that a compact connected 1-rectifiable set $S \subset \mathbb{R}^n$ S has the* epiperimetry property at scales less than R about $y \in S$ with respect to ϕ *if for every $x \in S \cap B(y, R)$ and every $0 < r < R$ the inequality*

$$\left|\frac{\mathcal{H}^1(S \cap B(x,r))}{2r} - 1\right| \leqslant \phi(r)$$

is satisfied.

The aim is to show that an almost minimizing set S has the epiperimetry property about points $y \in S$ with $\Theta^1(S,y) = 1$. This will follow from the properties of almost minimality and almost monotonicity.

Proposition 3. *If S is an almost minimizing set with respect to B, then for every $x \in S \setminus B$ the density $\Theta^1(S,x)$ exists and is larger than 1.*

Proof. Recall from Section 4 that $\mathcal{H}^1 \llcorner S$ is almost monotonic in $\mathbb{R}^n \setminus B$. This easily imply that $x \mapsto \Theta^1(S,x)$ is upper semicontinuous in $\mathbb{R}^n \setminus B$, see e.g. [3, Lemma 3.3]. On the other hand 1-rectifiable sets have their 1-density $\mathcal{H}^1$-almost everywhere equal to 1. As $\mathcal{H}^1$-negligible sets have empty interior, this completes the proof. ∎

We will now count the number of intersection points of an almost minimizing set S and circles centered on S.

Lemma 3. *Let S be a ξ-almost minimizing set with respect to B, let $x \in S \setminus B$ and $0 < r_0 < \operatorname{dist}(x,B)$. Assume that $\xi(r_0) < 1/8$ and that* $\operatorname{card}[S \cap \partial B(x,r_0)] \leqslant 2$. *Then*

$$\mathcal{L}^1\left(\{\rho \in [0,r_0] : \operatorname{card}[S \cap \partial B(x,\rho)] \geqslant 3\}\right) < \tfrac{3}{4} r_0.$$

Proof. Tchebysheff's inequality yields

$$\begin{aligned}\mathcal{L}^1&\left(\{\rho \in [0,r_0] : \operatorname{card}[S \cap \partial B(x,\rho)] \geqslant 3\}\right)\\ &\leqslant \frac{1}{3}\int_{\{\rho\in[0,r_0]:\operatorname{card}[S\cap\partial B(x,\rho)]\geqslant 3\}} \operatorname{card}[S \cap \partial B(x,\rho)]\, d\rho.\end{aligned}$$

Next it follows from coarea formula that

$$\begin{aligned}\int_{\{\rho\in[0,r_0]:\operatorname{card}[S\cap\partial B(x,\rho)]\geqslant 3\}} \operatorname{card}[S \cap \partial B(x,\rho)]\, d\rho &\leqslant \int_0^{r_0} \operatorname{card}[S \cap \partial B(x,\rho)]\, d\rho\\ &\leqslant \mathcal{H}^1(S \cap B(x,r_0)).\end{aligned}$$

Finally letting $\{c_1, c_2\}$ be the members of $S \cap \partial B(x,r_0)$ and applying the almost minimizing property (4) with $C := (S \setminus B(x,r_0)) \cup ([x,c_1] \cup [x,c_2])$, we obtain

$$\mathcal{H}^1(S \cap B(x,r_0)) \leqslant (1+\xi(r_0))\operatorname{card}[S \cap \partial B(x,r_0)]\, r_0 \leqslant (1+\xi(r_0))2r_0.$$

Consequently,

$$\begin{aligned}\mathcal{L}^1\left(\{\rho \in [0,r_0] : \operatorname{card}[S \cap \partial B(x,\rho)] \geqslant 3\}\right) &\leqslant \frac{1}{3}(1+\xi(r_0))2r_0\\ &< \frac{1}{3}\left(1+\frac{1}{8}\right)2r_0 \leqslant \frac{3}{4}r_0,\end{aligned}$$

what was announced. ∎

The importance of the last result lies in the existence of $\rho \in]0,r_0]$ such that $\operatorname{card}[S \cap \partial B(x,\rho)] \leqslant 2$ for all r_0 less than a certain value independent of x. The next proposition summarizes this.

