Multiple positive solutions for a nonlinear elliptic equation in weighted Sobolev space

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Abstract

In this paper, we consider the problem $(\mathcal{P}_\lambda)$ in the setting of a weighted Sobolev space $W^{1,p}(\Omega, \omega)$, where ω is a weight function defined on the unbounded domain Ω. The study is based on the variational methods and critical point theory. We show the existence of at least two nonnegative solutions, one with negative energy, the other one with energy which changes sign at a certain value of the positive parameter λ.

1 Introduction

In this paper, we are concerned with the problem of finding positive solutions to the following equation satisfied by the unknown scalar function u:

$$(\mathcal{P}_\lambda) \quad \begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) & = \quad f(x)|u|^{r-2}u \ \text{ in } \ \Omega, \\ \\ |\nabla u|^{p-2}\nabla u.\mathbf{n} + a(x)|u|^{p-2}u & = \quad \lambda b(x)|u|^{q-2}u \ \text{ on } \ \Gamma, \end{cases}$$

where $p < N$, $1 < q < p < r < p^* = \frac{Np}{N-p}$, $\mathbf{n}$ is the unit outward normal vector on Γ, $f : \Omega \longrightarrow \mathbb{R}$, $a, b : \Gamma \longrightarrow \mathbb{R}$ are given functions and λ is a real parameter. Here, Ω denotes an unbounded domain with noncompact and smooth boundary Γ.
This type of boundary value problems arises in a variety of situations: In the theory of nonlinear diffusion, in particular in the mathematical modeling of non-Newtonian

Received by the editors march 2004.
Communicated by P. Godin.
1991 *Mathematics Subject Classification :* 34B15.
Key words and phrases : Weighted Sobolev space, nonlinear boundary condition, Ekeland's principle, Palais-Smale condition.

fluids. For a discussion of some physical background, we refer the reader to [9]. From variational point of view, solutions of $(\mathcal{P}_\lambda)$ are critical points of the corresponding functional J_λ defined by:

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \frac{1}{p}\int_\Gamma a(x)|u|^p\,d\sigma - \frac{\lambda}{q}\int_\Gamma b(x)|u|^q\,d\sigma - \frac{1}{r}\int_\Omega f(x)|u|^r\,dx.$$

For unbounded domains with noncompact boundary, the arguments which can be used in the bounded case break down because of losses of compactness. This difficulty can be illustrated by the following: neither the embedding $W^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, nor the trace operator $W^{1,p}(\Omega) \hookrightarrow L^q(\Gamma)$ is compact. So to overcome it, we use a weighted Sobolev space as a variational framework of the problem.

The study of existence when the nonlinear term is placed in the equation has received considerable attention. For the Laplace operator, see for example [2, 6, 8]. Also see [3, 4, 5, 11, 13, 16] for the p-Laplacian. However, nonlinear boundary conditions have only been considered in recent years, see for example [6, 7, 11, 14, 15, 16]. In [16], the author considered a similar problem to $(\mathcal{P}_\lambda)$. After using the mountain-pass lemma, he proved that there exist at least two nonnegative solutions, one with positive energy, the other one with negative energy. Recently in [11], the author studied a problem with parameter in a bounded domain, namely the p-Laplacian with concave and convex nonlinearities on the right hand side in the equation. He showed the existence of two nonnegative solutions, the energy of the first is negative while the energy of the second changes sign at a cetain value of the parameter.

Here we consider problem $(\mathcal{P}_\lambda)$ in the unbounded domain Ω. Our results extend the corresponding results of [11] for a bounded domain. Even, they are generalizations for those of [16]. More precisely, our existence and multiplicity results are nonlocal with respect to the parameter λ. At the same time, our approach does not use the mountain-pass lemma.

The paper is organized as follows: In section 2, we set up the variational framework of the problem. In section 3 we verify that the functional associated to the problem $(\mathcal{P}_\lambda)$ satisfies the Palais-Smale condition. Section 4 is devoted to the behaviour of the energy corresponding to the positive solutions.

2 Variational setting

Let $1 \le p < \infty$. For a nonnegative measurable function ω, we define the weighted Lebesgue space $L^p(\Omega, \omega)$ by all measurable functions u which satisfies $\int_\Omega \omega|u|^p\,dx < \infty$ and associate with it the norm $\|u\|_{0,p,\omega} = (\int_\Omega \omega|u|^p\,dx)^{\frac{1}{p}}$. The weighted Sobolev space $W^{1,p}(\Omega; v_0, v_1)$ is defined as the space of all functions $u \in L^p(\Omega, v_0)$ such that all derivatives $(\frac{\partial u}{\partial x_i})_{i=1}^N$ belong to $L^p(\Omega, v_1)$. Equipped with its natural norm

$$\|u\|_{1,p,v_0,v_1} = (\textstyle\int_\Omega v_0|u|^p\,dx + \int_\Omega v_1|\nabla u|^p\,dx)^{\frac{1}{p}}\,,$$

$W^{1,p}(\Omega; v_0, v_1)$ is a Banach space. Here, we investigate the properties of $W^{1,p}(\Omega; v_0, v_1)$ via the following lemmas.

Lemma 1. (cf. Kufner & Opic [12], Pflüger [16, 17]). *Let $1 \le p \le r < \infty$, $\alpha, \beta \in \mathbb{R}$ and Ω is an unbounded domain in $\mathbb{R}^N$ with noncompact and smooth boundary Γ. We denote*

$$v_0(x) = (1+|x|)^{\beta-p}, \;\; v_1(x) = (1+|x|)^{\beta}, \;\; \textit{and } \omega(x) = (1+|x|)^{\alpha}.$$

1) The embedding of $W^{1,p}(\Omega; v_0, v_1)$ into $L^r(\Omega, \omega)$ is continuous respectively compact if and only if

$$\frac{N}{r} - \frac{N}{p} + 1 \ge 0 \;\; \textit{and} \;\; \frac{\alpha}{r} - \frac{\beta}{p} + \frac{N}{r} - \frac{N}{p} + 1 \le 0, \tag{1}$$

respectively

$$\frac{N}{r} - \frac{N}{p} + 1 > 0 \;\; \textit{and} \;\; \frac{\alpha}{r} - \frac{\beta}{p} + \frac{N}{r} - \frac{N}{p} + 1 < 0. \tag{2}$$

2) The embedding of $W^{1,p}(\Omega; v_0, v_1)$ into $L^r(\Gamma, \omega)$ is continuous respectively compact if and only if

$$\frac{N-1}{r} - \frac{N}{p} + 1 \ge 0 \;\; \textit{and} \;\; \frac{\alpha}{r} - \frac{\beta}{p} + \frac{N-1}{r} - \frac{N}{p} + 1 \le 0, \tag{3}$$

respectively

$$\frac{N-1}{r} - \frac{N}{p} + 1 > 0 \;\; \textit{and} \;\; \frac{\alpha}{r} - \frac{\beta}{p} + \frac{N-1}{r} - \frac{N}{p} + 1 < 0. \tag{4}$$

Remark 1. In the remaining of this text, for $\alpha_1, \alpha_2 \in \mathbb{R}$, we introduce the weight functions $\omega_1(x) = (1+|x|)^{\alpha_1}$ and $\omega_2(x) = (1+|x|)^{\alpha_2}$. The last theorem leads to the following:
(i) $W^{1,p}(\Omega; (1+|x|)^{-p}, 1)$ is compactly imbedded into $L^r(\Omega, \omega_1)$ for $\alpha_1 < -N + \frac{r}{p}(N-p)$ and $p < r < p^*$.
(ii) $W^{1,p}(\Omega; (1+|x|)^{-p}, 1)$ is imbedded into $L^p(\Gamma, \omega_2)$, continuously for $\alpha_2 = -p+1$ and compactly for $\alpha_2 < -p+1$.

