A non-abelian tensor product and universal central extension of Leibniz n-algebra

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Abstract

A non-abelian tensor product for Leibniz n-algebras is introduced as a generalization of the non-abelian tensor product for Leibniz algebras introduced by Kurdiani and Pirashvili. We use it to construct the universal central extension of a perfect Leibniz n-algebra.

1 Introduction

In 1973 Nambu [13] proposed a generalization of the classical Hamiltonian formalism where the Poisson bracket is replaced by a n-linear skew-symmetric bracket $\{\dots\}$ (the Nambu bracket) on the algebra of smooth functions on a manifold M. Within the framework of Nambu mechanics, the evolution of physical system is determined by $n-1$ functions $H_1, \dots, H_{n-1} \in C^\infty(\mathrm{M})$ and the equation of motion of an observable $f \in C^\infty(\mathrm{M})$ is given by $df/dt = \{H_1, \dots, H_{n-1}, f\}$.

These ideas inspired novel mathematical structures by extending the binary Lie bracket to a n-bracket (see [3], [4], [5], [14], [17]).

In the 90's Loday [9, 10] introduced a new kind of algebras, called Leibniz algebras, which are the non-skew-symmetric counterpart to Lie algebras. In brief, a Leibniz algebra $\mathfrak{g}$ is a K-vector space equipped with a bilinear bracket $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \forall x,y,z \in \mathfrak{g} \tag{1}$$

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Obviously, if this bracket satisfies $[x,x]=0, \forall x \in \mathfrak{g}$, then the Leibniz identity is the Jacobi identity and a Leibniz algebra is a Lie algebra.

In this context, was natural to extend this concept to Nambu algebras, so in 2002 Casas, Loday and Pirashvili [2] introduced the concept of Leibniz n-algebra, suggested by Takhtajan in [16], and developed a cohomology theory for this kind of algebras, which was complemented in [1] with a homology with trivial coefficients theory.

In this way, in section 3, we construct a type of non-abelian tensor product of Leibniz n-algebras (as a generalization of the non-abelian tensor product for Leibniz algebras introduced by Kurdiani and Pirashvili [7]) which is essential in order to construct the universal central extension of a perfect Leibniz n-algebra.

To summarize, for a Leibniz n-algebra $\mathcal{L}$ we define the Leibniz n-algebra $\mathcal{L}^{*n} := Coker(\delta_2 : \mathcal{L}^{\otimes(2n-1)} \to \mathcal{L}^{\otimes n})$ equipped with the bracket defined by formula (4) below. Then we achieve the exact sequence

$$0 \to {}_nHL_1(\mathcal{L}) \to \mathcal{L}^{*n} \stackrel{[-,\dots,-]}{\longrightarrow} \mathcal{L} \to {}_nHL_0(\mathcal{L}) \to 0$$

where ${}_nHL_\star(-)$ denotes the Leibniz homology with trivial coefficients for Leibniz n-algebras [1]. In case of perfect Leibniz n-algebras, that is $\mathcal{L} = [\mathcal{L}, \stackrel{n}{\dots}, \mathcal{L}]$, we have that ${}_nHL_0(\mathcal{L}) = 0$ and we proof that last sequence is the universal central extension of $\mathcal{L}$.

Previously we introduce in section 2 new concepts of Leibniz n-algebras as commutator n-sided ideal, derivations and semidirect product which are useful in section 3. Moreover we study the relationship between derivations and semidirect product achieving the exact sequence

$$0 \to Der(\mathcal{L}, \mathrm{M}) \to Der(\mathcal{K}, \mathrm{M}) \to Hom_{\mathcal{L}}(\mathcal{N}_{ab}, \mathrm{M})$$

associated to the exact sequence of Leibniz n-algebras $0 \to \mathcal{N} \to \mathcal{K} \to \mathcal{L} \to 0$, and the representatibility of derivation functor.

