Inclusion theorems for Cohen strongly summing multilinear operators

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Abstract

In this paper we investigate connections between the class of Cohen strongly summing multilinear operators and other classes of multilinear mappings, such as multiple summing and strongly summing mappings (in the sense of Dimant). As a consequence of our results, we show that if Y is a $\mathcal{L}_{p^*}$-space and $X_1, ..., X_m$ are $\mathcal{L}_p$-spaces ($1 < p < \infty$ and $1/p + 1/p^* = 1$), then every multiple p^*-summing m-linear operator is strongly p^*-summing.

Introduction

The concept of absolutely summing linear operators goes back to Grothendieck in the 1950s, but just in 1967 and 1968, the classical works of Pietsch [15] and Lindenstrauss and Pelczynski [7] clarified Grothendieck's precious ideas and contributed decisively to the vigorous development of the theory. Since Pietsch's paper [16] , several generalizations of absolutely summing operators to the multilinear setting have been investigated. For example, we can mention the classes of multiple summing -also called fully summing - and strongly summing multilinear mappings. The class of multiple summing mappings was firstly vaguely sketched by Ramanujan and Schock [17] , and introduced independently by Matos [8] and Bombal, Pérez-García and Villanueva [2, 14] , and exhaustively explored in the recent years (we mention, for example, [9, 11, 13, 18]). The class of strongly summing multilinear operators was introduced by Dimant in [6]. In this paper

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we will be interested in connections between the classes of multiple summing, strongly summing and Cohen strongly summing multilinear operators (this last class was introduced by Achour and the first named author in [1]).

The main goal of this paper is to translate, to the multilinear case, some results obtained by Cohen in [5] for linear operators. As a consequence of our multilinear results, we obtain a nice connection between the classes of multiple summing and strongly summing multilinear operators (in the sense of Dimant). Precisely, our main result states that if Y is a $\mathcal{L}_{p^*}$-space and $X_1, ..., X_m$ are $\mathcal{L}_p$- spaces ($1 < p < \infty$), then every multiple p^*-summing m-linear operator is strongly p^*-summing, where p^* is the conjugate of p. For other recent papers comparing different classes of multilinear mappings related to summability, we refer to [3, 4, 12].

This paper is organized as follows.

In section 1, we recall some notion and properties concerning Banach spaces and $\mathcal{L}_{p,\lambda}$-spaces.

In section 2, motived by the work of Cohen, we establish the relation between Cohen strongly m-linear and multiple p-summing m-linear operators acting on $\mathcal{L}_p$-spaces.

Section 3 and final section, contains the relationship between a multilinear operator T and its adjoint T^* for certain classes of summability. As consequence, we compare the notion of Cohen strongly p-summing m-linear operators with the concept of strongly p-summing m-linear operators, when the space Y^* is an $\mathcal{L}_p$-space. We end this section and the paper by announcing our main result.

1 Notation and preliminaries

We shall begin this section by recalling briefly some basic notations and terminology. Let X be a Banach space, then B_X is its closed unit ball and X^* its (topological) dual. Consider $1 \leq p \leq \infty$. We denote by $l_p(X)$ (resp. $l_p^n(X)$) the Banach space of all sequences (x_i) in X with the norm

$$\|(x_i)\|_{l_p(X)} = (\sum_1^\infty \|x_i\|^p)^{\frac{1}{p}} < \infty$$
$$(\text{resp. } \left\|(x_i)_{1\leq i\leq n}\right\|_{l_p^n(X)} = (\sum_1^n \|x_i\|^p)^{\frac{1}{p}})$$

and by $l_p^\omega(X)$ (resp. $l_p^{n\ \omega}(X)$) the Banach space of all sequences (x_i) in X with the norm

$$\|(x_n)\|_{l_p^\omega(X)} = \sup_{x^*\in B_{X^*}} (\sum_1^\infty |\langle x_i, x^*\rangle|^p)^{\frac{1}{p}} < \infty$$
$$(\text{resp. } \|(x_n)\|_{l_p^{n\ \omega}(X)} = \sup_{x^*\in B_{X^*}} (\sum_1^n |\langle x_i, x^*\rangle|^p)^{\frac{1}{p}}).$$

Let $1 \leq p < \infty$ and let $\lambda > 1$. A Banach space X is said to be an $\mathcal{L}_{p,\lambda}$-space if, every finite dimensional subspace $E \subset X$ is contained in a finite dimensional subspace $F \subset X$ for which there is an isomorphism $u : F \longrightarrow l_p^{\dim F}$ with $\|u\| \left\|u^{-1}\right\| < \lambda$. We say that X is an $\mathcal{L}_p$-space if it is an $\mathcal{L}_{p,\lambda}$-space for some $\lambda > 1$.