Lemma 2. (cf. Pflüger [16]). *Let $1 < p < N$. There exists two positive constants C_1 and C_2 such that for any u in W,*

$$\int_\Omega \frac{|u|^p}{(1+|x|)^p}\, d\,x \le C_1 \int_\Omega |\nabla u|^p\, d\,x + C_2 \int_\Gamma \frac{|u|^p}{(1+|x|)^{p-1}}\, d\,\sigma \tag{5}$$

Proof. Since $\mathcal{D}(\bar{\Omega})$ is a dense space in W with respect to its natural norm, then to establish this inequality, it suffices to prove it in $\mathcal{D}(\bar{\Omega})$. For this, for any $\mathbf{q}$ in $(\mathcal{D}(\bar{\Omega}))^N$ and u in $\mathcal{D}(\bar{\Omega})$, we have the Green formula, i.e

$$\int_\Omega (\mathbf{q}.\nabla u + u\mathrm{div}\mathbf{q})\, d\,x = \int_\Gamma \mathbf{q}.\mathbf{n} u\, d\,\sigma,$$

where $\mathbf{n}$ is the unit outward normal vector on Γ. Using this equality with u is replaced by $\frac{|u|^p}{(1+|x|)^p}$ and $\mathbf{q}$ by $\mathbf{x}$, we obtain:

$$\int_\Omega \left(\mathbf{x}.\nabla \left(\frac{|u|^p}{(1+|x|)^p} \right) + N \frac{|u|^p}{(1+|x|)^p} \right) d\,x = \int_\Gamma \mathbf{x}.\mathbf{n} \frac{|u|^p}{(1+|x|)^p}\, d\,\sigma. \tag{6}$$

On the other hand,

$$\left|\int_\Omega \mathbf{x}.\nabla\left(\frac{|u|^p}{(1+|x|)^p}\right)d\,x\right| \;\le\; p\int_\Omega \frac{|u|^{p-1}}{(1+|x|)^{p-1}}|\nabla u|\,d\,x + p\int_\Omega \frac{|u|^p}{(1+|x|)^p}\,d\,x,$$

thus, by Hölder's inequality, we may write

$$\begin{aligned}&\left|\int_\Omega \mathbf{x}.\nabla\left(\frac{|u|^p}{(1+|x|)^p}\right)d\,x\right|\\ &\qquad\le\; p\left(\int_\Omega \frac{|u|^p}{(1+|x|)^p}\,d\,x\right)^{\frac{p-1}{p}}\left(\int_\Omega |\nabla u|^p\,d\,x\right)^{\frac{1}{p}} + p\int_\Omega \frac{|u|^p}{(1+|x|)^p}\,d\,x\\ &\qquad\le\; p\varepsilon\int_\Omega \frac{|u|^p}{(1+|x|)^p}\,d\,x + pC_\varepsilon\int_\Omega |\nabla u|^p\,d\,x + p\int_\Omega \frac{|u|^p}{(1+|x|)^p}\,d\,x,\end{aligned}$$

where $\varepsilon > 0$ is an arbitrary real number. The previous inequality and (6) give us

$$\begin{aligned}N\int_\Omega \frac{|u|^p}{(1+|x|)^p}\,d\,x \;&\le\; \int_\Gamma \frac{|u|^p}{(1+|x|)^{p-1}}\,d\,\sigma + p\varepsilon\int_\Omega \frac{|u|^p}{(1+|x|)^p}\,d\,x + pC_\varepsilon\int_\Omega |\nabla u|^p\,d\,x\\ &+\; p\int_\Omega \frac{|u|^p}{(1+|x|)^p}\,d\,x\end{aligned}$$

and

$$(N-p(1+\varepsilon))\int_\Omega \frac{|u|^p}{(1+|x|)^p}\,d\,x \;\le\; \int_\Gamma \frac{|u|^p}{(1+|x|)^{p-1}}\,d\,\sigma + pC_\varepsilon\int_\Omega |\nabla u|^p\,d\,x.$$

The proof is complete. ■

In the sequel, W will denote the weighted Sobolev space $W^{1,p}(\Omega;(1+|x|)^{-p},1)$. It results from the Hardy-type inequality and the second assertion of Remark 1 that W can be equipped with the norm

$$\|u\|_W = \left[\int_\Omega |\nabla u|^p\,d\,x + \int_\Omega \frac{|u|^p}{(1+|x|)^{p-1}}\,d\,\sigma\right]^{\frac{1}{p}}.$$

Throughout this work, we make the following assumptions:
(i) There are constants $c_1, c_2 > 0$ such that

$$\frac{c_1}{(1+|x|)^{p-1}} \le a(x) \le \frac{c_2}{(1+|x|)^{p-1}}.$$

(ii) There exists a constant $c_3 > 0$ such that

$$0 < f \le c_3\omega_1.$$

(iii) $0 < b \in L^{\frac{p}{p-q}}(\Gamma, \omega_2^{\frac{q}{q-p}})$.
The first assumption leads to the equivalence between $\|u\|_W$ and

$$\|u\| = \left(\int_\Omega |\nabla u|^p\,d\,x + \int_\Gamma a(x)|u|^p\,d\,\sigma\right)^{\frac{1}{p}}. \tag{7}$$

So, W endowed by (7) is a Banach space.

3 The Palais-Smale condition

In this part, let us note $P(u) = \|u\|^p$, $Q(u) = \int_\Gamma b(x)|u|^q \, d\sigma$ and $R(u) = \int_\Omega f(x)|u|^r \, dx$. The energy functional corresponding to $(\mathcal{P}_\lambda)$ is given by:

$$J_\lambda(u) = \frac{1}{p}P(u) - \frac{\lambda}{q}Q(u) - \frac{1}{r}R(u). \tag{8}$$

We denote by N_f and N_b the corresponding Nemytskii operators, i.e

$$\begin{array}{rlll} N_f : & L^r(\Omega,\omega_1) & \longrightarrow L^{\frac{r}{r-1}}(\Omega,\omega_1^{\frac{1}{1-r}}) & \text{and} \quad N_b : \quad L^p(\Gamma,\omega_2) \longrightarrow L^{\frac{p}{p-1}}(\Gamma,\omega_2^{\frac{1}{1-p}}) \\ & u & \longmapsto f(x)|u|^{r-2}u & \qquad\qquad\qquad u \longmapsto b(x)|u|^{q-2}u \end{array}$$

Lemma 3. *The two operators defined before are bounded and continuous.*

Proof. We only prove the statement for N_f, since the arguments for N_b are similar. Let u in $L^r(\Omega,\omega_1)$. Then

$$\int_\Omega (f(x)|u|^{r-1})^{\frac{r}{r-1}}\omega_1^{\frac{1}{1-r}} \, dx \quad \leq \quad \int_\Omega k^{\frac{r}{1-r}}|u|^r\omega_1 \, dx,$$

which shows that N_f is bounded. The continuity of this operator follows from the usual properties of Nemytskii operators. Which ends the proof. ∎

Lemma 4. *The functional J_λ is Fréchet-differentiable on W.*

Proof. The directional derivative of J_λ in direction h in W is given by:

$$< J_\lambda'(u), h >= \frac{1}{p} < P'(u), h > -\frac{\lambda}{q} < Q'(u), h > -\frac{1}{r} < R'(u), h >,$$

where

$$\begin{aligned} < P'(u), h > &= p\left[\int_\Omega |\nabla u|^{p-2}\nabla u \nabla h \, dx + \int_\Gamma a(x)|u|^{p-2}uh \, d\sigma\right], \\ < Q'(u), h > &= q\int_\Gamma b(x)|u|^{q-2}uh \, d\sigma, \ < R'(u), h >= r\int_\Omega f(x)|u|^{r-2}uh \, dx. \end{aligned}$$