2 Preliminaries on Leibniz n-algebras

A *Leibniz n-algebra* is a K-vector space $\mathcal{L}$ equipped with a n- linear bracket $[-,\dots,-] : \mathcal{L}^{\otimes n} \to \mathcal{L}$ satisfying the following fundamental identity

$$[[x_1, x_2, \dots, x_n], y_1, y_2, \dots, y_{n-1}] = \sum_{i=1}^{n} [x_1, \dots, x_{i-1}, [x_i, y_1, y_2, \dots, y_{n-1}], x_{i+1}, \dots, x_n] \qquad (2)$$

A morphism of Leibniz n-algebras is a linear map preserving the n-bracket. So we have defined the category of Leibniz n-algebras, denoted by ${}_n\mathbf{Lb}$. In case $n=2$ the identity (2) is the Leibniz identity (1), so a Leibniz 2-algebra is a Leibniz algebra [10], and we use $\mathbf{Lb}$ instead of ${}_2\mathbf{Lb}$.

Leibniz $(n+1)$-algebras and Leibniz algebras are related by means of the Daletskii's functor [3] which assigns to a Leibniz $(n+1)$-algebra $\mathcal{L}$ the Leibniz algebra

$\mathcal{D}_n(\mathcal{L}) = \mathcal{L}^{\otimes n}$ with bracket

$$[a_1 \otimes \cdots \otimes a_n, b_1 \otimes \cdots \otimes b_n] := \sum_{i=1}^{n} a_1 \otimes \cdots \otimes [a_i, b_1, \ldots, b_n] \otimes \cdots \otimes a_n$$

Conversely, if $\mathcal{L}$ is a Leibniz algebra, then also is a Leibniz n-algebra under the following n-bracket [2]

$$[x_1, x_2, \ldots, x_n] := [x_1, [x_2, \ldots, [x_{n-1}, x_n], \ldots]] \tag{3}$$

Examples:

1. Examples of Leibniz algebras in [10] yield examples of Leibniz n-algebras with the bracket defined by equation (3).

2. A Lie triple system [8] is a vector space equipped with a bracket $[-,-,-]$ that satisfies the same identity (2) (particular case $n = 3$) and, instead of skew-symmetry, satisfies the conditions

$$[x, y, z] + [y, z, x] + [z, x, y] = 0$$

   and

$$[x, y, y] = 0.$$

   It is an easy exercise to verify that Lie triple systems are Leibniz 3-algebras.

3. Let g be a Leibniz algebra with involution σ. This means that σ is an automorphism of g and $\sigma^2 = id$. Then

$$\mathcal{L} := \{x \in \mathsf{g} \mid x + \sigma(x) = 0\}$$

   is a Leibniz 3-algebra with respect to the bracket

$$[x, y, z] := [x, [y, z]].$$

4. Let V be a (n+1)-dimensional vector space with basis $\{\vec{e_1}, \vec{e_2}, \ldots, \vec{e_{n+1}}\}$. Then we define $[\vec{x_1}, \vec{x_2}, \ldots, \vec{x_n}] := det(A)$, where A is the following matrix

$$\begin{pmatrix} \vec{e_1} & \vec{e_2} & \ldots & \vec{e_{n+1}} \\ x_{11} & x_{21} & \ldots & x_{(n+1)1} \\ x_{12} & x_{22} & \ldots & x_{(n+1)2} \\ \ldots & \ldots & \ldots & \ldots \\ x_{1n} & x_{2n} & \ldots & x_{(n+1)n} \end{pmatrix}$$

   Here $\vec{x_i} = x_{1i}\, \vec{e_1} + x_{2i}\, \vec{e_2} + \cdots + x_{(n+1)i}\, \vec{e_{n+1}}$. Easily one sees that V equipped with this bracket is a Leibniz n-algebra.

5. An associative trialgebra is a K-vector space A equipped with three binary operations: $\dashv, \perp, \vdash$ (called left, middle and right, respectively), satisfying eleven associative relations [12]. Then A can be endowed with a structure of Leibniz 3-algebra with respect to the bracket

$$\begin{aligned}[x, y, z] &= x \dashv (y \perp z) - (y \perp z) \vdash x - x \dashv (z \perp y) + (z \perp y) \vdash x \\ &= x \dashv (y \perp z - z \perp y) - (y \perp z - z \perp y) \vdash x\end{aligned}$$

   for all $x, y, z \in$ A.

