Classifications of Real Hypersurfaces in Complex Space Forms by means of Curvature Conditions

Miguel Ortega*

Abstract

It has been proved there are no semi-parallel real hypersurfaces in the complex projective space $\mathbb{C}P^n$, $n \geq 3$, and in any non-flat complex space form of complex dimension 2. Also, characterizations of geodesic hyperspheres and ruled real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$, have been obtained by considering some other curvature conditions. We generalize these results by studying two new conditions for real hypersurfaces in non-flat complex space forms. As a corollary, we extend the known characterizations to real hypersurfaces of type A_0 and A_1 and ruled real hypersurfaces in non-flat complex space forms. In particular, we prove that there are no semi-parallel real hypersurfaces in non-flat complex space forms of complex dimension at least 2.

1 Introduction

In Y. Tashiro and S. Tachibana's classical paper [11], we can find a proof for the non-existence of totally umbilical real hypersurfaces in non-flat complex space forms $\overline{M}^n(c)$, $n \geq 2$, of constant holomorphic sectional curvature $4c \neq 0$. This is closely related to the fact that there are no real hypersurfaces in $\overline{M}^n(c)$, $n \geq 2$, whose

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Weingarten endomorphism A is parallel ($\nabla A = 0$). In some way, the parallelism of the Weingarten endomorphism can be regarded as an expression involving a first order derivative of A. Thus, the action of the curvature operator R of the real hypersurface as a derivation on A, $R \cdot A$, is a second order derivative of A that naturally generalizes ∇A. Regarding this, we have been able to find in the literature the following:

1. S. Maeda studied in [5] semi-parallel real hypersurfaces in the complex projective space $\mathbb{C}P^n$, $n \geq 3$, i.e., real hypersurfaces satisfying $R \cdot A = 0$. His important result is the non-existence of such real hypersurfaces. But his proof does not hold for the complex hyperbolic space $\mathbb{C}H^n$ or for $n = 2$.

2. R. Niebergall and P. J. Ryan also obtained in [9] that there are no semi-parallel real hypersurfaces in non-flat complex space forms of complex dimension 2. R. Niebergall and P. J. Ryan's techniques are rather different from the ones used by S. Maeda, and they do not seem to work for $n \geq 3$.

   J. Berndt wrote a review of paper [9] (see MR 99d:53058), saying about the semiparallel condition, 'The correponding result [...] for $n > 2$ and $c > 0$ was obtained by S. Maeda [Math. Ann. 263 (1983), no. 4, 473-478; MR 85d:53025]. For $c < 0$ and $n > 2$ this problem is still open.' One of the main aims of this paper is to solve this problem.

   As a natural consequence of S. Maeda's paper, some authors studied weaker conditions than $R \cdot A = 0$. From now, for the sake of simplicity, we put $Q(X,Y)Z = (R(X,Y) \cdot A)Z$ for suitable tangent vectors X, Y, Z to the real hypersurface. Moreover, for the definition of the (local) structure vector field ξ, and for a description of real hypersurfaces of type A_0 and A_1, see the Preliminaries section.

3. T. Gotoh considered in [2] the expression $Q(X,Y)Z = 0$ for any tangent vectors X, Y, Z to the real hypersurface that are orthogonal to ξ, in $\mathbb{C}P^n$, $n \geq 3$. He obtained a characterization of the geodesic hyperspheres.

4. M. Kimura and S. Maeda studied $Q(X,Y)Z + Q(Y,Z)X + Q(Z,X)Y = 0$ for any tangent vectors X, Y, Z to the real hypersurface, in $\mathbb{C}P^n$, $n \geq 2$. For $n \geq 3$, they were able to characterize geodesic hyperspheres as the only real hypersurfaces satisfying this condition. However, for $n = 2$, they showed the surprising fact that a real hypersurface satisfies it if and only if its structure vector field ξ is principal.