P' is a continuous operator, then the operator Q' is the composition of

$$Q' : W \longrightarrow L^p(\Gamma,\omega_2) \xrightarrow{N_b} L^{\frac{p}{p-1}}(\Gamma,\omega_2^{\frac{1}{1-p}}) \xrightarrow{l} W',$$

with for any v in $L^{\frac{p}{p-1}}(\Gamma,\omega_2^{\frac{1}{1-p}})$ and h in W, $< l(v), h >= \int_\Gamma vh \, d\sigma$.
Furthermore, by the embedding of W into $L^p(\Gamma,\omega_2)$, we get

$$< l(v), h > \leq \left(\int_\Gamma |v|^{\frac{p}{p-1}}\omega_2^{\frac{1}{1-p}} \, d\sigma\right)^{1-\frac{1}{p}} \left(\int_\Gamma |h|^p\omega_2 \, d\sigma\right)^{\frac{1}{p}}.$$

In a similar way, R' is the composition of

$$R' : W \longrightarrow L^r(\Omega,\omega_1) \xrightarrow{N_f} L^{\frac{r}{r-1}}(\Omega,\omega_1^{\frac{1}{1-r}}) \xrightarrow{l} W',$$

with, for any v in $L^{\frac{r}{r-1}}(\Omega,\omega_1^{\frac{1}{1-r}})$, $< l(v), h >= \int_\Omega vh \, dx$. The compact embeddings of W into $L^r(\Omega,\omega_1)$ and $L^p(\Gamma,\omega_2)$ (see Remark 1) lead to that of Q' and R'. The lemma is proved. ∎

We introduce on $\mathbb{R}\times W$ the modified energy functional $\widetilde{J}_\lambda$ (cf. [10, 19]) defined by:

$$\tilde{J}_\lambda(t,u) := J_\lambda(tu).$$

Here, we recall that if $t(u)$ is a critical point for $t \longmapsto \widetilde{J}_\lambda(t,u)$ and $t(u)$ is smooth, then $t(u)u$ is a critical point for J_λ. By observing that the mapping $t \longmapsto \widetilde{J}_\lambda(t,u)$ is even, the study of critical points for $t \longmapsto \widetilde{J}_\lambda(t,u)$ can be restricted to $]0,\infty[$.

Lemma 5. *For any u in $W\setminus\{0\}$, there exists a unique $\lambda(u) > 0$ for which the functional $t \longmapsto \partial_t\tilde{J}_{\lambda(u)}(t,u)$ has a unique positive zero. Moreover for every $\lambda < \lambda(u)$ respectively $\lambda > \lambda(u)$, $t \longmapsto \partial_t\tilde{J}_\lambda(t,u)$ has exactly two positive zero respectively no zero.*

Proof. Let u be an element in $W\setminus\{0\}$. We have $\partial_t\tilde{J}_\lambda(t,u) = t^{q-1}\tilde{E}_\lambda(t,u)$ where $\tilde{E}_\lambda(t,u) = t^{p-q}P(u) - \lambda Q(u) - t^{r-q}R(u)$ and observe that

$$\begin{cases} \partial_t\tilde{J}_\lambda(t,u) &= 0 \\ \partial_{tt}\tilde{J}_\lambda(t,u) &= 0 \end{cases} \Longleftrightarrow \begin{cases} \tilde{E}_\lambda(t,u) &= 0 \\ \partial_t\tilde{E}_\lambda(t,u) &= 0. \end{cases}$$

The second equation of the right hand side, $\partial_t\tilde{E}_\lambda(t,u) = 0$, acquires the form

$$t^{p-q-1}((p-q)P(u) - (r-q)t^{r-p}R(u)) = 0 \tag{9}$$

and has one positive root

$$t(u) = \left[\frac{p-q}{r-q}\frac{P(u)}{R(u)}\right]^{\frac{1}{r-p}}. \tag{10}$$

It follows from (9) that $t \longmapsto \tilde{E}_\lambda(t,u)$ attains its maximum at $t(u)$. Moreover, it is increasing on $]0,t(u)[$, and decreasing on $]t(u),+\infty[$. Substituting $t(u)$ into $\widetilde{E}_\lambda(t,u)$, we obtain

$$\widetilde{E}_\lambda(t(u),u) = \frac{r-p}{r-q}P(u)\left[\frac{p-q}{r-q}\frac{P(u)}{R(u)}\right]^{\frac{p-q}{r-p}} - \lambda Q(u)$$

and the number of roots of $t \longmapsto \widetilde{E}_\lambda(t,u)$ depends on the sign of $\widetilde{E}_\lambda(t(u),u)$. More precisely

$$\begin{cases} t \longmapsto \widetilde{E}_\lambda(t,u) \text{ has two positive roots} \Longleftrightarrow \widetilde{E}_\lambda(t(u),u) > 0, \\ t \longmapsto \widetilde{E}_\lambda(t,u) \text{ has one positive root} \Longleftrightarrow \widetilde{E}_\lambda(t(u),u) = 0, \\ t \longmapsto \widetilde{E}_\lambda(t,u) \text{ has zero root} \Longleftrightarrow \widetilde{E}_\lambda(t(u),u) < 0. \end{cases}$$

We define now $\lambda(u)$ as the positive real number such that $\tilde{E}_{\lambda(u)}(t(u),u) = 0$, i.e

$$\lambda(u) = \frac{r-p}{r-q}\left[\frac{p-q}{r-q}\frac{P(u)}{R(u)}\right]^{\frac{p-q}{r-p}}\frac{P(u)}{Q(u)}, \tag{11}$$

and we have the following:

$$\begin{cases} t \longmapsto \widetilde{E}_\lambda(t,u) \text{ has two positive roots} \Longleftrightarrow \lambda < \lambda(u), \\ t \longmapsto \widetilde{E}_\lambda(t,u) \text{ has one positive root} \Longleftrightarrow \lambda = \lambda(u), \\ t \longmapsto \widetilde{E}_\lambda(t,u) \text{ has zero root} \Longleftrightarrow \lambda > \lambda(u). \end{cases}$$

The result is also true for the map $t \longmapsto \partial_t\widetilde{J}_\lambda(t,u)$. ∎

Remark 2. For $\lambda < \lambda(u)$, denoting the positive roots of $t \longmapsto \partial_t \tilde{J}_\lambda(t,u)$ by $\underline{t}(u,\lambda)$ and $\bar{t}(u,\lambda)$ ($\underline{t}(u,\lambda) < t(u) < \bar{t}(u,\lambda)$). It results from

$$\begin{aligned}\partial_t \tilde{E}_\lambda(\underline{t}(u,\lambda),u)\underline{t}(u,\lambda)^{q+1-p} &= (p-q)P(u) - (r-q)\underline{t}(u,\lambda)^{r-p}R(u)\\ &> (p-q)P(u) - (r-q)t(u)^{r-p}R(u) = 0,\end{aligned}$$

that $\partial_{tt}\tilde{J}_\lambda(\underline{t}(u,\lambda),u) > 0$. In a similar way, we get $\partial_{tt}\tilde{J}_\lambda(\bar{t}(u,\lambda),u) < 0$.