Let $\mathcal{L}$ be a Leibniz n-algebra. A subalgebra $\mathcal{K}$ of $\mathcal{L}$ is called *n-sided ideal* if $[l_1, l_2, \ldots, l_n] \in \mathcal{K}$ as soon as $l_i \in \mathcal{K}$ and $l_1, \ldots, l_{i-1}, l_{i+1}, \ldots, l_n \in \mathcal{L}$, for all $i = 1, 2, \ldots, n$. This definition ensures that the quotient $\mathcal{L}/\mathcal{K}$ is endowed with a well defined bracket induced naturally by the bracket in $\mathcal{L}$.

A *derivation* of a Leibniz n-algebra $\mathcal{L}$ is a linear map $d : \mathcal{L} \to \mathcal{L}$ for which the following identity holds:

$$d[x_1, \ldots, x_n] = \sum_{i=1}^{n} [x_1, \ldots, d(x_i), \ldots, x_n]$$

For instance, if we define the application $ad[y_2, \ldots, y_n] : \mathcal{L} \to \mathcal{L}, ad[y_2, \ldots, y_n](x) = [x, y_2, \ldots, y_n]$, fundamental identity (2) means that $ad[y_2, \ldots, y_n]$ is a derivation.

Let $\mathcal{M}$ and $\mathcal{P}$ be n-sided ideals of a Leibniz n-algebra $\mathcal{L}$. The *commutator ideal* of $\mathcal{M}$ and $\mathcal{P}$, denoted by $[\mathcal{M}, \mathcal{P}, \mathcal{L}^{n-2}]$, is the n-sided ideal of $\mathcal{L}$ spanned by the brackets $[l_1, \ldots, l_i, \ldots, l_j, \ldots, l_n]$ as soon as $l_i \in \mathcal{M}, l_j \in \mathcal{P}$ and $l_k \in \mathcal{L}$ for all k different to i, j. Obviously $[\mathcal{M}, \mathcal{P}, \mathcal{L}^{n-2}] \subset \mathcal{M} \cap \mathcal{P}$. In the particular case $\mathcal{M} = \mathcal{P} = \mathcal{L}$ we obtain the definition of *derived algebra* of a Leibniz n-algebra $\mathcal{L}$.

For a Leibniz n-algebra $\mathcal{L}$, we define its *centre* as the n-sided ideal

$$Z(\mathcal{L}) = \{l \in \mathcal{L} \mid [l_1, \ldots, l_{i-1}, l, l_{i+1}, \ldots, l_n] = 0, \forall l_i \in \mathcal{L}, i = 1, \ldots, \hat{i}, \ldots, n\}$$

The category ${}_n\mathbf{Lb}$ has zero object, products and coproducts and every morphism has image. From here, one can get the notion of centre (by Huq) [6] in a natural way. It is an easy exercise to show that $Z(\mathcal{L})$ coincides with this natural notion since is the maximal central subobject in the category ${}_n\mathbf{Lb}$.

An *abelian* Leibniz n-algebra is a Leibniz n-algebra with trivial bracket, that is, the commutator n-sided ideal $[\mathcal{L}^n] = [\mathcal{L}, \ldots, \mathcal{L}] = 0$. It is clear that a Leibniz n-algebra $\mathcal{L}$ is abelian if and only if $\mathcal{L} = Z(\mathcal{L})$. To any Leibniz n-algebra $\mathcal{L}$ we can associate its *largest abelian quotient* $\mathcal{L}_{ab}$, i. e., the abelianization functor works from ${}_n\mathbf{Lb}$ to K-vector spaces category; clearly the kernel of the projection map $\pi : \mathcal{L} \to \mathcal{L}_{ab}$ must contain the n-sided ideal $[\mathcal{L}^n]$. It is easy to verify that $\mathcal{L}_{ab} \cong \mathcal{L}/[\mathcal{L}^n]$.

An *abelian extension* of Leibniz n-algebras is an exact sequence $(\mathcal{K}) : 0 \to \mathrm{M} \xrightarrow{\kappa} \mathcal{K} \xrightarrow{\pi} \mathcal{L} \to 0$ of Leibniz n-algebras such that $[k_1, \ldots, k_n] = 0$ as soon as $k_i \in \mathrm{M}$ and $k_j \in \mathrm{M}$ for some $1 \leq i, j \leq n$ (i. e., $[\mathrm{M}, \mathrm{M}, \mathcal{K}^{n-2}] = 0$). Here $k_1, \ldots, k_n \in \mathcal{K}$. Clearly then M is an *abelian* Leibniz n-algebra. Let us observe that the converse is true only for $n = 2$.