5. On the other hand, Y. Matsuyama's point of view in [6] is a bit different. He considered the expression

   $$g(R(AX,Y)Z,W) - g(AR(X,Y)Z,W) = 0, \tag{1}$$

   for any X, Y, Z, W orthogonal to ξ, for real hypersurfaces in $\mathbb{C}P^n$, $n \geq 3$. In his theorem he included real hypersurfaces of type A_2 among those satisfying (1), but we check in Proposition 1 that this is not correct.

Throughout all these papers except [9], which is cited in our previous list in item 2, authors only deal with real hypersurfaces in $\mathbb{C}P^n$. In this note, we study real hypersurfaces in non-flat complex space forms by considering several conditions involving the curvature operator and the Weingarten endomorphism. Thus, we generalize the above mentioned results that we found in the literature. Firstly, in section 3 we prove the following

Theorem 1. *Let M be a connected real hypersurface in $\overline{M}^n(c)$, $n \geq 3$, $c \neq 0$. Then M satisfies (1) if and only if M is one of the following real hypersurfaces:*

a) a ruled real hypersurface;

b) an open subset of a real hypersurface of type A_0 or A_1.

Secondly, we consider

$$Q(X, Y)Z + Q(Z, X)Y + Q(Y, Z)X = 0, \tag{2}$$

for any X, Y, Z orthogonal to ξ, and

$$g(Q(X, Y)Z, W) + g(Q(Y, Z)X, W) + g(Q(Z, X)Y, W) = 0, \tag{3}$$

for any X, Y, Z, W orthogonal to ξ. Clearly, these two expressions generalize the conditions studied by T. Gotoh, S. Maeda and M. Kimura, and R. Niebergall and P. J. Ryan. We should point out that, at first sight, condition (3) is weaker than condition (2). In section 4 we prove the following theorem, showing that these two conditions are equivalent:

Theorem 2. *Let M be a connected real hypersurface in $\overline{M}^n(c)$, $n \geq 3$, $c \neq 0$. The following statements are pairwise equivalent:*

(i) M satisfies (2);

(ii) M satisfies (3);

(iii) M is either a ruled real hypersurface or an open subset of real hypersurface of type A_0 or A_1.

The last section is devoted to show the connection between our results and previous ones. In fact, in Corollary 1, we extend T. Gotoh and M. Kimura and S. Maeda's results to $\overline{M}^n(c)$, $n \geq 3$, obtaining a characterization of real hypersurfaces of type A_0 and A_1. Finally, in Corollary 2, we prove the non-existence of semi-parallel real hypersurfaces in $\overline{M}^n(c)$, $n \geq 2$, $c \neq 0$.

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2 Preliminaries

Let $\overline{M}^n(c)$ be a non-flat complex space form endowed with the metric g of constant holomorphic sectional curvature $4c \neq 0$ and complex dimension $n \geq 2$. For the sake of simplicity, if $c > 0$, we will only use $c = +1$, and we will call it the complex projective space $\mathbb{C}P^n$, and if $c < 0$, we just consider $c = -1$, so that we will call it the complex hyperbolic space $\mathbb{C}H^n$. Let M be a connected C^∞ real hypersurface in $\overline{M}^n(c)$, $c \neq 0$, $n \geq 2$, without boundary. Let N be a local unit normal vector field to M. If J is the almost complex structure of $\overline{M}^n(c)$, we define $\xi = -JN$. Usually, the vector field ξ is called the structure or the Reeb vector field of M. The distribution $T_p^o M = \{X \in T_pM : X \perp \xi_p\}$, $p \in M$, is called the holomorphic distribution on M. The Levi-Civita connection of $\overline{M}^n(c)$ and M will be denoted by $\bar{\nabla}$ and ∇, respectively. The Gauss and Weingarten formulae are

$$\begin{aligned} \bar{\nabla}_X Y &= \nabla_X Y + g(AX, Y)N, \\ \bar{\nabla}_X N &= -AX, \end{aligned} \tag{4}$$

for any $X, Y \in TM$. A local tangent vector field X is called principal if it is an eigenvector of A everywhere, and its associated eigenfunction is called principal curvature function. Given a point $p \in M$ and a principal curvature λ, we write $T_\lambda(p) = \{X \in T_pM : A_pX = \lambda(p)X\}$. This vector subspace is called the principal distribution associated with λ at p. The dimension of the principal distribution is known as the multiplicity of the principal curvature. The multiplicity of a principal distribution depends on the point, although there is a dense open subset on which it is locally constant.