Remark 3. By the embedding results of W into $L^r(\Omega,\omega_1)$ and $L^p(\Gamma,\omega_2)$, we find two positive constants C_Ω and C_Γ such that

$$R(u) \le kC_\Omega P(u)^{\frac{r}{p}} \text{ and } Q(u) \le \|b\|^{\frac{p-q}{p}}_{L^{\frac{p}{p-q}}(\Gamma,\omega_2^{\frac{q}{q-p}})} C_\Gamma^{\frac{q}{p}} P(u)^{\frac{q}{p}}.$$

We conclude that

$$\begin{aligned}\lambda(u) &= \frac{r-p}{r-q}\left[\frac{p-q}{r-q}\right]^{\frac{p-q}{r-p}}\left[\frac{P(u)^{\frac{r}{p}}}{R(u)}\right]^{\frac{p-q}{r-p}}\frac{P(u)^{\frac{q}{p}}}{Q(u)}\\ &\ge \frac{r-p}{r-q}\left[\frac{p-q}{r-q}\right]^{\frac{p-q}{r-p}}[kC_\Omega]^{\frac{r-p}{p-q}}\|b\|^{\frac{q-p}{p}}_{L^{\frac{p}{p-q}}(\Gamma,\omega_2^{\frac{q}{q-p}})} C_\Gamma^{\frac{-q}{p}} > 0.\end{aligned}$$

Let us define

$$\hat{\lambda} = \inf_{u\in W\setminus\{0\}} \lambda(u). \tag{12}$$

We remark that $u \longmapsto \lambda(u)$ is an homogeneous function and consequently

$$\hat{\lambda} = \inf_{u\in\mathbb{S}} \lambda(u) \tag{13}$$

where $\mathbb{S}$ is the unit sphere in W.

Lemma 6. *There exist two positive constants c_4 and c_5 such that for any positive real number t with $\partial_t\tilde{J}_\lambda(t,u) = 0$ and for u in $W \setminus \{0\}$, we have*

$$J_\lambda(tu) \ge c_4 P(tu) - c_5$$

.

Proof. Let $t > 0$. From $\partial_t\tilde{J}_\lambda(t,u) = 0$, we get

$R(tu) = P(tu) - \lambda Q(tu)$ which yields $J_\lambda(tu) = \left(\frac{1}{p} - \frac{1}{r}\right)P(tu) - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)Q(tu).$

Furthermore, invoking Young's inequality, we have

$$\begin{aligned}Q(tu) &\le \|b\|^{\frac{p-q}{p}}_{L^{\frac{p}{p-q}}(\Gamma,\omega_2^{\frac{q}{q-p}})} C_\Gamma^{\frac{q}{p}} P(tu)^{\frac{q}{p}}\\ &\le \|b\|^{\frac{p-q}{p}}_{L^{\frac{p}{p-q}}(\Gamma,\omega_2^{\frac{q}{q-p}})}\left(\frac{q}{p}\varepsilon^{\frac{p}{q}}P(tu) + C_\Gamma^{\frac{q}{p-q}}\varepsilon^{\frac{p}{q-p}}\right),\end{aligned}$$

where $\varepsilon > 0$ is an arbitrary real number. It follows that

$$\begin{aligned}J_\lambda(tu) &\ge \left(\frac{1}{p} - \frac{1}{r} - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)\frac{q}{p}\|b\|^{\frac{p-q}{p}}_{L^{\frac{p}{p-q}}(\Gamma,\omega_2^{\frac{q}{q-p}})}\varepsilon^{\frac{p}{q}}\right)P(tu)\\ &- \lambda\left(\frac{1}{q} - \frac{1}{r}\right)\|b\|^{\frac{p-q}{p}}_{L^{\frac{p}{p-q}}(\Gamma,\omega_2^{\frac{q}{q-p}})} C_\Gamma \varepsilon^{\frac{p}{q-p}}.\end{aligned}$$

The proof is achieved. ∎

Let us define for $\lambda \in]0, \hat{\lambda}[$,

$$\underline{\alpha}(\lambda) = \inf_{u \in W\setminus\{0\}} \tilde{J}_\lambda(\underline{t}(u,\lambda), u), \tag{14}$$

$$\overline{\alpha}(\lambda) = \inf_{u \in W\setminus\{0\}} \tilde{J}_\lambda(\overline{t}(u,\lambda), u). \tag{15}$$

Lemma 7. *For any $\lambda \in]0, \widehat{\lambda}[$, we have*

$$\underline{\alpha}(\lambda) = \inf_{u\in\mathbb{S}} \tilde{J}_\lambda(\underline{t}(u,\lambda), u), \;\; \overline{\alpha}(\lambda) = \inf_{u\in\mathbb{S}} \tilde{J}_\lambda(\overline{t}(u,\lambda), u), \tag{16}$$

where $\mathbb{S}$ is the unit sphere in W.

Proof.. For any real number $\gamma > 0$, we have

$$\begin{cases} \tilde{J}_\lambda(\gamma t, \frac{u}{\gamma}) & = & \tilde{J}_\lambda(t,u), \\ \partial_t \tilde{J}_\lambda(\gamma t, \frac{u}{\gamma}) & = & \frac{1}{\gamma}\partial_t \tilde{J}_\lambda(t,u), \\ \partial_{tt} \tilde{J}_\lambda(\gamma t, \frac{u}{\gamma}) & = & \frac{1}{\gamma^2}\partial_{tt} \tilde{J}_\lambda(t,u). \end{cases}$$

We shall prove the result for $\underline{\alpha}(\lambda)$ (similar argument for $\overline{\alpha}(\lambda)$).
By the characterization of the positive real number $\underline{t}(u,\lambda)$ (see Remark 2), it holds

$$\underline{t}\left(\frac{u}{\gamma}, \lambda\right) = \gamma \underline{t}(u,\lambda).$$

Therefore,

$$\tilde{J}_\lambda(\underline{t}(u,\lambda), u) = \tilde{J}_\lambda\left(\gamma \underline{t}\,(u,\lambda)\,, \frac{u}{\gamma}\right) = \tilde{J}_\lambda\left(\underline{t}\left(\frac{u}{\gamma}, \lambda\right), \frac{u}{\gamma}\right),$$

for any u in $W \setminus \{0\}$ and the result is proved by taking $\gamma = \|u\|$. ■

Lemma 8. *Let $\lambda \in]0, \widehat{\lambda}[$. Then*
(i) there exist $\{u_n\}_{n\geq 1}, \{v_n\}_{n\geq 1}$ in $\mathbb{S}$ such that

$$\begin{cases} \underline{\mathcal{J}}_\lambda(u_n) \to \underline{\alpha}(\lambda) \;\; as \;\; n \to +\infty, \\ \forall \varphi \in T_{u_n}\mathbb{S}, \underline{\mathcal{J}}'_\lambda(u_n)(\varphi) \leq \frac{1}{n}\|\varphi\|, \end{cases} , \begin{cases} \overline{\mathcal{J}}_\lambda(v_n) \to \overline{\alpha}(\lambda) \;\; as \;\; n \to +\infty, \\ \forall \varphi \in T_{v_n}\mathbb{S}, \overline{\mathcal{J}}'_\lambda(v_n)(\varphi) \leq \frac{1}{n}\|\varphi\|, \end{cases}$$

where $\underline{\mathcal{J}}_\lambda$ and $\overline{\mathcal{J}}_\lambda$ are the functionals defined on $\mathbb{S}$ by $\underline{\mathcal{J}}_\lambda(u) = J_\lambda(\underline{t}(u,\lambda)u)$ and $\overline{\mathcal{J}}_\lambda(u) = J_\lambda(\overline{t}(u,\lambda)u)$.