If $(\mathcal{K})$ is an abelian extension of Leibniz n-algebras, then M is equipped with n actions $[-, \ldots, -] : \mathcal{L}^{\otimes i} \otimes \mathrm{M} \otimes \mathcal{L}^{\otimes(n-1-i)} \to \mathrm{M}$, $0 \leq i \leq n-1$, satisfying $(2n-1)$ equations, which are obtained from (2) by letting exactly one of the variables $x_1, \ldots, x_n, y_1, \ldots, y_{n-1}$ be in M and all the others in $\mathcal{L}$ [2].

A *representation* of a Leibniz n-algebra $\mathcal{L}$ is a K-vector space M equipped with n actions of $[-, \ldots, -] : \mathcal{L}^{\otimes i} \otimes \mathrm{M} \otimes \mathcal{L}^{\otimes(n-1-i)} \to \mathrm{M}$, $1 \leq i \leq n-1$, satisfying these $(2n-1)$ axioms [2].

If we define the multilinear applications $\rho_i : \mathcal{L}^{\otimes n-1} \to \mathrm{End}_K(\mathrm{M})$ by

$$\rho_i(l_1, \ldots, l_{n-1})(m) = [l_1, \ldots, l_{i-1}, m, l_{i+1}, \ldots, l_{n-1}]$$

$1 \leq i \leq n$, then the axioms of representation can be expressed by the following identities [1]:

1. For $2 \leq k \leq n$,
$$\rho_k([l_1, \ldots, l_n], l_{n+1}, \ldots, l_{2n-2}) =$$
$$\sum_{i=1}^{n} \rho_i(l_1, \ldots, \hat{l_i}, \ldots, l_n) \cdot \rho_k(l_i, l_{n+1}, \ldots, l_{2n-2})$$

2. For $1 \leq k \leq n$,
$$[\rho_1(l_n, \ldots, l_{2n-2}), \rho_k(l_1, \ldots, l_{n-1})] =$$
$$\sum_{i=1}^{n-1} \rho_k(l_1, \ldots, l_{i-1}, [l_i, l_n, \ldots, l_{2n-2}], l_{i+1}, \ldots, l_{n-1})$$
being the bracket on $\mathrm{End}_K(\mathrm{M})$ the usual one for associative algebras.

A particular instance of representation is the case $\mathrm{M} = \mathcal{L}$, where the applications ρ_i are the adjoint representations

$$ad_i(l_1, \ldots, l_{n-1})(l) = [l_1, \ldots, l_{i-1}, l, l_{i+1}, \ldots, l_{n-1}]$$

If the components of the representation $ad : \mathcal{L}^{\otimes n-1} \to \mathrm{End}_K(\mathcal{L})$ are $ad = (ad_1, \ldots, ad_n)$, then $\mathrm{Ker}\ ad = \{l \in \mathcal{L} \mid ad_i(l_1, \ldots, l_{n-1})(l) = 0, \forall (l_1, \ldots, l_{n-1}) \in \mathcal{L}^{\otimes n-1}, 1 \leq i \leq n\}$, that is, $\mathrm{Ker}\ ad$ is the centre of $\mathcal{L}$.

Definition 1. *Let $\mathcal{L}$ be a Leibniz n-algebra and* M *a representation of $\mathcal{L}$. A derivation from $\mathcal{L}$ to* M *is a K-linear map $d : \mathcal{L} \to \mathrm{M}$ for which the following identity holds:*
$$d[l_1, \ldots, l_n] = \sum_{i=1}^{n} [l_1, \ldots, d(l_i), \ldots, l_n]$$

Notice that this property of d is compatible with n-linearity and the fundamental identity (2). We denote by $Der(\mathcal{L}, \mathrm{M})$ the K-vector space of all derivations from $\mathcal{L}$ to M. When $\mathcal{L}$ is regarded as representation of $\mathcal{L}$, then $Der(\mathcal{L}, \mathcal{L})$ coincides with $Der(\mathcal{L})$, the K-vector space of derivations of $\mathcal{L}$. If M is a trivial representation of $\mathcal{L}$, that is, the actions $[-, \ldots, -] : \mathcal{L}^{\otimes i} \otimes \mathrm{M} \otimes \mathcal{L}^{\otimes (n-1-i)} \to \mathrm{M}$, $1 \leq i \leq n-1$, are trivial, then a derivation $d : \mathcal{L} \to \mathrm{M}$ is a homomorphism of Leibniz n-algebras.

