Invariant Subspaces Of Toeplitz Operators And Uniform Algebras *

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Abstract

Let T_ϕ be a Toeplitz operator on the one variable Hardy space H^2. We show that if T_ϕ has a nontrivial invariant subspace in the set of invariant subspaces of T_z then ϕ belongs to H^∞. In fact, we also study such a problem for the several variables Hardy space H^2.

1 Introduction

Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. A probability measure m (on X) denotes a representing measure for some nonzero complex homomorphism. The abstract Hardy space $H^p = H^p(m)$, $1 \le p \le \infty$, determined by A is defined to be the closure of A in $L^p = L^p(m)$ when p is finite and to be the weak* closure of A in $L^\infty = L^\infty(m)$ when $p = \infty$.

Let P be the orthogonal projection from L^2 onto H^2. For ϕ in L^∞, put

$$T_\phi f = P(\phi f) \quad (f \in H^2)$$

and then T_ϕ is called a Toeplitz operator. In this paper, we are interested in invariant subspaces of Toeplitz operators. Put $\mathcal{A} = \{T_\phi\ ;\ \phi \in H^\infty\}$ and $\mathcal{A}^* = \{T_\phi^*\ ;\ \phi \in H^\infty\}$. Lat T_ϕ denotes the set of all invariant subspaces of T_ϕ, Lat $\mathcal{A} = \cap\ \{\text{Lat } T_\phi\ ;\ \phi \in H^\infty\}$ and Lat $\mathcal{A}^* = \cap\{\text{Lat } T_\phi^*; \phi \in H^\infty\}$. We don't know whether arbitrary T_ϕ has a nontrivial invariant subspace. When ϕ is in H^∞ and H^∞ has a nonconstant

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unimodular function q, T_ϕ has a nontrivial invariant subspace $M = qH^2$. Hence Lat $T_\phi \neq \{\langle 0\rangle,\ H^2\}$.

Let K be the orthogonal complement of $\bar{H}^2$ in L^2. Then $L^2 = H^2 \oplus \bar{K}$. $I(H^\infty)$ denotes the set of all unimodular functions in H^∞. A function in $I(H^\infty)$ is called an inner function. For a subset Y in $L^\infty, Y^\perp$ denotes $\{g \in L^1 \ ; \ \int g\bar{f}dm = 0 \quad (f \in Y)\}$.

In this paper we study the following four natural questions :

Question 1. *If* Lat $T_\phi \supseteq$ Lat $\mathcal{A}$ *then does* T_ϕ *belong to* $\mathcal{A}$ *?*

Question 2. *Suppose that* H^∞ *is a weak* closed maximal algebra in* L^∞. *If* Lat $T_\phi \subsetneq$ Lat $\mathcal{A}$ *then is* Lat $T_\phi = \{\langle 0\rangle,\ H^2\}$ *?*

Question 3. *Is* Lat $\mathcal{A}^* \cap$ Lat $\mathcal{A} = \{\langle 0\rangle, H^2\}$ *?*

Question 4. *Can we describe* Lat $T_\phi \cap$ Lat $\mathcal{A}$ *or equivalently* Lat $T_\phi \cap$ Lat $\mathcal{A}^*$?

In this paper, we will answer these four questions positively when A is the disc algebra. In fact, for Question 1 we can do it for more general uniform algebras. However for Question 2 we could not answer even for simple uniform algebras. Question 3 can be answered for almost all uniform algebras.

In this paper $H^p(D^n)$ denotes the Hardy space on the polydisc D^n and $H^p(\Omega)$ denotes the Hardy space on a finitely connected domain Ω. $L^p_a(D)$ denotes the Bergman space on D and put $N^2 = L^2(D) \ominus \{L^2_a(D) \oplus \bar{z}\bar{L}^2_a(D)\}$. H^p_0 denotes the set of $\{f \in H^p \ ; \ \int f dm = 0\}$. $H^p(\Gamma)$ denotes the usual Hardy space on the dual group $\hat{\Gamma}$ where Γ is an ordered subgroup of the reals.

