A note on Serre's condition for orientability of fibre bundles

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Let $F \xrightarrow{i} E \longrightarrow B$ be a fibre bundle with fibre F, and let R be a field. The fibre of the fibre bundle $F \to E \longrightarrow B$ is said to be totally non-homologous to zero in E with respect to R (see e.g. Leray [8] or Mimura, Toda [9]) if the fibre inclusion induces an epimorphism, $i^* : H^*(E;R) \to H^*(F;R)$, in cohomology with coefficients in R. Note (see e.g. [9]) that if B is path connected and F is totally non-homologous to zero with respect to R, then the system of local coefficient rings $\underline{H^*(F;R)}$ is trivial or, in other words, the fibre bundle is R-orientable. In this note, we restrict ourselves to smooth fibre bundles $F \to E \longrightarrow B$ with E and F closed connected manifolds.

If F is totally non-homologous to zero with respect to R, then (see e.g. [9, Chap. 3]) the Serre spectral sequence of the fibration $F \to E \xrightarrow{p} B$ collapses and the Leray-Hirsch theorem applies: $H^*(E;R)$ is free as an $\mathrm{Im}(p^*)$-module with a basis $\{e_\alpha\}$ such that $\{i^*(e_\alpha)\}$ is a homogeneous basis of $H^*(F;R)$ as an R-vector space.

This is one of the reasons why fibre bundles with fibre totally non-homologous to zero are very useful in many situations in topology. For instance, they can be traced behind the answer (given by Korbaš in [3]; see also Sankaran [10]) to the question of when a real flag manifold $\mathbb{R}F(n_1+\cdots+n_q) := O(n_1+\cdots+n_q)/O(n_1)\times\cdots\times O(n_q)$ with $q \geq 3$ possesses an almost complex structure. Or a very recent example: Korbaš and Lörinc in [7] succeeded in finding the $\mathbb{Z}_2$-cohomology cup-length and Lyusternik-Shnirel'man category of several infinite families of the real flag manifolds basically using the fact that the manifold $\mathbb{R}F(n_1+\cdots+n_q)$ can be expressed as the total space of a fibre bundle with fibre totally non-homologous to zero with respect to $\mathbb{Z}_2$. Indeed, an important rôle in their approach is played by a theorem (cf. Horanská, Korbaš [1, Lemma, p. 25]) which can be stated as follows: If for a

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fibre bundle (recall that we restrict ourselves to smooth fibre bundles with the total space and fibre closed connected manifolds) $F \to E \longrightarrow B$ the fibre F is totally non-homologous to zero with respect to R, then $\mathrm{cup}_R(E) \geq \mathrm{cup}_R(F) + \mathrm{cup}_R(B)$; here $\mathrm{cup}_R(X)$ denotes the R-cup-length of the space X.

Now, given a fibre bundle $F \to E \longrightarrow B$, a necessary condition for F being totally non-homologous to zero with respect to R is R-orientability of the fibre bundle. As was proved by J.-P. Serre (see [11], [9]), this necessary condition is satisfied (over all fields R) by any fibre bundle admitting a connected Lie group as its structure group.

We shall say that a fibre bundle $F \to E \longrightarrow B$ satisfies the Serre condition, if it admits a connected Lie group as its structure group. There are works (e.g. Kono, Shiga, and Tezuka's [2]), where the authors prove results on fibre bundles with fibre totally non-homologous to zero assuming that the Serre condition is satisfied. If one wishes to apply such results or one wants to compare them with other similar theorems, then one needs to know the strength of the requirement that a fibre bundle should satisfy the Serre condition. Our aim here is to show that the requirement is by no means weak.

It turns out (cf. [1], [2], [4], [5], [6]) that there are some spaces which, as fibres in fibre bundles, are R-TNHZ-nice (where TNHZ stands for totally non-homologous to zero) in the sense, that whenever a fibre bundle has one of these spaces as its fibre, then the fibre is totally non-homologous to zero with respect to R (and, as a consequence, the fibre bundle is R-orientable). To give a recent example: as was shown in [6], if n is odd, then the real Grassmann manifolds $\mathbb{R}G_{n,k} = O(n)/O(k) \times O(n-k)$ with $k = 2$ as well as the complex Grassmann manifolds $\mathbb{C}G_{n,k} = U(n)/U(k) \times U(n-k)$ with $k = 2$ are $\mathbb{Z}_2$-TNHZ-nice (even as fibres in Serre fibrations; but we do not consider these generalizations of fibre bundles here).

Inspired by this, we say that a space X is SC-nice (where SC stands for Serre condition) if any fibre bundle having X as fibre admits a connected Lie group as its structure group (as a consequence, the fibre bundle with fibre X is R-orientable for any field R).

We prove the following result in order to show that the Serre condition is not easy to satisfy, even if the fibre is as nice as $U(n)/U(k) \times U(n-k)$ or $Sp(n)/Sp(n-1) \times Sp(1)$.