Proposition 4. *Let S be a ξ-almost minimizing set with respect to B, let $x \in S \setminus B$ and $0 < r_0 < \operatorname{dist}(x, B)$. Assume that $\xi(r_0) < 1/8$ and that* $\operatorname{card}[S \cap \partial B(x, r_0)] \leqslant 2$. *Then there exists a sequence $(r_j)_{j \in \mathbb{N}^*}$ satisfying*

$$\lim_{j\to\infty} r_j = 0, \quad \operatorname{card}\left[S \cap \partial B(x, r_j)\right] \leqslant 2 \quad \textit{and} \quad \frac{r_j}{8} \leqslant r_{j+1} \leqslant r_j.$$

Proof. By Lemma 3, there exists $r_1 \in [\frac{1}{8}r_0, \frac{7}{8}r_0]$ such that $\operatorname{card}[S \cap \partial B(x, r_1)] \leqslant 2$. Then we obtain $r_2 \in [\frac{1}{8^2}r_1, \frac{7^2}{8^2}r_1]$ such that $\operatorname{card}[S \cap \partial B(x, r_2)] \leqslant 2$. Continuing to apply recursively Lemma 3 completes the proof. ∎

We have now obtained the existence of a sequence of "good radii" provided there is a good one to start with. The almost monotonicity will now give the existence of the starting "good radius".

Lemma 4. *Let μ be a Radon measure ζ-almost monotonic in U with $\Theta^1_*(\mu, x) \geqslant 1$ for μ-almost every $x \in U$. Assume that $\Theta^1(\mu, y) = 1$. Then*

$$(\forall \delta > 0)(\exists 0 < t < 1)\left(\exists 0 < r_0 < \tfrac{1}{3}\operatorname{dist}(y, B) : \zeta(r_0) \leqslant \tfrac{1}{8}\right)$$
$$(\forall x \in \operatorname{spt}\mu \cap B(y, tr_0)) : \frac{\mu(B(x, r_0))}{2r_0} \leqslant 1 + \delta.$$

Proof. Let $\delta > 0$. Assume that $0 < t < 1$ and $x \in \operatorname{spt}\mu \cap B(y, tr_0)$. Then we have

$$\begin{aligned}
\frac{\mu(B(x,r_0))}{2r_0} &\leqslant \frac{\mu(B(y, |x-y|_2 + r_0))}{2r_0} \\
&= \frac{\mu(B(y, |x-y|_2 + r_0))}{2(|x-y|_2 + r_0)}\left(1 + \frac{|x-y|_2}{r_0}\right) \\
&\leqslant e^{\zeta(tr_0 + r_0) - \zeta(|x-y|_2 + r_0)}\frac{\mu(B(y, tr_0 + r_0))}{2(tr_0 + r_0)}\left(1 + \frac{tr_0}{r_0}\right) \\
&\leqslant e^{\zeta((1+t)r_0)}\frac{\mu(B(y, (1+t)r_0))}{2((1+t)r_0)}(1+t).
\end{aligned}$$

Letting t tend to 0, the last term becomes $e^{\zeta(r_0)}$ which is smaller than $1 + \delta$ if r_0 is small enough. ∎

Proposition 5. *Let S be a ξ-almost minimizing set with respect to B and $y \in S \setminus B$ with $\Theta^1(S, y) = 1$. Then*

$$(\exists 0 < t < 1)\left(\exists 0 < r_0 < \tfrac{1}{3}\operatorname{dist}(y, B) : \zeta(r_0) < \tfrac{1}{8}\right)$$
$$(\forall x \in S \cap B(y, tr_0))(\exists r(x) \in \left[\tfrac{1}{4}r_0, r_0\right]) : \operatorname{card}\left[S \cap \partial B(x, r(x))\right] \leqslant 2,$$

ζ being the gauge associated with the almost monotonicity of $\mathcal{H}^1 \llcorner S$ in $\mathbb{R}^n \setminus B$.

Proof. Consider $\mu := \mathcal{H}^1 \llcorner S$ in Lemma 4. Taking $\delta := \frac{1}{8}$, we obtain $0 < t < 1$ and $0 < r_0 < \frac{1}{3}\operatorname{dist}(y, B)$ with $\zeta(r_0) < \frac{1}{8}$, such that

$$(\forall x \in S \cap B(y, \gamma r_0)) : \frac{\mathcal{H}^1(S \cap B(x, r_0))}{2r_0} \leqslant 1 + \frac{1}{8}.$$

Let $x \in S \cap B(y, tr_0)$. Then Tchebysheff's inequality and coarea formula imply

$$\mathcal{L}^1\left(\{\rho \in [0, r_0] \,:\, \mathrm{card}\,[S \cap \partial B(x,\rho)] \geqslant 3\}\right) \leqslant \frac{1}{3}\int_0^{r_0} \mathrm{card}\,[S \cap \partial B(x,\rho)]\; d\rho$$
$$\leqslant \frac{1}{3}\mathcal{H}^1(S \cap B(x, r_0)) < \frac{3}{4}r_0.$$

Therefore there exists $r(x) \in \left[\frac{1}{4}r_0, r_0\right]$ such that $\mathrm{card}\,[S \cap \partial B(x, r(x))] \leqslant 2$. ∎

Here is the main result of this section.