2 Lat $\mathcal{A} \subseteq$ Lat T_ϕ

In this section we study Question 1. Theorem 1 shows that Question 1 can be answered positively for very general uniform algebras.

Lemma 1. *Let* M *be a closed subspace of* H^2. $M \in$ Lat T_ϕ *if and only if* $\phi M \subset M \oplus \bar{K}$.

Proof. By definition of a Toeplitz operator, this is clear.

Lemma 2. *If* ϕ *is a function in* L^∞ *and* Lat $\mathcal{A} \subseteq$ Lat T_ϕ *then* $\phi = \phi_0 + \bar{k}_0$ *where* $\phi_0 \in H^2$ *and* $\bar{k}_0 \in \cap\{\bar{q}\bar{K} \ ; \ q \in I(H^\infty)\}$.

Proof. Since $L^2 = H^2 \oplus \bar{K}$, there exist $h \in H^2$ and $k \in K$ such that $\phi = h + \bar{k}$. If $q \in I(H^\infty)$ then $qH^2 \in$ Lat $\mathcal{A}$ and so by Lemma 1 $\phi q = qh + q\bar{k} \in qH^2 + \bar{K}$. Since $T_\phi q \in qH^2$ and $qh \in qH^2$, $P(q\bar{k}) \in qH^2$. Hence $q\bar{k} = q\ell + \bar{t}$ where $\ell \in H^2$ and $\bar{t} \in \bar{K}$. Therefore $\bar{k} = \ell + \overline{qt}$ and $\ell = \bar{k} - \overline{qt} \in H^2 \cap \bar{K} = \langle 0\rangle$. Hence $\ell = 0$ and $\bar{k} = \overline{qt}$. This implies that k belongs to $\bar{q}\bar{K}$ for any $q \in I(H^\infty)$.

Theorem 1. *Suppose that* $\cap\{\bar{q}\bar{K}\ ;\ q \in I(H^\infty)\} = \langle 0\rangle$. *If* ϕ *is a function in* L^∞ *and* Lat $\mathcal{A} \subseteq$ Lat T_ϕ *then* ϕ *belongs to* H^∞.

Proof. Lemmas 1 and 2 imply the theorem trivially.

Corollary 1. *Suppose that* $H^2 = H^2(\mathbf{T}^N)$. *If* ϕ *is a function in* L^∞ *and* Lat $\mathcal{A} \subseteq$ Lat T_ϕ *then* ϕ *belongs to* H^∞.

Proof. K is an invariant subspace under multiplications by the coordinates functions $z_1, \cdots, z_n$. $\cap\{z_1^{\ell_1}\cdots z_n^{\ell_n}K\ ;\ (\ell_1,\cdots,\ell_n) \geq (0,\cdots,0)\}$ is a reducing subspace and so $\cap\ z_n^{\ell_1}\cdots z_n^{\ell_n}K = \chi_E L^2$ for some characteristic function χ_E. Since $\chi_E L^2$ is orthogonal to $\bar{H}^2$, $\chi_E = 0$ and so $\langle 0\rangle = \cap \bar{z}_1^{\ell_1}\cdots\bar{z}_n^{\ell_n}\bar{K} = \cap\{\bar{q}\bar{K}\ ;\ q \in I(H^\infty)\}$.

Corollary 2. *Suppose that* $H^2 = H^2(\Omega)$. *If* ϕ *is a function in* L^∞ *and* Lat $\mathcal{A} \subseteq$ Lat T_ϕ *then* ϕ *belongs to* H^∞.

Proof. Let Z be the Ahlfors function for Ω then $\mid Z \mid = 1$ on $\partial\Omega = X$ (see [3]). $\bigcap_{n=0}^{\infty}\bar{Z}^n\bar{K}$ is invariant under the multiplications by Z and $\bar{H}^\infty$. Since H^∞ is a weak$*$ maximal subalgebra of L^∞, $\bigcap_{n=0}^{\infty}\bar{Z}^n\bar{K} = \chi_E L^2$. Since $\chi_E L^2$ is orthogonal to H^2, $\chi_E = 0$ and so $\cap\{\bar{q}\bar{K}\ ;\ q \in I(H^\infty)\} = \{0\}$.