Given a vector field X tangent to M on a neighbourhood of a point $p \in M$, we put $JX = \phi X + \eta(X)N$, where ϕX and $\eta(X)N$ are the tangential and the normal component of JX respectively. Thus, ϕ is a skew-symmetric tensor of type (1,1) and η is a 1-form on M. Furthermore, ξ is a locally defined vector field tangent to M. The set (ϕ, ξ, η, g) is called an almost contact metric structure on M, whose elementary properties are

$$\begin{gathered} \eta(X) = g(X, \xi), \quad \phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \nabla_X \xi = \phi AX \\ g(\phi X, Y) + g(X, \phi Y) = 0, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \end{gathered} \tag{5}$$

for any $X, Y \in TM$, where I denotes the identity transformation on TM. The Gauss equation of M is

$$\begin{gathered} R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ -2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{gathered} \tag{6}$$

for any $X, Y, Z \in TM$. The Codazzi equation of M is

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{7}$$

for any $X, Y \in TM$. Cecil and Ryan in [1] and Montiel in [7] classified the real hypersurfaces in complex space forms with at most two distinct principal curvatures at each point. Their results are summarized in the following

Theorem A. *Let M be a connected real hypersurface in $\overline{M}^n(c)$, $n \geq 3$, $c = \pm 1$, with at most two distinct principal curvatures at each point. Then M is an open subset of one of the following:*

1. *in $\mathbb{C}P^n$,*

   A_1) *a tube of radius $0 < r < \pi/2$ over a totally geodesic hyperplane $\mathbb{C}P^{n-1}$;*

2. *in $\mathbb{C}H^n$,*

   A_0) *a horosphere;*

   A_1) *a tube of radius $r > 0$ over a totally geodesic $\mathbb{C}H^k$, where $k = 0, n-1$;*

   B) *a tube of radius $\log((1+\sqrt{3})/\sqrt{2})$ over a totally geodesic $\mathbb{R}H^n$.*

Usually, tubes over a totally geodesic $\overline{M}^k(c)$, with $k \in \{1, \ldots, n-2\}$, are called real hypersurfaces of type A_2. A description of the horosphere A_0 can be found in [8]. Real hypersurfaces of type A_0, A_1 and A_2 are simply known as real hypersurfaces of type A. Finally, we recall that a real hypersurface in $\overline{M}^n(c)$ is said to be *ruled* if the distribution T^oM is integrable and its leaves are open subsets of totally geodesic hyperplanes $\overline{M}^{n-1}(c)$.

3 Proof of Theorem 1

The following lemma is part of the proof of Theorem 1, but it is interesting by itself. The fact that the Codazzi equation implies the non-existence of totally umbilical real hypersurfaces in $\overline{M}^n(c)$ also inspires it.

Lemma 1. *Let M be a connected real hypersurface in $\overline{M}^n(c)$, $n \geq 3$, $c \neq 0$. Suppose there exists a smooth function $\lambda : M \to \mathbb{R}$ such that $g(AX, Y) = \lambda g(X, Y)$ for any $X, Y \in T^oM$. Then λ is constant, and M is one of the following:*

a) if $\lambda = 0$: a ruled real hypersurface;

b) if $\lambda \neq 0$: an open subset of a real hypersurface of type A_0 or A_1.