Theorem 2. *Assume $\xi(r)$ is a gauge such that $\xi(r) \leqslant Cr^\gamma$ for $C > 0$ and $0 < \gamma \leqslant 1$. Then there exists a gauge $\phi(r) = C'r^\gamma$ having the following property. If S is a ξ-almost minimizing set with respect to B and if $y \in S \setminus B$ with $\Theta^1(S, y) = 1$, then there exists $0 < R$ such that*

(i) S has the epiperimetry property at scales lower than R about y with respect to the gauge $\phi(r)$,

(ii) for all $x \in S \cap B(y,R)$ and all $r \in]0, R]$: $\mathrm{card}\,[S \cap \partial B(x, r)] = 2$.

Proof. The second part follows from (i) and Proposition 1. It is thus enough to prove (i). Without loss of generality we can assume that $\xi(r) = Cr^\gamma$. Set

$$\zeta(r) := \int_0^r \frac{\xi(\rho)}{\rho}\, d\rho = \frac{C}{\gamma} r^\gamma$$

so that $\mathcal{H}^1 \llcorner S$ is ζ-almost monotonic in $\mathbb{R}^n \setminus B$ (Section 4). Define $\phi(r) := \kappa\zeta(r)$ for $\kappa \geqslant 1$ to be determined.

As $\Theta^1(S, x) = 1$, by Proposition 5, there exists $0 < t < 1$ and $0 < r_0 < \frac{1}{3}\mathrm{dist}(y, B)$, with $\zeta(r_0) < \frac{1}{8}$, such that

$$(\forall x \in S \cap B(y, tr_0))(\exists r(x) \in \left[\tfrac{1}{4}r_0, r_0\right]) : \mathrm{card}\,[S \cap \partial B(x, r(x))] \leqslant 2.$$

Set $R := \min\left(tr_0, \frac{1}{4}r_0\right)$.

Let $x \in S \cap B(y, R)$ and show that, if $0 < \rho \leqslant R$, then

$$-\phi(\rho) \leqslant \frac{\mathcal{H}^1(S \cap B(x,\rho))}{2\rho} - 1 \leqslant \phi(\rho). \tag{5}$$

The left inequality in (5) is a consequence of almost monotonicity (Definition 3) and the fact that $\Theta^1(S, x) \geqslant 1$ (Proposition 3). Indeed

$$1 - \phi(\rho) \leqslant 1 - \zeta(\rho) \leqslant e^{-\zeta(\rho)} \leqslant \frac{\mathcal{H}^1(S \cap B(x,\rho))}{2\rho}.$$

Let us establish the right hand inequality of (5) for $0 < \rho \leqslant r(x)$ (recall $r(x) \geqslant R$). It suffices to show that $\mathcal{H}^1(S \cap B(x, \rho)) \leqslant (1 + \zeta(\rho))2\rho$, $0 < \rho \leqslant r(x)$. Proceeding toward a contradiction, assume there exists a "bad radius" $0 < r^* \leqslant r(x)$ such that

$$\mathcal{H}^1(S \cap B(x, r^*)) > (1 + \zeta(r^*))2r^*. \tag{6}$$

Consider a sequence $(r_j)_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} r_j = 0$, $\operatorname{card}\left[S\cap\partial B(x,r_j)\right] \leqslant 2$, $\frac{r_j}{8} \leqslant r_{j+1} \leqslant r_j$ and $r_0 = r(x)$, for example that given by Proposition 4. The value $J := \max\left\{j\in\mathbb{N} \mid r^* \leqslant r_j\right\}$ is such that $\frac{r_J}{8} \leqslant r^* \leqslant r_J$. Consequently, inequality (6), almost monotonicity, almost minimality and the fact that $\xi \leqslant \zeta$ imply that

$$\begin{aligned} e^{\zeta(r^*)}(1+\phi(r^*)) &< e^{\zeta(r^*)}\frac{\mathcal{H}^1(S\cap B(x,r^*))}{2r^*} \\ &\leqslant e^{\zeta(r_J)}\frac{\mathcal{H}^1(S\cap B(x,r_J))}{2r_J} \\ &\leqslant e^{\zeta(r_J)}(1+\xi(r_J))\operatorname{card}\left[S\cap\partial B(x,r_J)\right] \\ &\leqslant e^{\zeta(r_J)}(1+\zeta(r_J)), \end{aligned}$$

a contradiction provided we choose κ sufficiently large. ∎

6 Regularity

6.1 Existence of approximation lines

Proposition 6. *Let $\xi(r)$ be a gauge such that $\xi(r) \leqslant Cr^\gamma$ for $C > 0$ and $0 < \gamma \leqslant 1$. Let $S \subset \mathbb{R}^n$ be a ξ-almost minimizing set with respect to B and let $y \in S\setminus B$ with $\Theta^1(S,y) = 1$. Let $R > 0$ be the radius associated with the epiperimetry property of S about y with respect to the gauge ϕ. Let $x \in S\cap B(y,R)$ and $0 < r \leqslant R$. Assume that $\phi(r) \leqslant \frac{1}{5}$. Then there exists $L_{x,r} \in G(n,1)$ such that*

$$d_{\mathcal{H}}\left(S\cap B(x,r), (x+L_{x,r})\cap B(x,r)\right) \leqslant 8r\sqrt{\phi(r)}.$$