Definition 2. *Let $\mathcal{L}$ be a Leibniz n-algebra and* M *a representation of $\mathcal{L}$. We define the semidirect product $\mathrm{M} \bowtie \mathcal{L}$ as the Leibniz n-algebra with underlying vector space $\mathrm{M} \oplus \mathcal{L}$ and bracket*

$$[(m_1, l_1), \ldots, (m_n, l_n)] = (\sum_{i=1}^{n} [l_1, \ldots, l_{i-1}, m_i, l_{i+1}, \ldots, l_n], [l_1, \ldots, l_n])$$

There is an obvious injective homomorphism of Leibniz n-algebras $i : \mathrm{M} \to \mathrm{M} \bowtie \mathcal{L}$ given by $i(m) = (m, 0), m \in \mathrm{M}$. There also is an obvious surjective homomorphism of Leibniz n-algebras $\pi : \mathrm{M} \bowtie \mathcal{L} \to \mathcal{L}$ given by $\pi(m, l) = l$. On the other hand, $i(\mathrm{M})$ is a n-sided ideal of $\mathrm{M} \bowtie \mathcal{L}$ with quotient $\mathcal{L}$, being the canonical projection π; thus the sequence $0 \to \mathrm{M} \xrightarrow{i} \mathrm{M} \bowtie \mathcal{L} \xrightarrow{\pi} \mathcal{L} \to 0$ is exact. Moreover $i(\mathrm{M})$ is a representation of $\mathrm{M} \bowtie \mathcal{L}$ via π, so the exact sequence is an abelian extension of Leibniz n-algebras which splits by means of $\sigma : \mathcal{L} \to \mathrm{M} \bowtie \mathcal{L}, \sigma(l) = (0, l), l \in \mathcal{L}$.

The projection $\theta : \mathrm{M} \bowtie \mathcal{L} \to \mathrm{M}, \theta(m, l) = m$, is a derivation, being M a representation of $\mathrm{M} \bowtie \mathcal{L}$ via π.

Theorem 1. *Let $\mathcal{L}$ be a Leibniz n-algebra and M a representation of $\mathcal{L}$. For every homomorphism of Leibniz n-algebras $f : \mathcal{Q} \to \mathcal{L}$ and every f-derivation $d : \mathcal{Q} \to \mathrm{M}$, there exists a unique homomorphism of Leibniz n-algebras $h : \mathcal{Q} \to \mathrm{M} \bowtie \mathcal{L}$ such that the following diagram is commutative*

$$\begin{tikzcd}
 & & \mathcal{Q} \arrow[dl, "d"'] \arrow[d, "h"] \arrow[dr, "f"] & & \\
0 \arrow[r] & \mathrm{M} \arrow[r, shift left, "i"] & \mathrm{M} \bowtie \mathcal{L} \arrow[l, shift left, "\theta"] \arrow[r, "\pi"] & \mathcal{L} \arrow[r] & 0
\end{tikzcd}$$

Conversely, every homomorphism of Leibniz n-algebras $h : \mathcal{Q} \to \mathrm{M} \bowtie \mathcal{L}$ determines a homomorphism of Leibniz n-algebras $f = \pi h : \mathcal{Q} \to \mathcal{L}$ and a f-derivation $d = \theta h : \mathcal{Q} \to \mathrm{M}$.

Proof. Define $h(x) = (d(x), f(x)), x \in \mathcal{Q}$. For converse, apply following lemma. ∎

Lemma 1. *Let $f : \mathcal{Q} \to \mathcal{L}$ be a homomorphism of Leibniz n-algebras and $d : \mathcal{L} \to \mathrm{M}$ a derivation, then $df : \mathcal{Q} \to \mathrm{M}$ is a derivation, being M a representation of $\mathcal{Q}$ via f.*

Corollary 1. *The set $Der(\mathcal{L}, \mathrm{M})$ is in one-to-one correspondence with the set of homomorphisms of Leibniz n-algebras $h : \mathcal{L} \to \mathrm{M} \bowtie \mathcal{L}$ such that $\pi h = 1_{\mathcal{L}}$.*