Proof. : Let G be a connected open subset of M on which ξ is globally defined. Set $G_0 = \{q \in G : \lambda(q) = 0\}$ and $G_1 = G \backslash G_0$. Clearly, G_1 is open. Let Γ be the set of interior points of G_0, and suppose that Γ is non-empty. Given $X, Y \in T^oM$, tangent to Γ, by (4) and (5), $g(\bar{\nabla}_X Y, N) = g(AX, Y) = 0$ and $g(\bar{\nabla}_X Y, \xi) = g(\nabla_X Y, \xi) = -g(Y, \nabla_X \xi) = -g(Y, \phi AX) = 0$. This means that $\bar{\nabla}_X Y \in T^oM$ for any $X, Y \in T^oM$. In other words, T^oM is integrable and its leaves are totally geodesic in $\overline{M}^n(c)$, that is, Γ is a ruled real hypersurface. Conversely, the very same computations show that a ruled real hypersurface satisfies $g(AX, Y) = 0$ for any $X, Y \in T^oM$. From now on, we make our computations on G_1. On it, we set

$A\xi = \mu\xi + U$ where $\mu = g(A\xi, \xi)$ and U is the component of $A\xi$ in T^oM. From our assumption, we obtain

$$A\xi = \mu\xi + U, \quad AU = \lambda U + \eta(AU)\xi, \quad AX = \lambda X \text{ for any } X \perp \{\xi, U\} \text{ on } G_1 \quad (8)$$

As $n \geq 3$, we can choose a unit vector field $X \in T^oM$ tangent to G_1 orthogonal to $\{\xi, U, \phi U\}$. By (5), (7) and (8),

$$\begin{aligned} 0 &= g((\nabla_X A)U - (\nabla_U A)X, \phi X) = g(\nabla_X(\lambda U + \eta(AU)\xi), \phi X) \\ &\quad -g(A\nabla_X U, \phi X) - g(\nabla_U(\lambda X), \phi X) + g(A\nabla_U X, \phi X) \\ &= \lambda g(\nabla_X U, \phi X) + \eta(AU)g(\phi AX, \phi X) - \lambda g(\nabla_X U, \phi X) \\ &\quad -\lambda g(\nabla_U X, \phi X) + \lambda g(\nabla_U X, \phi X) = \eta(AU)\lambda, \end{aligned}$$

and as $\lambda \neq 0$ on G_1, then, $0 = \eta(AU) = g(AU, \xi) = g(U, U)$, that is, $U = 0$ and therefore ξ is principal. Now, by (8), we see $AX = \lambda X$ for any $X \in T^oM$. Therefore, G_1 is locally congruent to one of the examples in Theorem A. All of them have constant principal curvatures, so λ must be locally constant on G_1. But the tube of radius $\log((1+\sqrt{3})/\sqrt{2})$ over a totally geodesic $\mathbb{R}H^n$ does not satisfy $AX = \lambda X$ for any $X \in T^oM$ (see [7]). Therefore, G_1 is locally congruent to a real hypersurface of type A_0 or A_1, and the function λ is locally constant on G_1. But G is connected, and the function λ is continuous on G, constant on G_0 and locally constant on G_1, so that λ is constant. This means we can repeat the reasoning in the whole M to obtain that M is either a ruled real hypersurface or a real hypersurface of type A_0 or A_1. ∎

Proposition 1. *Real hypersurfaces of type A_2 do not satisfy (1).*

Proof. : Any real hypersurface M of type A_2 has three distinct principal curvatures. If $c = +1$, then $\mu = 2\cot(2r)$, $\lambda_1 = \cot(r)$, $\lambda_2 = -\tan(r)$, where $r \in (0, \pi/2)$. If $c = -1$, then $\mu = 2\coth(2r)$, $\lambda_1 = \coth(r)$, $\lambda_2 = \tanh(r)$, where $r > 0$. Moreover, at each point $p \in M$, $T^o_pM = T_{\lambda_1}(p) \oplus T_{\lambda_2}(p)$, and $\phi T_{\lambda_1}(p) = T_{\lambda_1}(p)$, $\phi T_{\lambda_2}(p) = T_{\lambda_2}(p)$ ([1],[7],[8]). Now we choose unit vectors $X \in T_{\lambda_1}, W \in T_{\lambda_2}$, and we consider $Y = \phi W$, $Z = \phi X$. Bearing (6) in mind, we insert them in (1), obtaining $0 = \lambda_1 g(R(X,\phi W)\phi X, W) - \lambda_2 g(R(X,\phi W)\phi X, W) = (\lambda_1 - \lambda_2)g(R(X,\phi W)\phi X, W) = (\lambda_1 - \lambda_2)c$, so that $\lambda_1 - \lambda_2 = 0$, which is a contradiction. This concludes the proof. ∎