Corollary 3. *Let A be a Dirichlet algebra (see [4]). If* ϕ *is a function in* L^∞ *and* Lat $\mathcal{A} \subseteq$ Lat T_ϕ *then* ϕ *belongs to* H^∞.

Proof. Since H^∞ is a uniform algebra which has the annulus property ([2],[6]) on a totally disconnected space, by [2, Theorem 1] the set of quotients of inner functions is norm dense in the set of unimodular functions in L^∞. In this situation, $\bar{K} = \bar{H}_0^2$ and $Y = \cap\{\bar{q}\bar{K}\ ;\ q \in I(H^\infty)\} \subset \bar{H}^2$. $\bar{q}Y = Y$ for any q in $I(H^\infty)$ and so $\bar{q}_1 q_2 Y \subseteq Y$ for any q_1, q_2 in $I(H^\infty)$. Hence $\phi Y \subseteq Y$ for any unimodular function ϕ in L^∞. Hence $Y = \chi_E L^2$ for the characteristic function χ_E for some set E. Since $Y \subset \bar{H}^2$, Y must be $\{0\}$.

Proposition 1. *Suppose that* $H^2 = L_a^2(D)$, ϕ *is a function in* L^∞ *and* Lat $\mathcal{A} \subseteq$ Lat T_ϕ. *Then the following are valid.*

(1) ϕ *belongs to* $L_a^2(D) + N^2$.

(2) If $\phi = f + \ell$ *where* $f \in H^\infty$ *and* $\ell \in N^2$ *then* Lat $T_\ell \supseteq$ Lat $\mathcal{A}$.

Proof. (1) Since $zL_a^2 \in$ Lat T_ϕ by hypothesis, $\not{C} \in$ Lat $T_\phi^* =$ Lat $T_{\bar{\phi}}$ and so $\bar{\phi} = \bar{c} + \bar{k}$ where $c \in \not{C}$ and $k \in zL_a^2(D) + N^2$. Hence $\phi \in L_a^2(D) + N^2$. (2) If $\phi = f + \ell$ and $M \in$ Lat $\mathcal{A}$ then $\phi M \subset M + \bar{K}$. Hence $(f+\ell)g = fg + \ell g \in M + \bar{K}$ for any $g \in M$. Since $fg \in M$, $\ell g \in M + \bar{K}$ for any $g \in M$ and so $\ell M \subset M + \bar{K}$. Thus $M \in$ Lat T_ℓ.

A bounded operator B is called reflexive if whenever C is a bounded operator and Lat $B \subseteq$ Lat C then C belongs to the closed algebra (in weak operator topology) generated by B. When B is subnormal, it is known that B is reflexive [7]. Hence if f is a nonzero function in H^∞ and Lat $T_\phi \supseteq$ Lat T_f then T_ϕ belongs to the closed algebra generated by T_f. Hence T_ϕ belongs to $\mathcal{A}$. Usually Lat $\mathcal{A} \subsetneq$ Lat T_f and so this does not answer Question 1. However if there exists a function f in H^∞ such that Lat $T_f =$ Lat $\mathcal{A}$ then the above result about subnormal operators answers Question 1. Hence when $H^2 = H^2(T)$, if Lat $T_\phi \supseteq$ Lat $\mathcal{A}$ then T_ϕ belongs to $\mathcal{A}$ because Lat $T_z =$ Lat $\mathcal{A}$. Therefore Corollary 1 is not new for $N = 1$. Similarly Question 1 can be answered for $H^2 = L^2_a(D)$. Hence Proposition 1 is a very weak result.

3 Lat $T_\phi \subsetneq$ Lat $\mathcal{A}$

In this section we study Question 2. Theorem 2 shows that Question 2 can be answered positively for the disc algebra. In fact, it gives a few results for more general uniform algebras about Question 2.

Lemma 3. *Let Q be a function in $I(H^\infty)$. Then* $\bar{K} = \sum_{n=0}^{\infty} \oplus (\bar{K} \ominus \bar{Q}\bar{K})\bar{Q}^n \oplus \bigcap_{n=0}^{\infty} \bar{Q}^n \bar{K}$.