If we denote by ${}_n\mathbf{Leib}/\mathcal{L}$ the comma category over the Leibniz n-algebra $\mathcal{L}$, then there exists a natural equivalence between the functors

$$\begin{tikzcd}
{}_n\mathbf{Leib}/\mathcal{L} \arrow[d, shift right=3, "Der(-,\mathrm{M})"'] \arrow[d, shift left=3, "Hom_{{}_n\mathbf{Leib}/\mathcal{L}}(-,\mathrm{M} \bowtie \mathcal{L} \to \mathcal{L})"] \arrow[d, phantom, "\overset{\eta}{\Rightarrow}"] \\
\mathbf{Vect}_K
\end{tikzcd}$$

that is, the functor $Der(-, \mathrm{M})$ is representable.

Theorem 2. *Let $0 \to \mathcal{N} \to \mathcal{K} \to \mathcal{L} \to 0$ be an exact sequence of Leibniz n-algebras and let M be a representation of $\mathcal{L}$, then*

$$0 \to Der(\mathcal{L}, \mathrm{M}) \to Der(\mathcal{K}, \mathrm{M}) \to Hom_{\mathcal{L}}(\mathcal{N}_{ab}, \mathrm{M})$$

is natural exact sequence of K-vector spaces

Proof. Applying the left exact functor $Hom_{{}_n\mathbf{Leib}/\mathcal{L}}(-, \mathrm{M} \bowtie \mathcal{L} \to \mathcal{L})$ to the exact sequence, we obtain the exact sequence

$$0 \to Hom_{{}_n\mathbf{Leib}/\mathcal{L}}(\mathcal{L}, \mathrm{M} \bowtie \mathcal{L} \to \mathcal{L}) \to Hom_{{}_n\mathbf{Leib}/\mathcal{L}}(\mathcal{K}, \mathrm{M} \bowtie \mathcal{L} \to \mathcal{L})$$
$$\to Hom_{{}_n\mathbf{Leib}/\mathcal{L}}(\mathcal{N}, \mathrm{M} \bowtie \mathcal{L} \to \mathcal{L})$$

By natural equivalence η, this sequence is

$$0 \to Der(\mathcal{L}, \mathrm{M}) \to Der(\mathcal{K}, \mathrm{M}) \to Der(\mathcal{N}, \mathrm{M})$$

but $Der(\mathcal{N}, \mathrm{M}) \cong Hom(\mathcal{N}, \mathrm{M}) \cong Hom_{\mathcal{L}}(\mathcal{N}_{ab}, \mathrm{M})$, since M is a trivial representation of $\mathcal{N}$. ∎

Now we remember the (co)homology theory for Leibniz n-algebras developed in [1, 2].

Let $\mathcal{L}$ be a Leibniz n-algebra and let M be a representation of $\mathcal{L}$. Then $Hom(\mathcal{L}, M)$ is a $\mathcal{D}_{n-1}(\mathcal{L})$-representation as Leibniz algebras [2]. One defines the cochain complex ${}_nCL^*(\mathcal{L}, M)$ to be $CL^*(\mathcal{D}_{n-1}(\mathcal{L}), \mathrm{Hom}(\mathcal{L}, M))$. We also put ${}_nHL^*(\mathcal{L}, M) := H^*({}_nCL^*(\mathcal{L}, M))$. Thus, by definition one has ${}_nHL^*(\mathcal{L}, M) \cong HL^*(\mathcal{D}_{n-1}(\mathcal{L}), \mathrm{Hom}(\mathcal{L}, M))$. Here $CL^\star$ denotes the Leibniz complex and $HL^\star$ its homology, called Leibniz cohomology (see [10, 11] for more information).

In case $n = 2$, this cohomology theory gives ${}_2HL^m(\mathcal{L}, M) \cong HL^{m+1}(\mathcal{L}, M)$, $m \geq 1$ and ${}_2HL^0(\mathcal{L}, M) \cong Der(\mathcal{L}, M)$.

On the other hand, ${}_nHL^0(\mathcal{L}, M) \cong Der(\mathcal{L}, M)$ and ${}_nHL^1(\mathcal{L}, M) \cong \mathrm{Ext}(\mathcal{L}, M)$, where $\mathrm{Ext}(\mathcal{L}, M)$ denotes the set of isomorphism classes of abelian extensions of $\mathcal{L}$ by M [2].