Let now $m \in \mathbb{N}$ and let $X_1, ..., X_m, Y$ be Banach spaces over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). We will denote by $\mathcal{L}(X_1, ..., X_m; Y)$ the space of all continuous m-linear operators from $X_1 \times ... \times X_m$ into Y. If $Y = \mathbb{K}$, we write $\mathcal{L}(X_1, ..., X_m)$. In the case $X_1 = ... = X_m = X$, we will simply write $\mathcal{L}(^m X; Y)$.

We shall finish this section by announcing the definition of multilinear operators of *finite type*, as stated in [8].

Definition 1.1 [8]. *An m-linear operator $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is said to be of finite type if it is generated by the mappings of the form*

$$T_{y \otimes_{j=1}^m x_j^*} = x_1^* \otimes ... \otimes x_m^* \otimes y : \left(x^1, ..., x^m\right) \to x_1^*\left(x^1\right) ... x_m^*\left(x^m\right) y \tag{1.1}$$

for some non-zero $x_j^* \in X_j^*$ $(1 \leq j \leq m)$ and $y \in Y$.

The vector space of all m-linear operators of finite type is noted by $\mathcal{L}_f(X_1, ..., X_m; Y)$.

2 Multilinear operators on $\mathcal{L}_{p,\lambda}$-spaces

The goal of this section is to study the relationship between the classes of Cohen strongly p-summing and multiple p-summing multilinear operators acting on $\mathcal{L}_p$-spaces. We prove that the Banach space $\Pi_{p^*}^m(X_1, ..., X_m; Y)$ of all multiple p^*-summing m-linear operators from $X_1 \times ... \times X_m$ into Y is included in $\mathcal{D}_p^m(X_1, ..., X_m; Y)$, the space of all Cohen strongly p-summing m-linear operators, where X_j $(1 \leq j \leq m)$ is an $\mathcal{L}_p$-space and Y is a Banach space.

The following class of multilinear operators was introduced in [1] as an extension of the strongly p-summing operators introduced by Cohen in [5]. But for the convenience of the reader we start by recalling the linear case.

A linear operator T between two Banach spaces X, Y is strongly p-summing for $(1 < p \leq \infty)$ if there is a positive constant C such that for all $n \in \mathbb{N}$, $x_1, ..., x_n \in X$ and $y_1^*, ..., y_n^* \in Y^*$, we have

$$\left\| \left(\langle T(x_i), y_i^* \rangle \right)_{1 \leq i \leq n} \right\|_{l_1^n} \leq C \left\| (x_i) \right\|_{l_p^n(X)} \sup_{y \in B_Y} \left\| (y_i^*(y)) \right\|_{l_{p^*}^n}. \tag{2.1}$$

The smallest constant C which is noted by $d_p(u)$, such that the inequality (2.1) holds, is called the strongly p-summing norm on the space $\mathcal{D}_p(X, Y)$ of all strongly p-summing linear operators from X into Y, which is a Banach space. We have $\mathcal{D}_1(X, Y) = \mathcal{B}(X, Y)$.

Definition 2.1 [1]. *Let $1 \leq p < \infty$. An m-linear operator $T : X_1 \times ... \times X_m \longrightarrow Y$ (X_j, Y are arbitrary Banach spaces and $m \in \mathbb{N}^*$) is Cohen strongly p-summing if, and only if, there is a constant $C > 0$ such that for any $x_1^j, ..., x_n^j \in X_j$, $(j = 1, ..., m)$ and any $y_1^*, ..., y_n^* \in Y^*$, we have*

$$\left\| \left(\left\langle T\left(x_i^1, ..., x_i^m\right), y_i^* \right\rangle \right) \right\|_{l_1^n} \leq C \left(\sum_{i=1}^n \prod_{j=1}^m \left\| x_i^j \right\|_{X_j}^p \right)^{\frac{1}{p}} \sup_{y \in B_Y} \left\| (y_i^*(y)) \right\|_{l_{p^*}^n}. \tag{2.2}$$

The class of all Cohen strongly p-summing m-linear operators from $X_1 \times ... \times X_m$ into Y, which is denoted by $\mathcal{D}_p^m(X_1, ..., X_m; Y)$ is a Banach space with the norm $d_p^m(T)$ which is the smallest constant C such that the inequality (2.2) holds.