Proof. Since $|Q| = 1$ *a.e.* and $\bar{Q}\bar{K} \subset \bar{K}, \bar{Q}$ is an isometry on $\bar{K}$. Hence this is well known and called a Wold decomposition.

Theorem 2. *Suppose that* Lat $T_\phi \subsetneq$ Lat $\mathcal{A}$. *If $M \in$* Lat T_ϕ *and* $\cap \{\bar{Q}^n \bar{K}\ ;\ Q \in \mathcal{I}\} = \{0\}$ *for some subset $\mathcal{I}$ in $I(H^\infty)$ then there exists a nonconstant Q in $\mathcal{I}$ such that $M \cap (H^2 \ominus QH^2) \neq \langle 0 \rangle$ or $\phi M \subseteq M$.*

Proof. If $M \in$ Lat T_ϕ then by Lemma 1 there exist $f \in M,\ g \in M$ and $k \in K$ such that $\phi f = g + \bar{k}$. If $\phi M \not\subseteq M$ then we may assume that $k \neq 0$. For any fixed $Q \in \mathcal{I}$, by Lemma 3 $\bar{K} = \left\{ \sum_{n=0}^{\infty} \oplus (\bar{K} \ominus \bar{Q}\bar{K})\bar{Q}^n \right\} \oplus \bigcap_{n=0}^{\infty} \bar{Q}^n \bar{K}$ and so

$$\bar{k} = \sum_{n=0}^{\infty} k_n \bar{Q}^n + k_\infty$$

where $k_n \in \bar{K} \ominus \bar{Q}\bar{K}$ $(n = 0, 1, 2, \cdots)$ and $k_\infty \in \bigcap_{n=0}^{\infty} \bar{Q}^n \bar{K}$. Then $Q\bar{k} = Qk_0 + \sum_{n=1}^{\infty} k_n \bar{Q}^{n-1} + Qk_\infty$ and by Lemma 1 $Q\bar{k}$ belongs to $M + \bar{K}$ because $\phi f = g + \bar{k}$ and $QM \subset M$.

Suppose that there does not exist a nonconstant function Q in $\mathcal{I}$ such that $M \cap (H^2 \ominus QH^2) \neq \langle 0 \rangle$. Then we will get a contradiction. By what was proved above, Qk_0 belongs to $M \cap (H^2 \ominus QH^2) = \{0\}$. Hence $k_0 \equiv 0$. Next we consider $Q^2\bar{k}$ and then $k_1 \equiv 0$ follows. Proceeding similarly we can show that $\bar{k} = k_\infty$. By hypothesis, this implies that $\bar{k} \equiv 0$ because Q is arbitrary in $\mathcal{I}$. This contradiction implies that there exists $Q \in I$ such that $M \cap (H^2 \ominus QH^2) \neq \langle 0 \rangle$.

Corollary 4. *Suppose that* $H^2 = H^2(\mathbf{T}^N)$, ϕ *is a function in* L^∞ *and* $\mathrm{Lat}\ T_\phi \underset{\neq}{\subset} \mathrm{Lat}\ \mathcal{A}$. *If* $M \in \mathrm{Lat}\ T_\phi$ *and* $M \neq \langle 0 \rangle$ *then* M *contains a nonzero function which is* $(N-1)$*-variable. Hence if* $N = 1$ *then* $M = H^2$.

Proof. It is known that if $\phi M \subseteq M$ then $\phi \in H^\infty$. Hence we may assume that $\phi M \nsubseteq M$. Put $\mathcal{I} = \{z_1, \cdots, z_N\}$ then $\mathcal{I}$ satisfies the condition of Theorem 2. By Theorem 2, there exists z_j such that $1 \leq j \leq N$ and $(H^2 \ominus z_j H^2) \cap M \neq \{0\}$. Since $H^2 \ominus z_j H^2 = H^2(z'_j, T^{N-1})$ where $z = (z_j, z'_j)$, M contains a nonzero $(N-1)$-variable function.