Homology with trivial coefficients of a Leibniz n-algebra $\mathcal{L}$ is defined in [1] as the homology of the Leibniz complex ${}_nCL_\star(\mathcal{L}) := CL_\star(\mathcal{D}_{n-1}(\mathcal{L}), \mathcal{L})$, where $\mathcal{L}$ is endowed with a structure of $\mathcal{D}_{n-1}(\mathcal{L})$ symmetric corepresentation. We denote the homology groups of this complex by ${}_nHL_\star(\mathcal{L})$.

When $\mathcal{L}$ is a Leibniz 2-algebra, that is a Leibniz algebra, then we have that ${}_2HL_k(\mathcal{L}) \cong HL_{k+1}(\mathcal{L}), k \geq 1$. Particularly, ${}_2HL_0(\mathcal{L}) \cong HL_1(\mathcal{L}) \cong \mathcal{L}/[\mathcal{L}, \mathcal{L}] = \mathcal{L}_{ab}$. On the other hand, ${}_nHL_0(\mathcal{L}) = \mathcal{L}_{ab}$.

3 Universal central extensions of Leibniz n-algebras

Let $\mathcal{L}$ be a Leibniz n-algebra. We can endowed the tensor $\mathcal{L}^{\otimes n}$ with a structure of Leibniz n-algebra by means of the following bracket:

$$\begin{gathered}[x_{11} \otimes \cdots \otimes x_{n1}, x_{12} \otimes \cdots \otimes x_{n2}, \dots, x_{1n} \otimes \cdots \otimes x_{nn}] := \\ [x_{11}, [x_{12}, \dots, x_{n2}], \dots, [x_{1n}, \dots, x_{nn}]] \otimes x_{21} \otimes \cdots \otimes x_{n1} + \\ x_{11} \otimes [x_{21}, [x_{12}, \dots, x_{n2}], \dots, [x_{1n}, \dots, x_{nn}]] \otimes \cdots \otimes x_{n1} + \cdots + \\ x_{11} \otimes \cdots \otimes x_{(n-1)1} \otimes [x_{n1}, [x_{12}, \dots, x_{n2}], \dots, [x_{1n}, \dots, x_{nn}]]\end{gathered} \tag{4}$$

In particular case $n = 2$ we obtain a structure of Leibniz algebra on $\mathcal{L} \otimes \mathcal{L}$ which is the subject of [7].

Now we remember that the complex used in [1] in order to achieve the homology with trivial coefficients of a Leibniz n-algebra $\mathcal{L}$ is

$$\cdots \to \mathcal{L}^{\otimes k(n-1)+1} \xrightarrow{\delta_k} \mathcal{L}^{\otimes (k-1)(n-1)+1} \xrightarrow{\delta_{k-1}} \cdots \to \mathcal{L}^{\otimes 2n-1} \xrightarrow{\delta_2} \mathcal{L}^{\otimes n} \xrightarrow{\delta_1} \mathcal{L}$$

where the low differentials are

$$\begin{gathered}\delta_2(x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_{n-1}) = [x_1, \dots, x_n] \otimes y_1 \otimes \cdots \otimes y_{n-1} - \\ [x_1, y_1, \dots, y_{n-1}] \otimes x_2 \otimes \cdots \otimes x_n - \cdots - x_1 \otimes \dots x_{n-1} \otimes [x_n, y_1, \dots, y_{n-1}]\end{gathered}$$

and $\delta_1 : \mathcal{L}^{\otimes n} \to \mathcal{L}$ is the commutator map

$$\delta_1(x_1 \otimes \cdots \otimes x_n) = [x_1, \dots, x_n]$$

Definition 3. *For a Leibniz n-algebra $\mathcal{L}$, let be*

$$Z^{\rightarrow}(\mathcal{L}) = \{a \in \mathcal{L} \mid [x_1, a, \dots, x_n] = \dots = [x_1, x_2, \dots, a] = 0; x_1, \dots, x_n \in \mathcal{L}\}$$

Definition 4. *For a Leibniz n-algebra $\mathcal{L}$, let $\mathcal{L}^{ann}$ be the smallest n-sided ideal spanned by the elements of the form $[x_1, \dots, x_n], x_i \in \mathcal{L}, i = 1, \dots, n$ as soon as $x_i = x_j$.*