For $p = \infty$, (2.2) becomes

$$\left\| \left(\left\langle T\left(x_i^1, ..., x_i^m\right), y_i^* \right\rangle \right) \right\|_{l_1^n} \leq C \sup_{1 \leq i \leq n} \left(\prod_{j=1}^m \left\| x_i^j \right\|_{X_j} \right) \sup_{y \in B_Y} \left\| \left(y_i^*(y) \right) \right\|_{l_1^n}.$$

The following characterization can be found in [1, Theorem 2.4].

Theorem 2.2. *An m-linear operator $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is Cohen strongly p-summing $(1 < p \leq \infty)$ if, and only if, there exists a positive constant $C > 0$ and Radon probability measure μ on $B_{Y^{**}}$ such that for all $(x^1, ..., x^m) \in X_1 \times ... \times X_m$* and *$y^* \in Y^*$, we have*

$$\left| \left\langle T\left(x^1, ..., x^m\right), y^* \right\rangle \right| \leq C \prod_{j=1}^m \left\| x^j \right\| \left(\int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}. \tag{2.3}$$

Moreover, in this case
$d_p^m(T) = \inf\{C > 0 :$ *for all C verifying the inequality* (2.3)$\}$.

Before stating another definition, let us remark that the inequality (2.3) is equivalent to: for every $x^j \in B_{X_j}$ $(1 \leq j \leq m)$ and every $y^* \in Y^*$ we have

$$\left| \left\langle T\left(x^1, ..., x^m\right), y^* \right\rangle \right| \leq C \left(\int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}. \tag{2.4}$$

Definition 2.3 [2, 14]. *An m-linear operator $T : X_1 \times ... \times X_m \longrightarrow Y$ is multiple p-summing $(1 \leq p < \infty)$, if there is a constant $C > 0$ such that for any $x_1^j, ..., x_{n_j}^j \in X_j$ $(j = 1, ..., m)$, we have*

$$\left(\sum_{i_1, ..., i_m = 1}^{n_1, ..., n_m} \left\| T\left(x_{i_1}^1, ..., x_{i_m}^m\right) \right\|^p \right)^{\frac{1}{p}} \leq C \prod_{j=1}^m \left\| \left(x_{i_j}^j\right)_{i_j=1}^{n_j} \right\|_{l_p^{n_j\,\omega}(X_j)}. \tag{2.5}$$

As usual $\Pi_p^m(X_1, ..., X_m; Y)$ will stand for the Banach space of all multiple p-summing m-linear operators from $X_1 \times ... \times X_m$ into Y with its norm $\pi_p^m(T) = \inf\{C : C$ *verifies* (2.5)$\}$.

Proposition 2.4. *Let $r_1, ..., r_m \in \mathbb{N}^*$ and $1 < p \leq \infty$ be given. Let T be a multilinear operator from $l_p^{r_1} \times ... \times l_p^{r_m}$ into Y. Then, T belongs to $\Pi_{p^*}^m(l_p^{r_1}, ..., l_p^{r_m}; Y)$ and to $\mathcal{D}_p^m(l_p^{r_1}, ..., l_p^{r_m}; Y)$ with*

$$d_p^m(T) \leq \pi_{p^*}^m(T).$$

Proof. Consider a multilinear operator $T : l_p^{r_1} \times ... \times l_p^{r_m} \rightarrow Y$. It is clear that $T \in \mathcal{L}_f(X_1, ..., X_m; Y)$. Thus we have obviously that T belongs to $\Pi_{p^*}^m(l_p^{r_1}, ..., l_p^{r_m}; Y)$ and to $\mathcal{D}_p^m(l_p^{r_1}, ..., l_p^{r_m}; Y)$. Let now $\left(e_{k_j}\right)_{k_j=1}^{r_j}$ be the standard basis in $l_p^{r_j}$ $(1 \leq j \leq m)$. Since T is multiple p^*-summing, we have

$$
\begin{aligned}
&(\sum_{k_1,...,k_m=1}^{r_1,...,r_m} \left\| T\left(e_{k_1},...,e_{k_m}\right)\right\|^{p^*})^{\frac{1}{p^*}} \\
\leq\ & \pi_{p^*}^{m}\left(T\right)\prod_{j=1}^{m} \sup_{\left\|x_j^*\right\|_{l_{p^*}^{r_j}}=1} (\sum_{k_j=1}^{r_j} \left|x_j^*\left(e_{k_j}\right)\right|^{p^*})^{\frac{1}{p^*}} \\
\leq\ & \pi_{p^*}^{m}\left(T\right).
\end{aligned}
$$