Corollary 5. *Suppose that* $H^2 = H^2(\Omega)$, $\mathrm{Lat}\ T_\phi \underset{\neq}{\subset} \mathrm{Lat}\ \mathcal{A}$ *and* Z *is the Alfors function for* Ω *(see [3]). If* $M \in \mathrm{Lat}\ T_\phi$ *and* $M \neq \langle 0 \rangle$ *then* $M \cap (H^2 \ominus ZH^2) \neq \langle 0 \rangle$.

Proof. Put $\mathcal{I} = \{Z\}$ then $\mathcal{I}$ satisfies the condition of Theorem 2. It is known that if $\phi M \subseteq M$ then $\phi \in H^\infty$. Hence we may assume that $\phi M \nsubseteq M$.

Proposition 2. *If* T_ϕ *is subnormal and* $\mathrm{Lat}\ T_\phi \subseteq \mathrm{Lat}\ \mathcal{A}$ *then* T_ϕ *commutes with* $\mathcal{A}$ *and so* $T_\phi f = P(\phi_0 f)$ $(f \in H^\infty)$ *for some* ϕ_0 *in* H^2. *If* A *is a uniform algebra which approximates in modulus on* X *then* ϕ *belongs to* $H^2 \cap L^\infty$.

Proof. If T_ϕ is subnormal and $\mathrm{Lat}\ T_\phi \subseteq \mathrm{Lat}\ \mathcal{A}$ then it is known [7] that $\mathcal{A}$ is contained in the closed algebra generated by T_ϕ. Hence T_ϕ commutes with $\mathcal{A}$. Let $\phi_0 = T_\phi 1$ then $T_\phi f = T_\phi T_f 1 = T_f T_\phi 1 = P(\phi_0 f)$ for $f \in H^\infty$. Since $\|\phi_0 f\|_2 \leq \|T_\phi\| \|f\|_2$ $(f \in H^\infty)$,

$$\left| \int_X \phi_0 f \bar{g} dm \right| \leq \|\phi\|_\infty \|f\|_2 \|g\|_2 \quad (f, g \in H^\infty).$$

Hence

$$\left| \int_X \phi_0 \mid f \mid^2 dm \right| \leq \|\phi\|_\infty \|f^2\|_1.$$

Since A approximates in modulus on X, ϕ_0 belongs to $H^2 \cap L^\infty$. It is easy to see that $\phi = \phi_0$.

Corollary 6. *Suppose that* $H^2 = H^2(\mathbf{T}^N)$ *or* $H^2 = H^2(\Omega)$. *If* T_ϕ *is subnormal then* $\mathrm{Lat}\ T_\phi \underset{\neq}{\subset} \mathrm{Lat}\ \mathcal{A}$ *or* ϕ *belongs to* H^∞.

Proof. A uniform algebra A approximates in modulus on X, that is , for every positive continuous function g on X and $\varepsilon > 0$, there is an f in A with $|g - |f|| < \varepsilon$ if the set of unimodular elements of A separates points of X (see [6, Lemma 4.12]). Since the coordinate functions $z_1, \cdots, z_n$ separate T^N, the polydisc algebra approximates in modulus on T^N. If T_ϕ is subnormal on $H^2(T^N)$ and $\mathrm{Lat}\ T_\phi \subseteq \mathrm{Lat}\ \mathcal{A}$ then by Proposition 2 ϕ belongs to $H^2(T^N) \cap L^\infty = H^\infty(T^N)$. If $A = H^\infty(\Omega)$ then by [3, Lemma 4.8] $I(H^\infty(\Omega))$ separates $X =$ the maximal ideal space of $L^\infty(\partial D)$. Hence Corollary 6 for $H^2 = H^2(\Omega)$ follows from Proposition 2.

Proposition 3. *If* $\mathrm{Lat}\ T_\phi \subseteq$ *Lat* $\mathcal{A}$*, then* $\mathrm{Lat}\ T_\phi^* \cap \mathrm{Lat}\ T_\phi \subset \mathrm{Lat}\ \mathcal{A}^* \cap$ *Lat* $\mathcal{A}$.