Proof of Theorem 1. : By (6), equation (1) is equivalent to

$$\begin{aligned} 0 = c\{&g(X,Z)g(AY,W) - g(AX,Z)g(Y,W) + g(\phi Y,Z)g((\phi A - A\phi)X,W) \\ &-g(\phi AX,Z)g(\phi Y,W) - 2g(\phi AX,Y)g(\phi Z,W) + g(\phi X,Z)g(A\phi Y,W) \\ &+2g(\phi X,Y)g(A\phi Z,W)\} + g(AX,Z)g(A^2Y,W) - g(A^2X,Z)g(AY,W), \end{aligned} \quad (9)$$

for any $X, Y, Z, W \in T^oM$. Now, choose a point $p \in M$ and a connected open neighbourhood G of p in M such that the local vector field ξ is defined on G. Choose an orthonormal basis $\{e_1, \ldots, e_{2n-2}\}$ of T^oM on G. Let us define the smooth function $a : G \to \mathbb{R}$ by $a = \sum_{k=1}^{2n-2} g(Ae_k, e_k)$. We insert $X = e_k$ and $Z = \phi e_k$ in (9), and by summation over k we obtain

$$0 = g(\phi AY, W) + (2n-3)g(A\phi Y, W) - ag(\phi Y, W), \tag{10}$$

for any $Y, W \in T^oM$ tangent to G. If we exchange Y and W in this equation we obtain $0 = g(\phi AW, Y) + (2n-3)g(A\phi W, Y) - ag(\phi W, Y)$. We add this equation to (10), and we get $0 = (4-2n)g(\phi AY, W) + (2n-4)g(A\phi Y, W)$, and as $n \geq 3$, $g(\phi AY, W) = g(A\phi Y, W)$ for any $Y, W \in T^oM$. If we insert this expression in (10), then $0 = (2n-2)g(A\phi Y, W) - ag(\phi Y, W)$. Therefore, there exists a smooth function $\lambda : G \to \mathbb{R}$ such that $g(AY, W) = \lambda g(Y, W)$ for any $Y, W \in T^oM$ tangent to G. By Lemma 1, G is either a ruled real hypersurface or a real hypersurface of type A_0 or A_1. A simple reasoning of connectedness shows that M is a ruled real hypersurface or a real hypersurface of type A_0 or A_1. Suppose that M is a real hypersurface of type A_0 or A_1. Then there exists a constant λ such that $AX = \lambda X$ for any $X \in T^oM$. From this, condition (1) holds. If M is a ruled real hypersurface, by Lemma 1, then $g(AX, Y) = 0$ for any $X, Y \in T^oM$. That means that M satisfies equation (9). This concludes the proof. ∎

4 Proof of Theorem 2

Proof of Theorem 2. : We pointed out in the Introduction that (2) implies (3), so that we have statement (i) implies statement (ii). Now we prove that statement (ii) implies statement (iii). Let M be a connected real hypersurface in $\overline{M}^n(c)$, $n \geq 3$, satisfying (3). We should remember $(R(X,Y) \cdot A)Z = R(X,Y)AZ - AR(X,Y)Z$, so that by first Bianchi identity and (6), equation (3) is equivalent to

$$\begin{aligned} 0 = {} & g((\phi A + A\phi)X, Y)g(\phi Z, W) + g((\phi A + A\phi)Y, Z)g(\phi X, W) \\ & + g((\phi A + A\phi)Z, X)g(\phi Y, W) - 2g(\phi Z, X)g(\phi AY, W) \\ & - 2g(\phi X, Y)g(\phi AZ, W) - 2g(\phi Y, Z)g(\phi AX, W), \end{aligned} \tag{11}$$