Let $x_1^j,...,x_n^j$ be in $l_p^{r_j}$ $(1 \leq j \leq m)$ such that $x_i^j = \sum\limits_{k_j=1}^{r_j} a_{k_j,i}^j e_{k_j}$. Consider $y_1^*,...,y_n^*$ in Y^*. We have

$$
\begin{aligned}
&\sum_{i=1}^{n}\left|\left\langle T\left(x_i^1,...,x_i^m\right),y_i^*\right\rangle\right| \\
\leq\ & \sum_{i=1}^{n}\sum_{k_1,...,k_m=1}^{r_1,...,r_m}\left|\left\langle T\left(a_{k_1,i}^1 e_{k_1},...,a_{k_m,i}^m e_{k_m}\right),y_i^*\right\rangle\right| \\
\leq\ & \sum_{i=1}^{n}\sum_{k_1,...,k_m=1}^{r_1,...,r_m}\left|a_{k_1,i}^1...a_{k_m,i}^m\right|\left|\left\langle T\left(e_{k_1},...,e_{k_m}\right),y_i^*\right\rangle\right|.
\end{aligned}
$$

If $1 < p < \infty$, we have by Hölder's inequality (used twice)

$$
\begin{aligned}
&\sum_{i=1}^{n}\left|\left\langle T\left(x_i^1,...,x_i^m\right),y_i^*\right\rangle\right| \\
\leq\ & \sum_{i=1}^{n}(\sum_{k_1,...,k_m=1}^{r_1,...,r_m}\left|a_{k_1,i}^1...a_{k_m,i}^m\right|^{p})^{\frac{1}{p}}(\sum_{k_1,...,k_m=1}^{r_1,...,r_m}\left|\left\langle T\left(e_{k_1},...,e_{k_m}\right),y_i^*\right\rangle\right|^{p^*})^{\frac{1}{p^*}} \\
\leq\ & \sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|(\sum_{k_1,...,k_m=1}^{r_1,...,r_m}\left|\left\langle T\left(e_{k_1},...,e_{k_m}\right),y_i^*\right\rangle\right|^{p^*})^{\frac{1}{p^*}} \\
\leq\ & (\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|^{p})^{\frac{1}{p}}(\sum_{k_1,...,k_m=1}^{r_1,...,r_m}\left\|T\left(e_{k_1},...,e_{k_m}\right)\right\|^{p^*})^{\frac{1}{p^*}}\sup_{y\in B_Y}\left\|\left(y_i^*(y)\right)\right\|_{l_{p^*}^{n}} \\
\leq\ & \pi_{p^*}^{m}\left(T\right)(\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\|^{p})^{\frac{1}{p}}\sup_{y\in B_Y}\left\|\left(y_i^*(y)\right)\right\|_{l_{p^*}^{n}}.
\end{aligned}
$$

This implies that $d_p^m\left(T\right) \leq \pi_{p^*}^m\left(T\right)$. If $p = \infty$

$$
\begin{aligned}
&\sum_{i=1}^{n}\left|\left\langle T\left(x_i^1,...,x_i^m\right),y_i^*\right\rangle\right| \\
\leq\ & \sup_{1\leq i\leq n}\sup_{k_1,...,k_m}\left|a_{k_1,i}^1...a_{k_m,i}^m\right|(\sum_{i=1}^{n}\sum_{k_1,...,k_m=1}^{r_1,...,r_m}\left|\left\langle T\left(e_{k_1},...,e_{k_m}\right),y_i^*\right\rangle\right|) \\
\leq\ & \sup_{1\leq i\leq n}\prod_{j=1}^{m}\left\|x_i^j\right\|_{l_\infty^{r_j}}(\sum_{k_1,...,k_m=1}^{r_1,...,r_m}\left\|T\left(e_{k_1},...,e_{k_m}\right)\right\|)\sup_{y\in B_Y}\left\|\left(y_i^*(y)\right)\right\|_{l_1^{n}} \\
\leq\ & \pi_1^m\left(T\right)\sup_{1\leq i\leq n}\prod_{j=1}^{m}\left\|x_i^j\right\|_{l_\infty^{r_i}}\sup_{y\in B_Y}\left\|\left(y_i^*(y)\right)\right\|_{l_1^{n}}.
\end{aligned}
$$

We obtain, $d_\infty^m\left(T\right) \leq \pi_1^m\left(T\right)$. This completes the proof. ■

The following theorem is the principal result of this section.