Proof. Let $\{c_1, c_2\}$ be the members of $S\cap\partial B(x,r)$. We first show that

$$d_{\mathcal{H}}\left(S\cap B(x,r), [c_1,x]\cup[c_2,x]\right) \leqslant \sqrt{2}r\sqrt{\phi(r)}. \tag{7}$$

Let S_1 be the curve contained in $S\cap B(x,r)$ with endpoints x and c_1 and S_2 with endpoints x and c_1. We know that $\mathcal{H}^1(S_1) \leqslant (1+\phi(r))2r - r = r + 2r\phi(r)$ since $\mathcal{H}^1(S_2) \geqslant r$. Let z be a point of S_1 maximizing the distance to the line segment $[c_1,x]$ and let h be this distance. Let m be the middle of $[c_1,x]$. Let y be a point of the plane xc_1z at distance h of $[c_1,x]$ whose projection on this line segment is m. Set $l := \mathcal{H}^1([x,y]\cup[y,c_1])$. We have

$$l = \mathcal{H}^1([x,y]\cup[y,c_1]) \leqslant \mathcal{H}^1([x,z]\cup[z,c_1]) \leqslant \mathcal{H}^1(S_1).$$

Using Pythagoras' Theorem and the preceding inequalities,

$$h = \frac{1}{2}\sqrt{l^2-r^2} \leqslant \frac{1}{2}\sqrt{(r+2r\phi(r))^2-r^2} = r\sqrt{\phi(r)^2+\phi(r)} \leqslant \sqrt{2}r\sqrt{\phi(r)}$$

(the last inequality happens since $0 \leqslant \phi(r) \leqslant 1$). As h bounds the distance from any point of $[c_1,x]$ to S_1, we showed that $d_{\mathcal{H}}(S_1,[c_1,x]) \leqslant \sqrt{2}r\sqrt{\phi(r)}$. Applying again this argument to S_2, we obtain $d_{\mathcal{H}}\left(S_2, [c_2, x]\right) \leqslant \sqrt{2}r\sqrt{\phi(r)}$. So the same conclusion is true for $d_{\mathcal{H}}\left(S \cap B(x,r), [c_1,x] \cup [c_2,x]\right)$. This proves (7).

Now set $L_{x,r} := \operatorname{span}\{c_1 - x\}$ and show this line satisfies the desired inequality. According to (7), we have

$$d_{\mathcal{H}}\left(S \cap B(x,r), [c_1,x] \cup [c_2,x]\right) \leqslant \sqrt{2}r\sqrt{\phi(r)}.$$

Using this, let us estimate the quantity $d_{\mathcal{H}}\left([c_1,x] \cup [c_2,x], (x + L_{x,r}) \cap B(x,r)\right)$.

Define $l := \mathcal{H}^1\left([c_1,c_2]\right)$. Notice that the inequality of almost minimality (4) implies that

$$l = \mathcal{H}^1([c_1,c_2]) \geqslant \frac{\mathcal{H}^1(S \cap B(x,r))}{1+\phi(r)} \geqslant \frac{2r}{1+\phi(r)}.$$

In the rest of the proof, θ will be the angle opposite to $[c_1,c_2]$ in the triangle xc_1c_2 and will be $\beta = \pi - \theta$.

According to the "generalized Pythagoras' Formula",

$$\cos\beta = \frac{l^2 - 2r^2}{2r^2} \geqslant \frac{\left(\frac{2r}{1+\phi(r)}\right)^2 - 2r^2}{2r^2} = \frac{2 - (1+\phi(r))^2}{(1+\phi(r))^2}.$$

Thus,

$$\sin\beta = \sqrt{1-\cos^2\beta} \leqslant \sqrt{1 - \left(\frac{2-(1+\phi(r))^2}{(1+\phi(r))^2}\right)^2} = \frac{2\sqrt{(1+\phi(r))^2 - 1}}{(1+\phi(r))^2} \leqslant 4\sqrt{\phi(r)}$$

(the last inequality comes from the fact that $(1+\phi(r))^2 \geqslant 1$ and $0 \leqslant \phi(r) \leqslant \frac{1}{5}$). This implies that

$$d_{\mathcal{H}}\left([c_1,x] \cup [c_2,x], (x + L_{x,r}) \cap B(x,r)\right) = r\sin\beta \leqslant 4r\sqrt{\phi(r)}$$

and the statement follows from triangular inequality. ∎

6.2 Behavior of the approximation lines

The behavior of the approximation lines is governed by the following two properties.