for any $X, Y, Z, W \in T^oM$. Given a point $p \in M$, we choose an orthonormal basis $\{e_1, \ldots, e_{2n-2}\}$ of T^oM defined on an open neighbourhood G of p. Define the function $a : G \to \mathbb{R}$ by $a = (1/2)\sum_{k=1}^{2n-2}\{g(Ae_k, e_k) + g(A\phi e_k, \phi e_k)\}$. If we take $Y = e_k$ and $Z = \phi e_k$ in (11), summing up over k we obtain $0 = -2g(A\phi X, W) - (4n-6)g(\phi AX, W) + 2ag(\phi X, W)$ and therefore we get

$$0 = g(A\phi X, W) + (2m-3)g(\phi AX, W) - ag(\phi X, W),$$

for any $X, W \in T^oM$ on G. This equation is equal to equation (10), so we can repeat the proof of Theorem 1 to obtain that M is locally congruent to either a ruled real hypersurface or a real hypersurface of type A_0 or A_1.

Now we should check that statement (iii) implies statement (i). Firstly, if M is real hypersurface of type A_0 or A_1, there exists a real constant a such that $AX = aX$ for any $X \in T^oM$. Then, it is easy to check that equation (2) is satisfied. Secondly, let M be a ruled real hypersurface. By the above reasoning, statement (i) is equivalent to the following equation

$$\begin{aligned}0 = g((\phi A + A\phi)X, Y)g(\phi Z, W) + g((\phi A + A\phi)Y, Z)g(\phi X, W)\\ +g((\phi A + A\phi)Z, X)g(\phi Y, W) - 2g(\phi Z, X)g(\phi AY, W)\\ -2g(\phi X, Y)g(\phi AZ, W) - 2g(\phi Y, Z)g(\phi AX, W),\end{aligned} \tag{12}$$

for any $X, Y, Z \in T^oM$ and any $W \in TM$. By Lemma 1, $g(AX, Y) = 0$ for any $X, Y \in T^oM$. Therefore, if $X, Y, Z, W \in T^oM$, equation (12) is satisfied. If we choose $X, Y, Z \in T^oM$ and $W = \xi$, by (5) and Lemma 1, equation (12) is satisfied. This finishes the proof. ∎

Remark 1. All real hypersurfaces in $\overline{M}^2(c)$ satisfy (2) and (3). Indeed, let M be a real hypersurface in $\overline{M}^2(c)$. Take a unit vector field $X \in T^oM$. Any other $Y, Z \in T^oM$ can be written $Y = aX + b\phi X$ and $Z = \alpha X + \beta\phi X$, where a, b, α, β are suitable functions defined on a certain open subset of M. Then, by elementary properties of R,

$$\begin{aligned}&Q(X, Y)Z + Q(Y, Z)X + Q(Z, X)Y = R(X, Y)AZ + R(Y, Z)AX + R(Z, X)AY\\ &= b\alpha R(X, \phi X)AX + b\beta R(X, \phi X)A\phi X + a\beta R(X, \phi X)AX\\ &+b\alpha R(\phi X, X)AX + a\beta R(\phi X, X)AX + b\beta R(\phi X, X)A\phi X = 0.\end{aligned}$$

This means that the hypothesis $n \geq 3$ cannot be removed from Theorem 2.

5 Further results

Corollary 1. *Let M be a connected real hypersurface in $\overline{M}^n(c)$, $n \geq 3$, $c \neq 0$. The following statements are pairwise equivalent:*

1. *M satisfies $Q(X, Y)Z = 0$ for any $X, Y, Z \in T^oM$;*
2. *M satisfies $Q(X, Y)Z + Q(Y, Z)X + Q(Z, X)Y = 0$ for any $X, Y, Z \in TM$;*
3. *M is an open subset of a real hypersurface of type A_0 or A_1.*