Theorem 2.5. *Fix $m \in \mathbb{N}^*$. Let $1 < p \leq \infty$ and let X_j $(1 \leq j \leq m)$ be an $\mathcal{L}_{p,\lambda}$-space. Then*

$$\Pi_{p^*}^m(X_1, ..., X_m; Y) \subset \mathcal{D}_p^m(X_1, ..., X_m; Y) \text{ and } d_p^m(T) \leq \pi_{p^*}^m(T)\,\lambda^m.$$

Proof. Let $n \in \mathbb{N}^*$,$x_1^j, ..., x_n^j$ be in X_j and $T \in \Pi_{p^*}^m(X_1, ..., X_m; Y)$. Since X_j is an $\mathcal{L}_{p,\lambda}$-space, then there exists a finite dimensional subspace $M_j \subset X_j$ which contains the linear subspace spanned by $x_1^j, ..., x_n^j$ and an invertible operator $S_j : l_p^{r_j} \rightarrow M_j$ $(\dim M_j = r_j)$ such that $\|S_j\| \left\|S_j^{-1}\right\| \leq \lambda$. Consider the following diagram

$$\begin{array}{llllll}
X_1 & \times ... \times & X_m & \xrightarrow{T} & & Y \\
\uparrow i_1 & & \uparrow i_m & & & \widetilde{T} \uparrow \\
M_1 & \times ... \times & M_m & \underset{(S_1, ..., S_m)}{\longleftarrow} & l_p^{r_1} & \times ... \times \quad l_p^{r_m} \\
\uparrow k_1 & & \uparrow k_m & & & \\
\operatorname{span}\{x_1^1, ..., x_n^1\} & \times ... \times & \operatorname{span}\{x_1^m, ..., x_n^m\} & & &
\end{array}$$

where i_j and k_j are the canonical inclusion mappings and the operator $\widetilde{T}$ is defined by the equality $\widetilde{T} = T(i_1 \circ S_1, ..., i_m \circ S_m)$. Since $T \in \Pi_{p^*}^m(X_1, ..., X_m; Y)$, it follows that $\pi_{p^*}^m\left(\widetilde{T}\right) \leq \pi_{p^*}^m(T) \prod_{j=1}^m \|S_j\| \|i_j\|$. Therefore, using Proposition 2.4, we have that $\widetilde{T} \in \mathcal{D}_p^m(l_p^{r_1}, ..., l_p^{r_m}; Y)$ and $d_p^m\left(\widetilde{T}\right) \leq \pi_{p^*}^m\left(\widetilde{T}\right) \leq \pi_{p^*}^m(T) \prod_{j=1}^m \|S_j\|$. If we let $z_i^j = S_j^{-1} x_i^j$ in $l_p^{r_j}$ and $y_1^*, ..., y_n^*$ in Y^*, we have

$$\begin{aligned}
& \sum_{i=1}^n \left|\left\langle T(x_i^1, ..., x_i^m), y_i^*)\right\rangle\right| \\
= \; & \sum_{i=1}^n \left|\left\langle \widetilde{T}(z_i^1, ..., z_i^m), y_i^*)\right\rangle\right| \\
\leq \; & d_p^m\left(\widetilde{T}\right) (\sum_{i=1}^n \prod_{j=1}^m \left\|z_i^j\right\|_{l_p^{r_j}}^p)^{\frac{1}{p}} \sup_{y \in B_Y} \|(y_i^*(y))\|_{l_{p^*}^n} \\
\leq \; & \pi_{p^*}^m(T) \prod_{j=1}^m \|S_j\| (\sum_{i=1}^n \prod_{j=1}^m \left\|z_i^j\right\|_{l_p^{r_j}}^p)^{\frac{1}{p}} \sup_{y \in B_Y} \|(y_i^*(y))\|_{l_{p^*}^n}.
\end{aligned}$$