(ii) For any $u \in W \setminus \{0\}$, $\tilde{J}_\lambda(\underline{t}(u,\lambda), u) < 0$ and $t \longmapsto \tilde{J}_\lambda(t,u)$ is increasing on $]\underline{t}(u,\lambda), \overline{t}(u,\lambda)[$.

(iii) $\liminf_{n\to+\infty} \|\underline{t}(u_n,\lambda)u_n\| > 0$ *and* $\liminf_{n\to+\infty} \|\overline{t}(u_n,\lambda)u_n\| > 0.$

Proof. We start by proving the first assertion. By Remark 2 and by the implicit function theorem, it follows, for any $u \in W \setminus \{0\}$, that $\underline{t}(u, \lambda)$ is a $\mathcal{C}^1$ function with respect to its first variable u. Let us remember that the $\mathcal{C}^1$ functional defined on $\mathbb{S}$ by $\underline{\mathcal{J}}_\lambda(u) = \tilde{J}_\lambda(\underline{t}(u, \lambda), u) = J_\lambda(\underline{t}(u, \lambda)u)$ is bounded below. Therefore, it satifies the Ekeland variational principle on the complete manifold $\mathbb{S}$, i.e there exists a sequence $\{u_n\}_{n\geq 1}$ in $\mathbb{S}$ such that

$$\begin{cases} \underline{\mathcal{J}}_\lambda(u_n) \to \underline{\alpha}(\lambda) \ \text{ as } \ n \to +\infty, \\ \\ \forall \varphi \in T_{u_n}\mathbb{S}, \underline{\mathcal{J}}'_\lambda(u_n)(\varphi) \leq \frac{1}{n}\|\varphi\|. \end{cases}$$

Similar result can be proved with the point $\overline{t}(u, \lambda)$.

Here, we establish the point (ii). First, let us remember the two properties: $\tilde{E}_\lambda(\underline{t}(u, \lambda), u) = 0$ and $t \longmapsto \tilde{E}_\lambda(t, u)$ is increasing on $]0, t(u)[$, which together with $0 < \underline{t}(u, \lambda) < t(u)$, imply that $\tilde{E}_\lambda(t, u) < 0$ on $]0, \underline{t}(u, \lambda)[$ and > 0 on $]\underline{t}(u, \lambda), t(u)[$. Taking into account that $\partial_t \tilde{J}_\lambda(t, u) = t^{q-1}\tilde{E}_\lambda(t, u)$, we conclude that $t \longmapsto \tilde{J}_\lambda(t, u)$ is decreasing on $]0, \underline{t}(u, \lambda)[$ and increasing on $]\underline{t}(u, \lambda), t(u)[$, and hence $\tilde{J}_\lambda(\underline{t}(u, \lambda), u) < 0$. For the second, we have also the two properties: $\tilde{E}_\lambda(\overline{t}(u, \lambda), u) = 0$ and $t \longmapsto \tilde{E}_\lambda(t, u)$ is decreasing on $]t(u), +\infty[$, which together with $\overline{t}(u, \lambda) > t(u)$, give us $\tilde{E}_\lambda(t, u) > 0$ on $]t(u), \overline{t}(u, \lambda)[$. At the same time, $\tilde{E}_\lambda(t, u) > 0$ on $]\underline{t}(u, \lambda), t(u)]$. Therefore $t \longmapsto \tilde{J}_\lambda(t, u)$ is increasing on $]\underline{t}(u, \lambda)], \overline{t}(u, \lambda)[$ and the proof is complete.
Arguing by contradiction in order to prove the last assertion. If the first point is not true, then for any $n \geq 1$, there exists a subsequence of $\{u_n\}_{n\geq 1}$, still denoted by $\{u_n\}_{n\geq 1}$, such that $\|\underline{t}(u_n, \lambda)u_n\| \to 0$.
It follows that $\tilde{J}_\lambda(\underline{t}(u_n, \lambda), u_n) \to 0$ as $n \to +\infty$ and $\underline{\alpha}(\lambda) = 0$. The contradiction comes now from the fact that $\inf\limits_{u \in W\setminus\{0\}} \tilde{J}_\lambda(\underline{t}(u, \lambda), u) = 0$ and $\tilde{J}_\lambda(\underline{t}(u, \lambda), u) < 0$.
For the second, we consider for any $n \geq 1$, a subsequence of $\{u_n\}_{n\geq 1}$, still denoted by $\{u_n\}_{n\geq 1}$, such that $\|\overline{t}(u_n, \lambda)u_n\| \to 0$. ¿From $\partial_t \tilde{J}_\lambda(\overline{t}(u_n, \lambda), u_n) = 0$ and $\partial_{tt}\tilde{J}_\lambda(\overline{t}(u_n, \lambda), u_n) < 0$, it results

$$\begin{cases} P(\overline{t}(u_n, \lambda)u_n) - \lambda Q(\overline{t}(u_n, \lambda)u_n) - R(\overline{t}(u_n, \lambda)u_n) = 0, \\ \\ (p-1)P(\overline{t}(u_n, \lambda)u_n) - \lambda(q-1)Q(\overline{t}(u_n, \lambda)u_n) - (r-1)R(\overline{t}(u_n, \lambda)u_n) < 0, \end{cases}$$

which give

$$(p-q)P(\overline{t}(u_n, \lambda)u_n) \leq (r-q)R(\overline{t}(u_n, \lambda)u_n).$$

Combining it with the compactness of the embedding of W into $L^r(\Omega, \omega_1)$, we find

$$(p-q)P(\overline{t}(u_n, \lambda)u_n) \leq (r-q)kC_\Omega P(\overline{t}(u_n, \lambda)u_n)^{\frac{r}{p}},$$

and hence,

$$(p-q) \leq (r-q)kC_\Omega P(\overline{t}(u_n, \lambda)u_n)^{\frac{r}{p}-1} \to 0 \ \text{ as } \ n \to +\infty.$$

This contradicts $q < p$. The proof is complete. ■

Before proving that $\{\underline{t}(u_n, \lambda)u_n\}_{n\geq 1}$ and $\{\overline{t}(u_n, \lambda)u_n\}_{n\geq 1}$ are Palais-Smale sequences for the functionals $J_\lambda(\underline{t}(u, \lambda)u)$ and $J_\lambda(\overline{t}(u, \lambda)u)$ respectively, we establish a result which will be useful in the sequel.

Remark 4. We consider the mapping

$$\begin{array}{rll} \alpha: & W\setminus\{0\} & \longrightarrow \mathbb{R}_+^* \times \mathbb{S} \\ & u & \longmapsto \alpha(u) = (\alpha_1(u), \alpha_2(u)) \end{array}$$

with $(\alpha_1(u), \alpha_2(u)) = \left(\|u\|, \frac{u}{\|u\|}\right)$.
Let $\varphi \in W$ be arbitrary choosen. The real number

$$\begin{aligned} \alpha_1'(u)(\varphi) &= \|u\|^{1-p}\left(\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi\, d\,x + \int_\Gamma a(x)|u|^{p-2}u\varphi\, d\,\sigma\right) \\ &\le \|u\|^{1-p}\left[\left(\int_\Omega |\nabla u|^p\, d\,x\right)^{\frac{p-1}{p}}\left(\int_\Omega |\nabla\varphi|^p\, d\,x\right)^{\frac{1}{p}} + \right. \\ &\qquad \left.\left(\int_\Gamma a(x)|u|^p\, d\,\sigma\right)^{\frac{p-1}{p}}\left(\int_\Gamma a(x)|\varphi|^p\, d\,\sigma\right)^{\frac{1}{p}}\right] \\ &\qquad\qquad \le \|u\|^{1-p}\|u\|^{p-1}\|\varphi\| = \|\varphi\|. \end{aligned}$$