Proof. If $M \in \mathrm{Lat}\ T_\phi^*$ then $M^\perp \in \mathrm{Lat}\ T_\phi$ and so $M^\perp \in \mathrm{Lat}\ \mathcal{A}$ because $\mathrm{Lat}\ T_\phi \subseteq \mathrm{Lat}\ \mathcal{A}$. Hence $M \in \mathrm{Lat}\ \mathcal{A}^*$ and so $\mathrm{Lat}\ T_\phi^* \subseteq \mathrm{Lat}\ \mathcal{A}^*$.

By Proposition 3, when Let $\mathcal{A}^* \cap \text{Lat } \mathcal{A} = \{\langle 0 \rangle, H^2\}$, if Lat $T_\phi \subsetneq$ Lat $\mathcal{A}$ then T_ϕ does not have a nontrivial reducing subspace. Hence if T_ϕ is normal then Lat $T_\phi \not\subset$ Lat $\mathcal{A}$. Therefore it is important to know that Lat $\mathcal{A}^* \cap$Lat $\mathcal{A} = \{\langle 0 \rangle, H^2\}$, that is, $\mathcal{A}$ is irreducible.

4 Lat $\mathcal{A}^* \cap$ Lat $\mathcal{A}$

In this section we study Question 3. Theorem 3 shows that Question 3 can be answered positively for usual uniform algebras. Recall $\mathcal{A}^* = \{T_\phi^* \ ; \ \phi \in H^\infty\}$.

Theorem 3. *If $M \in$ Lat $\mathcal{A}^* \cap$ Lat $\mathcal{A}$ then $M \subset \chi_E L^2 \subset M + \bar{K}$ where $E = \cup\{\text{supp } f \ ; \ f \in M\}$. Hence if $E = X$ then $M = H^2$.*

Proof. If $\phi \in L^\infty$ then by the Stone-Weierstrass theorem for any $\varepsilon > 0$ there exist $f_1, \cdots, f_n$ and $g_1, \cdots, g_n$ in H^∞ such that $\| \phi - \sum_{j=1}^{n} f_j \bar{g}_j \|_\infty < \varepsilon$. Since $T_{f_j \bar{g}_j} M \subset M$ for $j = 1, \cdots, n$, $T_\phi M \subset M$. By Lemma 1 $\phi M \subset M \oplus \bar{K}$. Thus $\chi_E L^2 \subset M \oplus \bar{K}$. If $E = X$ then $L^2 = M \oplus \bar{K}$ and so $M = H^2$.

Corollary 7. *Suppose that there does not exist a nonzero function in H^2 such that $m(\{x \in X \ ; \ f(x) = 0\}) \neq 0$. If $M \in$ Lat $\mathcal{A}^* \cap$ Lat $\mathcal{A}$ then $M = \langle 0 \rangle$ or H^2.*

5 Lat $T_\phi \cap$ Lat $\mathcal{A}$

In this section we study Question 4. We don't know whether Lat $T_\phi \neq \{\langle 0 \rangle, H^2\}$. However we show that Lat $T_\phi \cap$ Lat $\mathcal{A} = \{\langle 0 \rangle, H^2\}$ if $\phi \notin H^\infty$ and $H^2 = H^2(\mathbf{T})$. For any M in Lat T_ϕ, put

$$K_M = \{k \in K \ ; \ \bar{k} = \phi f - g \text{ for some } f \text{and } g \in M\},$$

then $K_M \subseteq K$ and $\phi M \subset M + \bar{K}_M$ (see Lemma 1).

Theorem 4. *If $M \in$ Lat $T_\phi \cap$ Lat $\mathcal{A}$ then $K_M \times (H^2 \ominus M) \subseteq (H^\infty)^\perp$ and $T_k^*(H^\infty) \subseteq M$ for any k in K_M.*

Proof. By the remark above, if $M \in$ Lat $T_\phi \cap$ Lat $\mathcal{A}$ then $\phi M \subset M + \bar{K}_M$. If $k \in K_M$ then by its definition there exist f and g such that $\phi f = g + \bar{k}$. For any $\ell \in H^\infty$, $\phi f \ell = g\ell + \bar{k}\ell \in M + \bar{K}_M$ and so $P(\bar{k}\ell) \in M$. Since

$$\bar{k}\ell = P(\bar{k}\ell) + (I - P)(\bar{k}\ell) \in M + \bar{K}_M,$$

if $s \in H^2 \ominus M$ then $\langle \bar{k}\ell, s\rangle = \langle P(\bar{k}\ell), s\rangle = 0$. Hence ks belongs to $(H^\infty)^\perp$ and so $K_M \times (H^2 \ominus M) \subseteq (H^\infty)^\perp$. The above proof implies that $T_k^*(H^\infty) \subseteq M$.