Since $z_i^j = S_j^{-1} x_i^j$, we obtain

$$\begin{aligned}
& \sum_{i=1}^n \left|\left\langle T(x_i^1, ..., x_i^m), y_i^*)\right\rangle\right| \\
\leq \; & \pi_{p^*}^m(T)\,\lambda^m (\sum_{i=1}^n \prod_{j=1}^m \left\|x_i^j\right\|_{X_j}^p)^{\frac{1}{p}} \sup_{y \in B_Y} \|(y_i^*(y))\|_{l_{p^*}^n}.
\end{aligned}$$

Therefore, T belongs to $\mathcal{D}_p^m(X_1, ..., X_m; Y)$ and

$$d_p^m(T) \leq \pi_{p^*}^m(T)\,\lambda^m$$

which finishes the proof. ■

3 Comparison of $\mathcal{D}_p^m(X_1,...,X_m;Y)$ and $\mathcal{L}_s^p(X_1,...,X_m;Y)$

In [8], the adjoint of an m-linear operator is defined as follows: let $X_1,...,X_m,Y$ be Banach spaces. If $T \in \mathcal{L}(X_1,...,X_m;Y)$, we define the *adjoint* of T by

$$T^* : Y^* \to \mathcal{L}(X_1,...,X_m),\ y^* \to T^*(y^*) : X_1 \times ... \times X_m \to \mathbb{K} \tag{3.1}$$

with $T^*(y^*)(x^1,...,x^m) = y^*(T(x^1,...,x^m))$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). A natural question is to study the connection between multilinear operators and their adjoints for different classes of summability. If $X_1,...,X_m$ are $\mathcal{L}_\infty$-spaces, Y is an infinite-dimensional Hilbert space and $T \in \mathcal{L}(X_1,...,X_m;Y)$, Pellegrino and Souza [10, Theorem 2.4] have shown that if T^* is almost summing, then T is absolutely $(1,2)$-summing multilinear operator. Here, we show that, T belongs to the class of Cohen strongly p-summing m-linear operators if, and only if, its adjoint T^* belongs to the class of absolutely p^*-summing linear operators. On the other hand, if T^* is strongly p^*-summing linear operator then T is strongly p-summing m-linear operator. As consequence, if Y is a Banach space such that Y^* is an $\mathcal{L}_p$-space then, the space $\mathcal{D}_p^m(X_1,...,X_m;Y)$ is included in $\mathcal{L}_s^{p^*}(X_1,...,X_m;Y)$, the space of all strongly p^*-summing m-linear operators, which we define below.

In the next result, we characterize the class of Cohen strongly m-linear operators by using the adjoint operator like that given by Cohen in the linear case [5].

Theorem 3.1. *Let $1 < p \leq \infty$. Let $T \in \mathcal{L}(X_1,...,X_m;Y)$ and T^* its adjoint. Then T belongs to $\mathcal{D}_p^m(X_1,...,X_m;Y)$ if, and only if, T^* belongs to $\pi_{p^*}(Y^*,\mathcal{L}(X_1,..., X_m))$ and we have $d_p^m(T) = \pi_{p^*}(T^*)$.*

Proof. Suppose that $T \in \mathcal{D}_p^m(X_1,...,X_m;Y)$. By (2.3) we have

$$\left|\left\langle T(x^1,...,x^m), y^*\right\rangle\right| \leq d_p^m(T)\prod_{j=1}^m \left\|x^j\right\| \left(\int_{B_{Y^{**}}} |y^{**}(y^*)|^{p^*} d\mu(y^{**})\right)^{\frac{1}{p^*}}$$

for all $x^j \in X_j$ ($1 \leq j \leq m$) and $y^* \in Y^*$. Taking the supremum over all sequences $(x^j)_{1\leq j\leq m}$ with $\|x^j\| \leq 1$, we obtain

$$\sup_{\|x^1\|,...,\|x^m\|\leq 1} \left|T^*(y^*)(x^1,...,x^m)\right| \leq d_p^m(T)\left(\int_{B_{Y^{**}}} |y^{**}(y^*)|^{p^*} d\mu(y^{**})\right)^{\frac{1}{p^*}}.$$

Then

$$\|T^*(y^*)\| \leq d_p^m(T)\left(\int_{B_{Y^{**}}} |y^{**}(y^*)|^{p^*} d\mu(y^{**})\right)^{\frac{1}{p^*}}.$$