Proposition 7. *Let $\xi(r)$ be a gauge such that $\xi(r) \leqslant Cr^\gamma$ for $C > 0$ and $0 < \gamma \leqslant 1$. Let $S \subset \mathbb{R}^n$ be a ξ-almost minimizing set with respect to B and let $y \in S \setminus B$ with $\Theta^1(S,y) = 1$. Let $R > 0$ be the radius associated with the epiperimetry property of S about y with gauge ζ. Let $x \in S \cap B(y,R)$ and $0 < r \leqslant R$. Assume that $\phi(R) \leqslant \frac{1}{5}$. Then the angle ψ between the lines $L_{x,r}$ and $L_{x,R}$ satisfies*

$$|\sin\psi| \leqslant 8\left(1 + \frac{R}{r}\right)\sqrt{\phi(R)}.$$

Proof. Let $y_0 \in (x + L_{x,r}) \cap \partial B(x,r)$. Let us consider the right-angled triangle whose angle is ψ, the hypotenuse is r and the side opposite to ψ is $\operatorname{dist}(y_0, x + L_{x,R})$. By Proposition 6, there exists $x_0 \in B(x,r) \cap S$ such that $|y_0 - x_0|_2 \leqslant 8r\sqrt{\phi(r)}$ and $y_1 \in (x + L_{x,R}) \cap B(x,r)$ such that $|y_1 - x_0|_2 \leqslant 8R\sqrt{\phi(R)}$. This shows that $\operatorname{dist}(y_0, x + L_{x,R}) \leqslant 8r\sqrt{\phi(r)} + 8R\sqrt{\phi(R)}$ and the conclusion follows. ■

Proposition 8. *Let $\xi(r)$ be a gauge such that $\xi(r) \leqslant Cr^\gamma$ for $C > 0$ and $0 < \gamma \leqslant 1$. Let $S \subset \mathbb{R}^n$ be a ξ-almost minimizing set with respect to B and let $y \in S \setminus B$ with $\Theta^1(S,y) = 1$. Let $R > 0$ be the radius associated with the epiperimetry property of S about y with gauge ϕ. Let $x \in S \cap B(y,R)$ and $0 < r \leqslant R$. Assume that $\phi(r) \leqslant \frac{1}{5}$ and that that $B(x,r) \subset B(y,R)$. Then, for all $0 < \rho \leqslant r$ and all $x_1, x_2 \in (B(x,r) \cap S) \setminus U(x,\rho)$, the angle ψ between the lines $L_{x_1,r}$ and $L_{x_2,r}$ satisfies*

$$|\sin\psi| \leqslant 32\frac{r}{\rho}\sqrt{\phi(r)}.$$

Proof. Let L be the line passing through x_1 and x. This line meets $x_1 + L_{x_1,r}$ and $x + L_{x,r}$. If θ_1 is the angle between L and $x_1 + L_{x_1,r}$, if θ_2 is the angle between L and $x + L_{x,r}$ and if ψ_1 is the angle between $L_{x_1,r}$ and $L_{x,r}$, we have that $|\psi_1| \leqslant |\theta_1| + |\theta_2|$. Consequently, $|\sin\psi_1| \leqslant |\sin\theta_1| + |\sin\theta_2|$. As $x_1 \in B(x,r) \cap S$, by Proposition 6, there exists $y_1 \in x + L_{x,r}$ such that $|x_1 - y_1|_2 \leqslant 8r\sqrt{\phi(r)}$. So, if $p(x_1)$ is the orthogonal projection of x_1 onto $x + L_{x,r}$,

$$|\sin\theta_1| = \frac{|p(x) - x_1|_2}{|x - x_1|_2} \leqslant \frac{8r\sqrt{\phi(r)}}{|x - x_1|_2} \leqslant \frac{8r\sqrt{\phi(r)}}{\rho}.$$

With same estimate on $|\sin\theta_2|$, we obtain that $|\sin\psi_1| \leqslant 16\frac{r}{\rho}\sqrt{\phi(r)}$. Swapping x_1 and x_2, we find that $|\sin\psi_2| \leqslant 16\frac{r}{\rho}\sqrt{\phi(r)}$, where ψ_2 is the angle between $L_{x_2,r}$ and $L_{x,r}$. Since $|\psi| \leqslant |\psi_1| + |\psi_2|$, we have $|\sin\psi| \leqslant |\sin\psi_1| + |\sin\psi_2|$ hence $|\sin\psi| \leqslant 32\frac{r}{\rho}\sqrt{\phi(r)}$. ■

6.3 Limit lines

Here is the result claiming that the lines of approximation stabilize as the scale tends to 0.