Next, $\alpha_2'(u)(\varphi)$ belongs to $T_u\mathbb{S}$ and

$$\alpha_2'(u)(\varphi) = \frac{\varphi}{\|u\|} - \frac{u}{\|u\|^2}\alpha_1'(u)(\varphi) \le 2\frac{\|\varphi\|}{\|u\|}$$

Now, we are able to prove

Lemma 9. *Let $\{u_n\}_{n\ge1}$ be as above. The sequences $\{\underline{t}(u_n,\lambda)u_n\}_{n\ge1}$ and $\{\overline{t}(u_n,\lambda)u_n\}_{n\ge1}$ are Palais-Smale sequences for the functionals $J_\lambda(\underline{t}(u,\lambda)u)$ and $J_\lambda(\overline{t}(u,\lambda)u)$ respectively.*

Proof. We only establish the statement for $J_\lambda(\underline{t}(u,\lambda)u)$, the arguments are similar for $J_\lambda(\overline{t}(u,\lambda)u)$. Since $\underline{\mathcal{J}}_\lambda(u_n) = J_\lambda(\underline{t}(u_n,\lambda)u_n) \to \underline{\alpha}(\lambda)$ as $n\to\infty$, then it remains to prove that $\|J_\lambda'(\underline{t}(u_n,\lambda)u_n)\|_* \to 0$ as $n\to\infty$, where $\|.\|_*$ denotes the norm on the dual space W'.
Let us remember that for any ψ in $T_{u_n}\mathbb{S}$,

$$\begin{aligned} |\underline{\mathcal{J}}_\lambda'(u_n)(\psi)| &= |\partial_t\tilde{J}_\lambda(\underline{t}(u_n,\lambda),u_n)\underline{t}'(u_n,\lambda)(\psi) + \partial_u\tilde{J}_\lambda(\underline{t}(u_n,\lambda),u_n)(\psi)| \\ &= |\partial_u\tilde{J}_\lambda(\underline{t}(u_n,\lambda),u_n)(\psi)| \le \frac{1}{n}\|\psi\|, \end{aligned} \tag{17}$$

where $\underline{t}'(u_n,\lambda)$ denotes the derivative of $\underline{t}(.,\lambda)$ with respect to its first variable at the point (u_n,λ).
Considering $\underline{v}_n = \underline{t}(u_n,\lambda)u_n$, from $u_n\in\mathbb{S}$, we get

$$J_\lambda(\underline{v}_n) = \widetilde{J}_\lambda(\underline{t}(u_n,\lambda),u_n) = \widetilde{J}_\lambda\left(\|\underline{v}_n\|, \frac{\underline{v}_n}{\|\underline{v}_n\|}\right) = \widetilde{J}_\lambda\circ\alpha(\underline{v}_n).$$

Therefore, for any φ in W,

$$\begin{aligned} |J_\lambda'(\underline{t}(u_n,\lambda)u_n)(\varphi)| &= |J_\lambda'(\underline{v}_n)(\varphi)| \\ &= |\partial_t\widetilde{J}_\lambda(\alpha(\underline{v}_n))\alpha_1'(\underline{v}_n)(\varphi) + \partial_u\widetilde{J}_\lambda(\alpha(\underline{v}_n))(\alpha_2'(\underline{v}_n)(\varphi))| \\ &= |\partial_u\widetilde{J}_\lambda(\underline{t}(u_n,\lambda),u_n)(\varphi_n^2)| = |\underline{\mathcal{J}}_\lambda'(u_n)(\varphi_n^2)|, \end{aligned}$$

with $\varphi_n^2 = \alpha_2'(\underline{t}(u_n, \lambda), u_n)(\varphi)$ belongs to $T_{u_n}\mathbb{S}$. By virtue of Remark 4, we have

$$\|\varphi_n^2\| = \|\alpha_2'(\underline{t}(u_n, \lambda), u_n)(\varphi)\| \leq 2\frac{\|\varphi\|}{\|\underline{v}_n\|} = 2\frac{\|\varphi\|}{\underline{t}(u_n, \lambda)}$$

and by the last assertion of Lemma 8, the sequence $\{\underline{t}(u_n, \lambda)\}_{n\geq 1}$ is bounded below by a positive constant $c(\lambda)$. Consequently, for all $n \in \mathbb{N}$, we get from (17)

$$|J_\lambda'((\underline{t}(u_n, \lambda)u_n)(\varphi)| \leq \frac{2}{n}\frac{\|\varphi\|}{c(\lambda)}$$

and the proof is complete. ■

In this section, we verify that the functional J_λ satisfies the Palais-Smale condition. In order to do this, we need the following lemma:

Lemma 10. (cf. Diaz [9], Lemma 4.10) *Let $x, y \in \mathbb{R}^N$. If $p \geq 2$, then it holds:*

$$|x-y|^p \leq C(|x|^{p-2}x - |y|^{p-2}y)(x-y). \tag{18}$$

If $1 < p < 2$, then it holds:

$$|x-y|^2 \leq C(|x|^{p-2}x - |y|^{p-2}y)(x-y)(|x|+|y|)^{2-p}. \tag{19}$$

Now, we may state the main result of this section.

Theorem 1. *Let $1 < q < p < r < 2^*$ and $\lambda \in]0, \widehat{\lambda}[$. Then the problem $(\mathcal{P}_\lambda)$ has at least two nonnegative weak solutions.*

Proof. We must show the existence of a subsequence of $\{\underline{t}(u_n, \lambda)u_n\}_{n\geq 1}$ which converges strongly in W (similar argument for $\{\overline{t}(u_n, \lambda)u_n\}_{n\geq 1}$). First, as a consequence of Lemma 6, the sequence $\{\underline{t}(u_n, \lambda)u_n\}_{n\geq 1}$ is bounded in W. Next, we show that $\{\underline{U}_n\}_{n\geq 1} = \{\underline{t}(u_n, \lambda)u_n\}_{n\geq 1}$ is a Cauchy sequence. In the case $p \geq 2$, we obtain:

$$\begin{aligned}
\|\underline{U}_n - \underline{U}_k\|^p &= \int_\Omega |\nabla(\underline{U}_n - \underline{U}_k)|^p\,dx + \int_\Gamma a(x)|\underline{U}_n - \underline{U}_k|^p\,d\sigma \\
&\leq C\,(< J_\lambda'(\underline{U}_n), \underline{U}_n - \underline{U}_k > - < J_\lambda'(\underline{U}_k), \underline{U}_n - \underline{U}_k >) \\
&+ \frac{\lambda}{q} < Q'(\underline{U}_n) - Q'(\underline{U}_k), \underline{U}_n - \underline{U}_k > + \frac{1}{r} < R'(\underline{U}_n) - R'(\underline{U}_k), \underline{U}_n - \underline{U}_k > \\
&\leq C\left(\|J_\lambda'(\underline{U}_n)\|_* + \|J_\lambda'(\underline{U}_k)\|_* + \frac{\lambda}{q}\|Q'(\underline{U}_n) - Q'(\underline{U}_k)\|_*\right. \\
&+ \left.\frac{1}{r}\|R'(\underline{U}_n) - R'(\underline{U}_k)\|_*\right)\|\underline{U}_n - \underline{U}_k\|.
\end{aligned}$$

Now, from $\|J_\lambda'(\underline{U}_n)\|_* \to 0$ as $n \to \infty$, we obtain, using the compactness of Q' and R', the existence of a subsequence of $\{\underline{U}_n\}_{n\geq 1}$, still denoted by $\{\underline{U}_n\}_{n\geq 1}$, that converges strongly to $\underline{U}$ in W.