By Pietsch Domination Theorem, $T^* \in \pi_{p^*}(Y^*, \mathcal{L}(X_1, ..., X_m))$ and we have $\pi_{p^*}(T^*) \leq d_p^m(T)$.
To prove the converse, suppose that $T^* \in \pi_{p^*}(Y^*, \mathcal{L}(X_1, ..., X_m))$. We have

$$\begin{aligned} \left|\left\langle T\left(x^1, ..., x^m\right), y^*\right\rangle\right| &= \left|T^*(y^*)\left(x^1, ..., x^m\right)\right| \\ &\leq \|T^*(y^*)\| \prod_{j=1}^m \left\|x^j\right\|. \end{aligned}$$

Using Pietsch Domination Theorem for p^*-summing linear operators, we obtain

$$\left|\left\langle T\left(x^1, ..., x^m\right), y^*\right\rangle\right| \leq \pi_{p^*}(T^*) \prod_{j=1}^m \left\|x^j\right\| \left(\int_{B_{Y^{**}}} |y^{**}(y^*)|^{p^*} d\mu(y^{**})\right)^{\frac{1}{p^*}}.$$

Thus by (2.3) T is Cohen strongly p-summing and $d_p^m(T) \leq \pi_{p^*}(T^*)$. ∎

We also need the definition of strongly p-summing multilinear operators introduced by Dimant in [6].

Definition 3.2 [6]. *Let $1 \leq p \leq \infty$ and $T \in \mathcal{L}(X_1, ..., X_m; Y)$. The operator T is strongly p-summing if there exists a positive constant C such that for every $x_1^j, ..., x_n^j \in X_j$ $(j = 1, ..., m)$ we have*

$$\left(\sum_{i=1}^n \left\|T\left(x_i^1, ..., x_i^m\right)\right\|^p\right)^{\frac{1}{p}} \leq C \sup_{\Phi \in B_{\mathcal{L}(X_1,...,X_m)}} \left(\sum_{i=1}^n \left|\Phi\left(x_i^1, ..., x_i^m\right)\right|^p\right)^{\frac{1}{p}}. \qquad (3.1)$$

The class of all strongly p-summing m-linear operators from $X_1 \times ... \times X_m$ into Y, which is denoted by $\mathcal{L}_s^p(X_1, ..., X_m; Y)$ is a Banach space with the norm $\|T\|_{ss,p}$ which is the smallest constant C such that the inequality (3.1) holds.

We add that, this definition does not coincide with Cohen's concept for $m = 1$.
This kind of multilinear operators verify some analogous properties of those in the linear case, in particular the Pietsch Domination Theorem.

Theorem 3.3 [6, Proposition 1.2]. *Let $T \in \mathcal{L}(X_1, ..., X_m; Y)$. The following assertions are equivalent.*
(i) *The operator T is strongly p-summing.*
(ii) *There exist a regular probability measure μ on $(B_{\mathcal{L}(X_1,...,X_m)}, \omega^*)$ and a constant $C > 0$ such that for every $\left(x^1, ..., x^m\right)$ in $X_1 \times ... \times X_m$, we have*

$$\left\|T\left(x^1, ..., x^m\right)\right\| \leq C\left(\int_{B_{\mathcal{L}(X_1,...,X_m)}} \left|\Phi\left(x^1, ..., x^m\right)\right|^p d\mu(\Phi)\right)^{\frac{1}{p}}. \qquad (3.2)$$

We have the following result.

Theorem 3.4. *Let $1 < p \leq \infty$. If $T \in \mathcal{L}(X_1, ..., X_m; Y)$* is *such that T^* is a Cohen strongly p^*-summing linear operator, then T is strongly p-summing multilinear operator.*

Proof. Suppose that T^* is strongly p^*-summing linear operator. We have by (2.1)

$$\left|\sum_{i=1}^{n}\left\langle T^{*}\left(y_{i}^{*}\right), z_{i}^{*}\right\rangle\right| \leq d_{p^{*}}\left(T^{*}\right)(\sum_{i=1}^{n}\left\|y_{i}^{*}\right\|^{p^{*}})^{\frac{1}{p^{*}}} \sup_{\Phi \in B_{\mathcal{L}(X_{1},\ldots,X_{m})}} (\sum_{i=1}^{n}\left|z_{i}^{*}(\Phi)\right|^{p})^{\frac{1}{p}}.$$