Proposition 9 (existence). *Let $\xi(r)$ be a gauge such that $\xi(r) \leqslant Cr^\gamma$ for $C > 0$ and $0 < \gamma \leqslant 1$. Let $S \subset \mathbb{R}^n$ be a ξ-almost minimizing set with respect to B and let $y \in S \setminus B$ with $\Theta^1(S,y) = 1$. Let $R > 0$ be the radius associated with the epiperimetry property of S about y with respect to the gauge ϕ. Let $x \in S \cap B(y,R)$. Then there exists $L_x \in G(n,1)$ such that, for all $0 < r \leqslant R$ with $\phi(r) \leqslant \frac{1}{5}$, we have*

$$d_{\mathcal{H}}(L_x \cap B(0,1), L_{x,r} \cap B(0,1)) \leqslant Er^{\frac{\gamma}{2}},$$

$E > 0$ being a real number depending only on γ.

Proof. Define the sequence $(r_j)_{j\in\mathbb{N}}$ by $r_j := \frac{r}{2^j}$. Set $\phi(r) = C'r^\gamma$ like in Theorem 2. Then Proposition 7 implies that

$$d_{\mathcal{H}}\left(L_{x,r_{j+l}} \cap B(0,1), L_{x,r_j} \cap B(0,1)\right) \leqslant 24\sqrt{C'}\sum_{k=j}^{j+l} r_k^{\frac{\gamma}{2}} = 24\sqrt{C'}r^{\frac{\gamma}{2}}\sum_{k=j}^{j+l}\left(2^{-\frac{\gamma}{2}}\right)^k. \quad (8)$$

Since $0 < 2^{-\frac{\gamma}{2}} < 1$, $(L_{x,r_j} \cap B(0,1))_{j\in\mathbb{N}}$ is a Cauchy sequence in $(\mathcal{K}(B(0,1)), d_{\mathcal{H}})$ which is complete. So there exists $L_x \in G(n,1)$ such that $\lim_{j\to\infty} L_{x,r_j} = L_x$. Finally, taking $j = 0$ and letting $l \to \infty$, we obtain the result with $E := \frac{24\sqrt{C'}}{1-2^{-\frac{\gamma}{2}}}$. ∎

Proposition 10 (oscillations). *Let $\xi(r)$ be a gauge such that $\xi(r) \leqslant Cr^\gamma$ for $C > 0$ and $0 < \gamma \leqslant 1$. Let $S \subset \mathbb{R}^n$ be a ξ-almost minimizing set with respect to B and let $y \in S \setminus B$ with $\Theta^1(S,y) = 1$. Let $R > 0$ be the radius associated with the epiperimetry property of S about y with respect to the gauge ϕ. Let $x_1, x_2 \in S \cap B(y,R)$ and set $r :=$ dist(x_1,x_2). Assume that $0 < r \leqslant R$ and $\phi(r) \leqslant \frac{1}{5}$. Then*

$$d_{\mathcal{H}}\left(L_{x_1} \cap B(0,1), L_{x_2} \cap B(0,1)\right) \leqslant F|x_1 - x_2|_2^{\frac{\gamma}{2}},$$

$F > 0$ being a real number depending only on γ.

Proof. By Proposition 8 applied with $r = \rho$ and $x = x_1$ and Proposition 9,

$$\begin{aligned} d_{\mathcal{H}}\left(L_{x_1} \cap B(0,1), L_{x_2} \cap B(0,1)\right) &\leqslant d_{\mathcal{H}}\left(L_{x_1} \cap B(0,1), L_{x_1,r} \cap B(0,1)\right) \\ &\quad + d_{\mathcal{H}}\left(L_{x_1,r} \cap B(0,1), L_{x_2,r} \cap B(0,1)\right) \\ &\quad + d_{\mathcal{H}}\left(L_{x_2,r} \cap B(0,1), L_{x_2} \cap B(0,1)\right) \\ &\leqslant Er^{\frac{\gamma}{2}} + 32r^{\frac{\gamma}{2}} + Er^{\frac{\gamma}{2}}, \\ &= Fr^{\frac{\gamma}{2}} \end{aligned}$$

where $F := 2E + 32$. ∎

6.4 Representation by a graph above the limit line

We will show that a minimizing set is the graph of a function of class $\mathcal{C}^{1,\frac{\gamma}{2}}$ above the "stabilized line" about each point of 1-density equal to 1. We start by obtaining this function. The following definition will be useful in this way.