Proposition. *Neither the complex Grassmann manifold* $\mathbb{C}G_{n,k} = U(n)/U(k) \times U(n-k)$ *nor the quaternionic projective space* $\mathbb{H}G_{n,1} = Sp(n)/Sp(n-1) \times Sp(1)$ *is SC-nice.*

Proof. Similarly as Korbaš [4, Remark (f)], we take the involution T on $\mathbb{C}G_{n,k}$ induced by complex conjugation. Then in cohomology (with integers $\mathbb{Z}$ as coefficients), $T^*(c_1(\gamma)) = -c_1(\gamma)$, where $c_1(\gamma)$ is the first Chern class of the canonical k-plane bundle γ over $\mathbb{C}G_{n,k}$. In addition to this, one has the antipodal involution a on the m-sphere S^m ($m \geq 1$), and so one can construct the fibre bundle

$$\mathbb{C}G_{n,k} \to (\mathbb{C}G_{n,k} \times S^m)/(T \times a) \longrightarrow \mathbb{R}P^m, \qquad (*)$$

with structure group $\mathbb{Z}_2$.

In cohomology with coefficients in $\mathbb{Z}$, or in $\mathbb{Z}_q$, where q is an odd prime, we have $T^*(c_1(\gamma)) = -c_1(\gamma)$, and as a consequence, the fibre in (*) cannot be totally non-homologous to zero with respect to the field $\mathbb{Z}_q$. In particular, it is so for coefficients in $\mathbb{Z}_p$ for any odd prime p not dividing $n!$ (recall that $n!$ is the order of the Weyl group $W(U(n))$). Let us fix one p of this type.

Now, let us suppose that the fibre bundle (*) admits a connected structure group. Then the fibre bundle is orientable over the field $\mathbb{Z}_p$. As a consequence, by a theorem of Shiga and Tezuka [12], the fibre in the fibre bundle (*) is totally non-homologous to zero with respect to $\mathbb{Z}_p$. This contradiction shows that the fibre bundle (*) does not admit a connected Lie group as its structure group. We have proved that $\mathbb{C}G_{n,k}$ is not SC-nice.

To deal with the quaternionic case, we take the involution T on the quaternionic projective space $\mathbb{H}G_{n,1} = \mathbb{H}P^{n-1}$ induced by quaternionic conjugation. Then in cohomology, $T^*(e(\gamma)) = -e(\gamma) \in H^4(\mathbb{H}P^{n-1};\mathbb{Z})$, where $e(\gamma)$ is the Euler class of the (realification of the) canonical line bundle γ over $\mathbb{H}P^{n-1}$. The reason is that if γ^* is the quaternionic conjugate bundle to γ, then the orientation of (the realification of) γ^* is opposite to the orientation of (the realification of) γ. Indeed, if $(\vec{x})$ is a basis of the quaternionic fibre of the line bundle γ over some point, then we take the real basis $(\vec{x}, i\vec{x}, j\vec{x}, k\vec{x})$ to be positive in the realification of the fibre. At the same time, in the fibre of γ^* over the same point, we take the same quaternionic basis $(\vec{x})$. But then the corresponding real basis in the realification of the fibre (which, when we forget about the orientation, is the same as the realification of the fibre of γ) is $(\vec{x}, -i\vec{x}, -j\vec{x}, -k\vec{x})$, hence it is a negative basis there. As a consequence, $e(\gamma^*) = -e(\gamma)$.

Note that the Euler class of the canonical line bundle over $\mathbb{H}P^{n-1}$ does not vanish (in fact it can be taken as a generator of the integral cohomology ring of $\mathbb{H}P^{n-1}$).

Then, similarly as in the complex case, we take the antipodal involution a on the m-sphere S^m $(m \geq 1)$, and so we construct the fibre bundle

$$\mathbb{H}P^{n-1} \to (\mathbb{H}P^{n-1} \times S^m)/(T \times a) \longrightarrow \mathbb{R}P^m, \qquad (**)$$

with structure group $\mathbb{Z}_2$.

In cohomology with coefficients in $\mathbb{Z}$, or in $\mathbb{Z}_q$, where q is an odd prime, we have $T^*(e(\gamma)) = -e(\gamma)$, and as a consequence the fibre here cannot be totally non-homologous to zero. In particular, this is so for coefficients in $\mathbb{Z}_p$ for any odd prime p not dividing $2^n n!$ (where $2^n n!$ is the order of the Weyl group $W(Sp(n))$). Again, as in the complex case, let us fix some p of this type.

Suppose that the fibre bundle (**) admits a connected structure group. Then the fibre bundle is orientable over $\mathbb{Z}_p$. Therefore, by the result of Shiga and Tezuka quoted above (it applies, because the rank of $Sp(n)$ is the same as the rank of $Sp(1) \times Sp(n-1)$), the fibre in our fibre bundle is totally non-homologous to zero with respect to $\mathbb{Z}_p$. This contradiction shows that neither $\mathbb{H}P^{n-1}$ is SC-nice. The proposition is proved. ■

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