Let now $x_{1}^{j},...,x_{n}^{j} \in X_{j}$ ($1 \leq j \leq m$). We consider the linear operator $T_{x_{i}^{1},...,x_{i}^{m}}$: $\mathcal{L}\left(X_{1},...,X_{m}\right) \rightarrow \mathbb{K}$ defined by

$$T_{x_{i}^{1},...,x_{i}^{m}}(\Phi)=\Phi\left(x_{i}^{1},...,x_{i}^{m}\right).$$

We obtain

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\left\langle T\left(x_{i}^{1},...,x_{i}^{m}\right), y_{i}^{*}\right\rangle\right| \\
=&\left|\sum_{i=1}^{n}\left\langle T^{*}\left(y_{i}^{*}\right), T_{x_{i}^{1},...,x_{i}^{m}}\right\rangle\right| \\
\leq & d_{p^{*}}\left(T^{*}\right)\left(\sum_{i=1}^{n}\left\|y_{i}^{*}\right\|^{p^{*}}\right)^{\frac{1}{p^{*}}} \sup_{\Phi \in B_{\mathcal{L}(X_{1},...,X_{m})}}(\sum_{i=1}^{n}\left|T_{x_{i}^{1},...,x_{i}^{m}}(\Phi)\right|^{p})^{\frac{1}{p}} \\
=& d_{p^{*}}\left(T^{*}\right)\left(\sum_{i=1}^{n}\left\|y_{i}^{*}\right\|^{p^{*}}\right)^{\frac{1}{p^{*}}} \sup_{\Phi \in B_{\mathcal{L}(X_{1},...,X_{m})}}(\sum_{i=1}^{n}\left|\Phi\left(x_{i}^{1},...,x_{i}^{m}\right)\right|^{p})^{\frac{1}{p}}.
\end{aligned}$$

Taking the supremum over all sequences $\left(y_{i}^{*}\right)_{1 \leq i \leq n}$ with $(\sum\limits_{1}^{n}\left\|y_{i}^{*}\right\|^{p^{*}})^{\frac{1}{p^{*}}} \leq 1$, we obtain

$$\begin{aligned}
&(\sum_{i=1}^{n}\left\|T\left(x_{i}^{1},...,x_{i}^{m}\right)\right\|^{p})^{\frac{1}{p}} \\
=& \sup\left\{\left|\sum_{i=1}^{n}\left\langle T\left(x_{i}^{1},...,x_{i}^{m}\right), y_{i}^{*}\right\rangle\right| : (\sum_{1}^{n}\left\|y_{i}^{*}\right\|^{p^{*}})^{\frac{1}{p^{*}}} \leq 1\right\} \\
\leq & d_{p^{*}}\left(T^{*}\right) \sup_{\Phi \in B_{\mathcal{L}(X_{1},...,X_{m})}}(\sum_{i=1}^{n}\left|\Phi\left(x_{i}^{1},...,x_{i}^{m}\right)\right|^{p})^{\frac{1}{p}}.
\end{aligned}$$

Then, T is strongly p-summing and we have $\|T\|_{ss,p} \leq d_{p^{*}}\left(T^{*}\right)$. ∎

Finally, if Y^{*} is an $\mathcal{L}_{p}$-space we can give the following comparison between the classes of Cohen strongly p-summing and strongly p-summing m-linear operators.

Corollary 3.5. *Let $1<p \leq \infty$. If Y^{*} is an $\mathcal{L}_{p}$-space then*

$$\mathcal{D}_{p}^{m}(X_{1},...,X_{m};Y) \subset \mathcal{L}_{s}^{p^{*}}(X_{1},...,X_{m};Y).$$

Proof. Let $T \in \mathcal{D}_{p}^{m}(X_{1},...,X_{m};Y)$. From Theorem 3.1, we know that T^{*} is p^{*}-summing. Since Y^{*} is an an $\mathcal{L}_{p}$-space, from [5, Theorem 3.2.3] we conclude that T^{*} is (Cohen) strongly p-summing. According to Theorem 3.4, we obtain $T \in \mathcal{L}_{s}^{p^{*}}(X_{1},...,X_{m};Y)$. ∎

Our main result is the following corollary, that is a straightforward consequence of Corollary 3.5 and Theorem 2.5.

Corollary 3.6. *Let* $1 < p < \infty$. *Suppose that* X_j $(1 \leq j \leq m)$ *is an* $\mathcal{L}_p$*-space and* Y *is an* $\mathcal{L}_{p^*}$*-space. Then*

$$\Pi_{p^*}^m(X_1, ..., X_m; Y) \subset \mathcal{L}_s^{p^*}(X_1, ..., X_m; Y).$$

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