Definition 5. *Given $S \subset \mathbb{R}^n$, $x_0 \in S$, $r_0 > 0$, $\rho_0 > 0$, $\sigma > 0$ and $W_0 \in G(n,m)$ we set*

$$G(S, x_0, r_0, \rho_0, \sigma, W_0) := \{x \in S \cap B(x_0, r_0) \mid \forall \rho \in]0, \rho_0] : \\ S \cap B(x,\rho) \subset B(x + W_0, \sigma\rho)\}.$$

Lemma 5. *Define $G := G(S, x_0, r_0, \rho_0, \sigma, W_0)$. Choose $0 < \sigma < 1$ and $2r_0 \leqslant \rho_0$. Then there exists a function $u : p_{W_0}(G) \to W_0^\perp$ such that $G =$ graph(u) and*

$$\operatorname{lip} u \leqslant \frac{\sigma}{\sqrt{1-\sigma^2}}.$$

Proof. See [3, Lemma 8.2]. ∎

Proposition 11. *Let $\xi(r)$ be a gauge such that $\xi(r) \leqslant Cr^{\gamma}$ for $C > 0$ and $0 < \gamma \leqslant 1$. Let $S \subset \mathbb{R}^n$ be a ξ-almost minimizing set with respect to B and let $y \in S \setminus B$ with $\Theta^1(S, y) = 1$. Let $R > 0$ be the radius associated with the epiperimetry property of S about y with respect to the gauge ϕ. Then, given $0 < \sigma < 1$, there exists $r_0 = r_0(R, \gamma, \sigma)$ and $\rho_0 = \rho_0(R, \gamma, \sigma) \geqslant 2r_0$ such that for all $x_0 \in S \cap B(y, R)$:*

$$S \cap B(x_0, r_0) = G(S, x_0, r_0, \rho_0, \sigma, L_{x_0}).$$

Proof. Let ϕ be the gauge associated with the epiperimetry property. Take $0 < \sigma < 1$. Set

$$r_0 := \frac{1}{2} \min\left(\left(\frac{\sigma}{H}\right)^{\frac{2}{\gamma}}, R, \inf \phi^{-1}\left(\left\{\tfrac{1}{5}\right\}\right)\right)$$

where $H := 4 + E + \frac{F}{2^{\frac{\gamma}{2}}}$, E and F being constants obtained at Section 6.3. Finally, simply set $\rho_0 := 2r_0$.

Choose $x_0 \in S \cap B(y, R)$, $x \in S \cap B(x_0, r_0)$ and $\rho \in]0, \rho_0]$. We are going to show that $S \cap B(x, \rho) \subset B(x + L_{x_0}, \sigma\rho)$. We have that

$$\begin{aligned}
&d_{\mathcal{H}}\left(L_{x,\rho} \cap B(0,1), L_{x_0} \cap B(0,1)\right) \\
&\leqslant d_{\mathcal{H}}\left(L_{x,\rho} \cap B(0,1), L_x \cap B(0,1)\right) + d_{\mathcal{H}}\left(L_x \cap B(0,1), L_{x_0} \cap B(0,1)\right) \\
&\leqslant E\rho^{\frac{\gamma}{2}} + F\,|x - x_0|_2^{\frac{\gamma}{2}} \\
&\leqslant E\rho_0^{\frac{\gamma}{2}} + Fr_0^{\frac{\gamma}{2}} \\
&= \left(E + \frac{F}{2^{\frac{\gamma}{2}}}\right)\rho_0^{\frac{\gamma}{2}}.
\end{aligned}$$

In $B(x, \rho)$, this yields

$$d_{\mathcal{H}}\left((x + L_{x,\rho}) \cap B(x,\rho), (x + L_{x_0}) \cap B(x,\rho)\right) \leqslant \left(E + \frac{F}{2^{\frac{\gamma}{2}}}\right)\rho\rho_0^{\frac{\gamma}{2}}.$$

Moreover, Proposition 6 says that

$$d_{\mathcal{H}}\left(S \cap B(x,\rho), (x + L_{x,\rho}) \cap B(x,\rho)\right) \leqslant 4\rho^{1+\frac{\gamma}{2}} \leqslant 4\rho\rho_0^{\frac{\gamma}{2}}.$$