Proof. : We begin by checking that statement 1 and statement 3 are equivalent. If a real hypersurface M satisfies statement 1, it satisfies statement (i) of Theorem 2, so that it is either a ruled real hypersurface or of type A_0 or A_1. If M is of type A_0 or A_1, similar computations as Theorem 2 show they satisfy statement 1. Now, suppose that M is a ruled real hypersurface. For details in the rest of the proof, see paper [4]. We know that any ruled real hypersurface is orientable and its Weingarten endomorphism is of the form

$$A\xi = \mu\xi + U \quad AU = |U|^2\xi, \quad AX = 0 \text{ for any } X \perp \{\xi, U\}, \tag{13}$$

where μ is a smooth function on M, and $U \in T^oM$ is not a unit vector in general. The set of points where the vector U vanishes cannot have interior points, that is to say, ξ cannot be principal on any open subset of the real hypersurface. By (5),

(6) and (13), we compute $-(R(U,\phi U)\cdot A)\phi U = -R(U,\phi U)A\phi U + AR(U,\phi U)\phi U = AR(U,\phi U)\phi U = 4cAU \neq 0$. This shows that no ruled real hypersurface satisfies statement 1.

Now we check that statement 2 is equivalent to statement 3. If M is a real hypersurface satisfying statement 2, by Theorem 2, M is either a ruled real hypersurface or a real hypersurface of type A_0 or A_1. If M is of type A_0 or A_1, we only have to realize that any of these real hypersurfaces satisfies that there exist two real constants λ, μ such that $AX = \lambda X + \mu\eta(X)\xi$ for any $X \in TM$. By first Bianchi's identity and (6), a long but straightforward computation shows our assertion. Next, suppose that M is a ruled real hypersurface. As $n \geq 3$, we can choose a unit vector $X \in T^oM$ orthogonal to $\{U, \phi U\}$. By (13), first Bianchi identity,(5) and (6), $Q(X,\phi X)\xi + Q(\xi, X)\phi X + Q(\phi X, \xi)X = R(X,\phi X)A\xi + R(\xi, X)A\phi X + R(\phi X, \xi)AX = \mu R(X,\phi X)\xi + R(X,\phi X)U = R(X,\phi X)U = -2c\phi U \neq 0$. Therefore, no ruled real hypersurfaces satisfies statement 2. This concludes the proof. ■

Remark 2. One may wonder if hypothesis $n \geq 3$ is necessary in Corollary 1. If $n = 2$, we can repeat the proof of Theorem 3 in [3] to obtain that a real hypersurface in $\overline{M}^2(c)$ satisfies statement 2 if and only if ξ is principal. The author does not know what happens for statement 1.

Corollary 2. *There are no real hypersurfaces in $\overline{M}^n(c)$, $n \geq 2$, $c \neq 0$, such that $R \cdot A = 0$.*

Proof. : We have already pointed out that the case $n = 2$ has been studied by R. Niebergall and P. J. Ryan in [9]. So, we study the case $n \geq 3$. Suppose that there exists a real hypersurface in $\overline{M}^n(c)$ satisfying $(R(X,Y)\cdot A)Z = 0$ for any $X, Y, Z \in TM$. Then it satisfies statement 1 in Corollary 1. Therefore, it is locally congruent to a real hypersurface of type A_0 or A_1. In that case, there exist two distinct real constants λ, μ such that $AX = \lambda X$ for any $X \in T^oM$ and $A\xi = \mu\xi$. From this and (6), given $X \in T^oM$, $0 = (R(X,\xi)\cdot A)\xi = R(X,\xi)A\xi - AR(X,\xi)\xi = (\mu I - A)R(X,\xi)\xi = (c+\mu\lambda)(\mu - \lambda)X$, so the only possibility is $0 = c + \mu\lambda$. But if $c = +1$, then $\mu = 2\cot(2r)$ and $\lambda = \cot(r)$ for a certain $0 < r < \pi/2$, so that this equation does not hold. If $c = -1$, then either $\mu = 2\coth(2r)$ and $\lambda = \coth(r)$, or $\mu = 2\coth(2r)$ and $\lambda = \tanh(r)$ for a certain $r > 0$, or $\mu = 2$ and $\lambda = 1$. Again, this last equation is not satisfied. This concludes the proof. ■

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Departamento de Geometría y Topología
Facultad de Ciencias, Universidad de Granada
18071 Granada (Spain)
email : miortega@ugr.es