For $1 < p < 2$, invoking (19) and Hölder's inequality, we get:

$$\begin{aligned}
&\left[\int_\Omega |\nabla(\underline{U}_n - \underline{U}_k)|^p \, dx\right]^{\frac{2}{p}} \\
&\quad \leq C \left[\int_\Omega \left[\left(|\nabla \underline{U}_n|^{p-2}\nabla \underline{U}_n - |\nabla \underline{U}_k|^{p-2}\nabla \underline{U}_k\right)(\nabla \underline{U}_n - \nabla \underline{U}_k)\right]^{\frac{p}{2}}\right. \\
&\qquad\qquad\qquad \left. [|\nabla \underline{U}_n| + |\nabla \underline{U}_k|]^{\frac{p(2-p)}{2}} \, dx\right]^{\frac{2}{p}} \\
&\quad \leq C \left[\int_\Omega \left[\left(|\nabla \underline{U}_n|^{p-2}\nabla \underline{U}_n - |\nabla \underline{U}_k|^{p-2}\nabla \underline{U}_k\right)(\nabla \underline{U}_n - \nabla \underline{U}_k)\right] dx\right] \\
&\qquad\qquad\qquad \times \left[\int_\Omega [|\nabla \underline{U}_n| + |\nabla \underline{U}_k|]^p \, dx\right]^{\frac{2-p}{p}}. \qquad (20)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left[\int_\Gamma a(x)|\underline{U}_n - \underline{U}_k|^p \, d\sigma\right]^{\frac{2}{p}} &\leq C\left[\int_\Gamma a(x)\left[(|\underline{U}_n|^{p-2}\underline{U}_n - |\underline{U}_k|^{p-2}\underline{U}_k)(\underline{U}_n - \underline{U}_k)\right] d\sigma\right] \\
&\times \left[\int_\Omega [|\underline{U}_n| + |\underline{U}_k|]^p \, dx\right]^{\frac{2-p}{p}}. \qquad (21)
\end{aligned}$$

By the boundeness of the sequence $\{\underline{U}_n\}_{n\geq 1}$ and by the Young inequality, we can find a constant $C_3 > 0$ such that

$$\begin{aligned}
\|\underline{U}_n - \underline{U}_k\|^2 &\leq 2^{\frac{2}{p}-1}\left(\left[\int_\Omega |\nabla(\underline{U}_n - \underline{U}_k)|^p \, dx\right]^{\frac{2}{p}} + \left[\int_\Gamma a(x)|\underline{U}_n - \underline{U}_k|^p \, d\sigma\right]^{\frac{2}{p}}\right) \\
&\leq 2C_3\,[< J'_\lambda(\underline{U}_n) - J'_\lambda(\underline{U}_k), \underline{U}_n - \underline{U}_k > \\
&\qquad + \frac{\lambda}{q} < Q'(\underline{U}_n) - Q'(\underline{U}_k), \underline{U}_n - \underline{U}_k > \\
&\qquad\quad + \frac{1}{r} < R'(\underline{U}_n) - R'(\underline{U}_k), \underline{U}_n - \underline{U}_k > \Big].
\end{aligned}$$

By similar argument to the one used in the proof of the first case $p \geq 2$, we obtain the convergence of a subsequence of $\{\underline{U}_n\}_{n\geq 1}$ to $\underline{U}$ in W, which leads, by the third assertion of Lemma 8, to $0 < \lim_{n\to+\infty} \|\underline{U}_n\| = \|\underline{U}\|$. Next, it follows from $J_\lambda(\underline{U}_n) = J_\lambda(|\underline{U}_n|)$ that the sequence $\{\underline{U}_n\}_{n\geq 1}$ can be repalced by the nonnegative one, $\{|\underline{U}_n|\}_{n\geq 1}$, and the existence of $|\underline{U}| \geq 0$ as a weak solution of $(\mathcal{P}_\lambda)$ is proved. ■

Remark 5. It results from the convergence of $\{\overline{U}_n\}_{n\geq 1} = \{\overline{t}(u_n, \lambda)u_n\}_{n\geq 1}$ to $\overline{U}$ in W that $\overline{t}(u_n, \lambda) = \|\overline{U}_n\| \to \|\overline{U}\| > 0$ as $n \to \infty$. By the caracterization of the positive real number $\overline{t}(u_n, \lambda)$ (see Remark 2), which together with $u_n \to \overline{u} = \frac{\overline{U}}{\|\overline{U}\|}$ as $n \to \infty$ give us $\overline{t}(\overline{u}, \lambda) = \|\overline{U}\|$.

4 Behaviour of the energy

In this section, we examine the behaviour of the energy of $\underline{U}$ and $\overline{U}$.

Theorem 2. *Let* $1 < q < p < r < 2^*$*. Then*

$$(i)\;\; J_\lambda(\underline{U}) < 0 \;\; \textit{for} \;\; \lambda \in]0, \widehat{\lambda}[$$

and

$$(ii) \begin{cases} J_\lambda(\overline{U}) > 0 \;\; \textit{for} \;\; \lambda \in]0, \lambda_0[, \\ J_\lambda(\overline{U}) = 0 \;\; \textit{for} \;\; \lambda = \lambda_0, \\ J_\lambda(\overline{U}) < 0 \;\; \textit{for} \;\; \lambda \in]\lambda_0, \widehat{\lambda}[\end{cases}$$

where $\lambda_0 = \dfrac{q}{r}\left(\dfrac{r}{p}\right)^{\frac{r-q}{r-p}} \widehat{\lambda}.$

Proof. Note that (i) arises automatically from the second point of Lemma 8. For (ii), let u be an arbitrary element of $W \setminus \{0\}$ et let us write

$$\widetilde{J}_\lambda(t,u) = t^q \widetilde{F}_\lambda(t,u) \;\; \text{where} \;\; \widetilde{F}_\lambda(t,u) = t^{p-q}\frac{P(u)}{p} - \frac{\lambda}{q}Q(u) - t^{r-q}\frac{R(u)}{r}.$$

The equation $\partial_t \widetilde{F}_\lambda(t,u) = 0$ acquires the form

$$t^{p-q-1}\left(\frac{p-q}{p}P(u) - \frac{r-q}{r}t^{r-p}R(u)\right) = 0 \tag{22}$$

and has one positive root

$$t_0(u) = \left(\frac{r}{p}\right)^{\frac{1}{r-p}} \left[\frac{p-q}{r-q}\frac{P(u)}{R(u)}\right]^{\frac{1}{r-p}} = \left(\frac{r}{p}\right)^{\frac{1}{r-p}} t(u) \tag{23}$$

with $t(u)$ is defined by (10). We observe that for any real numbers r, p such that $1 < p < r$, $\left(\frac{r}{p}\right)^{\frac{1}{r-p}} > 1$ and hence $t_0(u) > t(u)$.
It results from that (22) that $t \longmapsto \widetilde{F}_\lambda(t,u)$ attains its maximum at $t_0(u)$. Moreover, it is increasing on $]0, t_0(u)[$ and decreasing on $]t_0(u), +\infty[$.
Inserting $t_0(u)$ into $\widetilde{F}_\lambda(t,u)$, we get

$$\widetilde{F}_\lambda(t_0(u),u) = \left[\frac{p-q}{r-q}\frac{P(u)}{R(u)}\right]^{\frac{p-q}{r-p}} \left(\frac{r}{p}\right)^{\frac{p-q}{r-p}} \frac{r-p}{p(r-q)}P(u) - \frac{\lambda}{q}Q(u),$$

and the number of roots of $t \longmapsto \widetilde{F}_\lambda(t,u)$ depends on the sign of $\widetilde{F}_\lambda(t_0(u),u)$. More precisely,

$$\begin{cases} t \longmapsto \widetilde{F}_\lambda(t,u) \;\; \text{has two positive roots} \iff \widetilde{F}_\lambda(t_0(u),u) > 0, \\ t \longmapsto \widetilde{F}_\lambda(t,u) \;\; \text{has one positive root} \iff \widetilde{F}_\lambda(t_0(u),u) = 0, \\ t \longmapsto \widetilde{F}_\lambda(t,u) \;\; \text{has zero root} \iff \widetilde{F}_\lambda(t_0(u),u) < 0. \end{cases}$$