But then

$$\begin{aligned}
&d_{\mathcal{H}}\left(S \cap B(x,\rho), (x + L_{x_0}) \cap B(x,\rho)\right) \\
&\leqslant d_{\mathcal{H}}\left(S \cap B(x,\rho), (x + L_{x,\rho}) \cap B(x,\rho)\right) \\
&\qquad + d_{\mathcal{H}}\left((x + L_{x,\rho}) \cap B(x,\rho), (x + L_{x_0}) \cap B(x,\rho)\right) \\
&\leqslant \left(4 + E + \frac{F}{2^{\frac{\gamma}{2}}}\right)\rho\rho_0^{\frac{\gamma}{2}} \\
&= H\rho\rho_0^{\frac{\gamma}{2}} \\
&\leqslant H\rho\left(\left(\frac{\sigma}{H}\right)^{\frac{2}{\gamma}}\right)^{\frac{\gamma}{2}} \\
&= \rho\sigma.
\end{aligned}$$

Therefore

$$S \cap B(x,\rho) \subset B((x + L_{x_0}) \cap B(x,\rho), \sigma\rho)$$

and, consequently, $S \cap B(x,\rho) \subset B((x + L_{x_0}), \sigma\rho)$. ∎

We need the following lemma whose easy proof is left to the reader.

Lemma 6. *Let $\Omega \subset \mathbb{R}$ be open. Given $u : \Omega \to \mathbb{R}^N$ such that*

- $\operatorname{lip} u < +\infty$,
- *there exists $C > 0$ and $0 < \eta \leqslant 1$ such that, if u is derivable at $t_1, t_2 \in \Omega$, we have*

$$\left|u'(t_1) - u'(t_2)\right|_2 \leqslant C|t_1 - t_2|^\eta,$$

then u is $\mathcal{C}^{1,\eta}$ on Ω.

Now let us proof the main result.

Theorem 3. *Let $S \subset \mathbb{R}^n$ be compact, connected and 1-rectifiable. Assume that there exists a finite set $B \subset \mathbb{R}^n$ and a gauge $\xi(r) \leqslant Cr^\gamma$, $C > 0$ and $0 < \gamma \leqslant 1$, such that S is ξ-almost minimizing with respect to B. Let $x \in S \setminus B$ with $\Theta^1(S,x) = 1$. Then there exists $r > 0$ such that $S \cap B(x,r)$ is a simple curve of class $\mathcal{C}^{1,\frac{\gamma}{2}}$.*

Proof. Let $R > 0$ be the radius associated with the epiperimetry property of S about x. Choose $\sigma \in]0,1[$ and consider the radius $r_0 > 0$ given by preceding proposition. So we have that $S \cap B(x,r_0) = G(S,x,r_0,\rho_0,\sigma,L_x)$ for a certain ρ_0.

According to Lemma 5, there exists a function

$$u : p_{L_x}(S \cap B(x,r_0)) \to L_x^\perp$$

such that $S \cap B(x,r_0) = \operatorname{graph}(u)$ and $\operatorname{lip} u$ is a finite constant which depends only on σ.

By Rademacher's Theorem [5, 3.1.6], the function u is derivable almost everywhere.

If $u'(t)$ exists, then it is easy to see that $(1,u'(t)) \in L_{(t,u(t))}$. Consequently, if u is derivable at t_1 and t_2, we have

$$\begin{aligned}
\left|u'(t_1) - u'(t_2)\right|_2 &\leqslant \left(1 + (\operatorname{lip} u)^2\right) d_{\mathcal{H}}\left(L_{(t_1,u(t_1))} \cap B(0,1), L_{(t_2,u(t_2))} \cap B(0,1)\right) \\
&\leqslant \left(1 + (\operatorname{lip} u)^2\right) F|(t_1,u(t_1)) - (t_2,u(t_2))|_2^{\frac{\gamma}{2}} \\
&\leqslant \left(1 + (\operatorname{lip} u)^2\right)^{1+\frac{\gamma}{4}} F|t_1 - t_2|^{\frac{\gamma}{2}}.
\end{aligned}$$

Moreover the definition domain of u, $p_{L_x}(S \cap B(x,r_0))$, is a closed bounded interval, say $[a_1,a_2]$. So, thanks to Lemma 6 applied to the $n-1$ components of u, u has class $\mathcal{C}^{1,\frac{\gamma}{2}}$ on $]a_1,a_2[$.

A short computation shows that

$$\left(1 + (\operatorname{lip} u)^2\right)^{-\frac{1}{2}} r_0 \leqslant |p_{L_x}(x) - a_i| \leqslant r_0,\ 1 \leqslant i \leqslant 2.$$

Setting $\Delta := \left(1 + (\operatorname{lip} u)^2\right)^{-\frac{1}{2}} r_0$, value which depends only on R and γ, we get that

$$S \cap B(x,\Delta) \subset \operatorname{graph}\left(u|_{[p_{L_x}(x)-\Delta, p_{L_x}(x)+\Delta]}\right)$$

is the graph of a function of class $\mathcal{C}^{1,\frac{\gamma}{2}}$. Then it suffices to put $r := \min(R,\Delta)$. ∎

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