Corollary 8. *Suppose that $H^2 = H^2(\Omega)$, $\mathbb{C} \backslash \Omega$ has n components and $\phi \notin H^\infty$. If $M \in$ Lat $T_\phi \cap$ Lat $\mathcal{A}$ then $\dim(H^2 \ominus M) \leq n$.*

Proof. By Theorem 4

$$K_M \times (H^2 \ominus M) \subseteq (H^\infty)^\perp \cap (\bar{H}^\infty)^\perp = (H^\infty + \bar{H}^\infty)^\perp \cap L^1$$

and $\dim(H^\infty + \bar{H}^\infty)^\perp \cap L^1 = n$ because $\mathbb{C}\backslash\Omega$ has n components. If $K_M = \langle 0 \rangle$ then $\phi M \subset M$. It is known [4] that L^∞ is generated by ϕ and H^∞ in the weak* topology. Hence $M \in \text{Lat } \mathcal{A} \cap \text{Lat } \mathcal{A}^* = \{\langle 0 \rangle, H^2\}$ by Corollary 7 and so $M = H^2$. It is clear that if $K_M \neq \langle 0 \rangle$ then $\dim(H^2 \ominus M) \leq n$.

Corollary 9. *If* $H^2 = H^2(\mathbf{T})$ *and* $\phi \notin H^\infty$ *then* $\text{Lat } T_\phi \cap \text{Lat } \mathcal{A} = \{\langle 0 \rangle, H^2\}$.

Proof. When Ω is the open unit disc, $H^2(\Omega) = H^2(\mathbf{T})$ and so by Corollary 8 $\text{Lat } T_\phi \cap \text{Lat } \mathcal{A} = \{\langle 0 \rangle, H^2\}$.

Corollary 10. *Let A be a Dirichlet algebra. If* $\phi \notin H^\infty$ *then* $\text{Lat } T_\phi \cap \text{Lat } \mathcal{A} = \{\langle 0 \rangle, H^2\}$.

Proof. It is known that $(\bar{H}^\infty)^\perp \cap (H^\infty)^\perp = \langle 0 \rangle$. The corollary is a result of Theorem 4.

In general, it seems to be difficult to describe $\text{Lat } T_\phi \cap \text{Lat } \mathcal{A}$. When $H^2 = H^2(\Omega)$ and $\bar{\phi} \in H^\infty$, $\text{Lat } T_\phi \cap \text{Lat } \mathcal{A} = \{\langle 0 \rangle, H^2\}$ by Corollary 8. In fact, if $M \in \text{Lat } T_\phi \cap \text{Lat } \mathcal{A}$ then $\bar{\phi}(H^2 \ominus M) \subseteq H^2 \ominus M$. Since $\dim(H^2 \ominus M) < \infty$ by Corollary 8, M must be equal to H^2. When $H^2 = H^2(T^2)$ and $\phi = \bar{z}$, $\text{Lat } T_\phi \cap \text{Lat } \mathcal{A} = \{\langle 0 \rangle, qH^2(w,T); q = q(w) \text{ is a one variable inner function}\}$ where z and w are the independent variables on T^2. In fact, if $M \in \text{Lat } T_\phi \cap \text{Lat } \mathcal{A}$ then $T_z^* M_1$ is orthogonal to M where $M_1 = M \ominus zM$. Since $T_z^* M_1 \subset M$, $T_z^* M_1 = \langle 0 \rangle$ and so $M_1 \subset H^2(w,T)$. Corollary 10 shows that $\text{Lat } \mathcal{A}^* \cap \text{Lat } \mathcal{A} = \{\langle 0 \rangle, H^2\}$ if A is a Dirichlet algebra.

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