We define now $\lambda_0(u)$ as the positive real number such that $\widetilde{F}_{\lambda_0(u)}(t_0(u),u) = 0$, i.e

$$\lambda_0(u) = \frac{q}{r}\left(\frac{r}{p}\right)^{\frac{r-q}{r-p}} \lambda(u),$$

where $\lambda(u)$ is defined by (11), and we have the following:

$$\begin{cases} \widetilde{F}_\lambda(t_0(u),u) > 0 \Longleftrightarrow \lambda < \lambda_0(u), \\ \widetilde{F}_\lambda(t_0(u),u) = 0 \Longleftrightarrow \lambda = \lambda_0(u), \\ \widetilde{F}_\lambda(t_0(u),u) < 0 \Longleftrightarrow \lambda > \lambda_0(u). \end{cases}$$

Taking into account that $\widetilde{J}_\lambda(t,u) = t^q \widetilde{F}_\lambda(t,u)$, we obtain

$$\begin{cases} \widetilde{J}_\lambda(t_0(u),u) > 0 \Longleftrightarrow \lambda < \lambda_0(u), \\ \widetilde{J}_\lambda(t_0(u),u) = 0 \Longleftrightarrow \lambda = \lambda_0(u), \\ \widetilde{J}_\lambda(t_0(u),u) < 0 \Longleftrightarrow \lambda > \lambda_0(u). \end{cases}$$

We have to prove that $\lambda_0(u) < \lambda(u)$. To do this, let us introduce the decreasing function

$$\begin{array}{lll}]0,1[& \longrightarrow & \mathbb{R} \\ t & \longmapsto & \frac{-\ln t}{1-t} \end{array}$$

which permits us to conclude that

$$\ln \frac{1}{x} > \frac{1-x}{1-y} \ln \frac{1}{y}$$

for every real numbers x, y such that $0 < x < y < 1$. In particular, taking $x = \frac{q}{r}$ and $y = \frac{p}{r}$, we get

$$\frac{r}{q} > \left(\frac{r}{p}\right)^{\frac{r-q}{r-p}}$$

and hence $\lambda_0(u) < \lambda(u)$. Consequently,

$$\lambda_0 := \inf_{u \in W \setminus \{0\}} \lambda_0(u) < \widehat{\lambda}.$$

The growth properties of $t \longmapsto \widetilde{F}_{\lambda_0(u)}(t,u)$ imply that $\partial_{tt}\widetilde{F}_{\lambda_0(u)}(t_0(u),u) < 0$, which, together with $\widetilde{F}_{\lambda_0(u)}(t_0(u),u) = \partial_t \widetilde{F}_{\lambda_0(u)}(t_0(u),u) = 0$ give us $\partial_t \widetilde{J}_{\lambda_0(u)}(t_0(u),u) = 0$ and $\partial_{tt}\widetilde{J}_{\lambda_0(u)}(t_0(u),u) < 0$. Using the caracterization of the positive real number $\overline{t}(u, \lambda_0(u))$ (see Remark 2), we obtain

$$\overline{t}(u, \lambda_0(u)) = t_0(u).$$

First, we show the first assertion of (ii). Fixing a $\lambda \in]0, \lambda_0[$, by the definition of λ_0, we have $\widetilde{J}_\lambda(t_0(u),u) > 0$ for any $u \in W \setminus \{0\}$. On the other hand, $t \longmapsto \widetilde{J}_\lambda(t,u)$ attains its maximum at $\overline{t}(u,\lambda)$ therefore for any $u \in W \setminus \{0\}$, $\widetilde{J}_\lambda(\overline{t}(u,\lambda),u) \geq \widetilde{J}_\lambda(t_0(u),u) > 0$ and according to Remark 5, $J_\lambda(\overline{U}) = \widetilde{J}_\lambda(\overline{t}(\overline{u},\lambda),\overline{u}) > 0$.
For the case $\lambda = \lambda_0$, we have

$$J_{\lambda_0}(\overline{U}) = \widetilde{J}_{\lambda_0}(\overline{t}(\overline{u},\lambda_0),\overline{u}) = \overline{\alpha}(\lambda_0) := \inf_{u \in \mathbb{S}} \widetilde{J}_{\lambda_0}(\overline{t}(u,\lambda_0),u).$$

On the other hand, the lower semicontinuity of the functional defined on $W \setminus \{0\}$ by $u \longmapsto \frac{P(u)^{\frac{p-q}{r-p}+1}}{R(u)^{\frac{p-q}{r-p}} Q(u)}$ with

$$\begin{aligned} \lambda_0 &= \frac{q}{r}\left(\frac{r}{p}\right)^{\frac{r-q}{r-p}} \frac{r-p}{r-q} \left[\frac{p-q}{r-q}\right]^{\frac{p-q}{r-p}} \inf_{u \in W\setminus\{0\}} \frac{P(u)^{\frac{p-q}{r-p}+1}}{R(u)^{\frac{p-q}{r-p}} Q(u)} \\ &= \frac{q}{r}\left(\frac{r}{p}\right)^{\frac{r-q}{r-p}} \frac{r-p}{r-q} \left[\frac{p-q}{r-q}\right]^{\frac{p-q}{r-p}} \inf_{u \in W\setminus\{0\}, R(u)^{\frac{p-q}{r-p}} Q(u) = 1} P(u)^{\frac{p-q}{r-p}+1} \end{aligned}$$

leads to the existence of $u^* \in W \setminus \{0\}$ such that $\lambda_0 = \lambda_0(u^*)$. This implies that

$$J_{\lambda_0}(\overline{U}) \leq \widetilde{J}_{\lambda_0(u^*)}(\overline{t}(u^*, \lambda_0(u^*)), u^*) = \widetilde{J}_{\lambda_0(u^*)}(t_0(u^*), u^*) = 0.$$

Moreover, since $t \longmapsto \widetilde{J}_{\lambda_0}(t, \overline{u})$ attains its maximum at $\overline{t}(\overline{u}, \lambda_0)$ and $\lambda \longmapsto \widetilde{J}_\lambda(t, u)$ is a decreasing function on $\mathbb{R}_+^*$ then

$$J_{\lambda_0}(\overline{U}) = \widetilde{J}_{\lambda_0}(\overline{t}(\overline{u}, \lambda_0), \overline{u}) \geq \widetilde{J}_{\lambda_0}(t_0(\overline{u}), \overline{u}) \geq \widetilde{J}_{\lambda_0(\overline{u})}(t_0(\overline{u}), \overline{u}) = 0.$$

Finally, assume that $\lambda_0 < \lambda < \widehat{\lambda}$. Since for any $u \in W \setminus \{0\}$, $J_\lambda(u)$ is a decreasing function with respect to λ, then $J_\lambda(\overline{U}) < J_{\lambda_0}(\overline{U}) = 0$. ■

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