Lemma 2. *$Z^{\rightarrow}(\mathcal{L})$ is a n-sided ideal of $\mathcal{L}$. Moreover it is verified that*

$$[\mathcal{L}, Z^{\rightarrow}(\mathcal{L}), \dots, \mathcal{L}] = \dots = [\mathcal{L}, \mathcal{L}, \dots, Z^{\rightarrow}(\mathcal{L})] = 0$$

and

$$[Z^{\rightarrow}(\mathcal{L}), \mathcal{L}, \dots, \mathcal{L}] \subseteq \mathcal{L}^{ann}$$

Proof: The proof is straightforward and we leave it to the reader. ■

Lemma 3. *The image of the differential $\delta_2 : \mathcal{L}^{\otimes 2n-1} \to \mathcal{L}^{\otimes n}$ is an abelian n-sided ideal of $\mathcal{L}^{\otimes n}$. Moreover $Im\ \delta_2 \subset Z^{\rightarrow}(\mathcal{L}^{\otimes n})$.*

Proof: The proof only uses the fundamental identity (2), the structure on $\mathcal{L}^{\otimes n}$ given by identity (4) and lemma 2. ■

Now we consider the vector space

$$\mathcal{L}^{*n} = \mathcal{L} * \overset{n}{\dots} * \mathcal{L} := Coker(\delta_2 : \mathcal{L}^{\otimes(2n-1)} \to \mathcal{L}^{\otimes n})$$

which is equipped with a structure of Leibniz n-algebra induced by the bracket (4) defined on $\mathcal{L}^{\otimes n}$. We denote by $x_1 * \dots * x_n$ the image of $x_1 \otimes \dots \otimes x_n \in \mathcal{L}^{\otimes n}$ into $\mathcal{L}^{*n}$. Since

$$\begin{gathered}[x_1, \dots, x_n] * y_2 * \dots * y_n = \\ [x_1, y_2, \dots, y_n] * x_2 * \dots * x_n + \dots + \\ x_1 * \dots * x_{n-1} * [x_n, y_2, \dots, y_n]\end{gathered}$$

we see that

$$\begin{gathered}[x_{11} * \dots * x_{n1}, x_{12} * \dots * x_{n2}, \dots, x_{1n} * \dots * x_{nn}] = \\ [x_{11}, \dots, x_{n1}] * [x_{12}, \dots, x_{n2}] * \dots * [x_{1n}, \dots x_{nn}] \end{gathered} \tag{5}$$

Having in mind the definition of homology with trivial coefficients one has the exact sequence of Leibniz n-algebras

$$0 \to {}_nHL_1(\mathcal{L}) \to \mathcal{L}^{*n} \xrightarrow{[-,\dots,-]} \mathcal{L} \to {}_nHL_0(\mathcal{L}) \to 0 \tag{6}$$

Here ${}_nHL_0(\mathcal{L})$ and ${}_nHL_1(\mathcal{L})$ are abelian Leibniz n-algebras. Moreover one can show that ${}_nHL_1(\mathcal{L})$ is a central subalgebra of $\mathcal{L}^{*n}$.

Proposition 1. *Let $\mathcal{L}$ be a free Leibniz n-algebra, then the homomorphism*

$$[-, \dots, -] : \mathcal{L}^{*n} \to \mathcal{L}$$

is injective.

Proof. In (6) ${}_nHL_1(\mathcal{L}) = 0$ (see theorem 2 [1]). ■

Given n-sided ideals $\mathcal{M}'_i$ of $\mathcal{L}$ such that $\mathcal{M}'_i \subseteq \mathcal{M}_i$, being $\mathcal{M}_i$ n-sided ideals of $\mathcal{L}$, $i = 1, \dots, n$, then there exists a canonical homomorphism $i : \mathcal{M}'_1 * \cdots * \mathcal{M}'_n \to \mathcal{M}_1 * \cdots * \mathcal{M}_n$, where $\mathcal{M}_1 * \cdots * \mathcal{M}_n$ means the smallest ideal of $\mathcal{L} * \overset{n}{\dots} * \mathcal{L}$ spanned by the elements $m_1 * \cdots * m_n$ with $m_i \in \mathcal{M}_i, i = 1, \dots, n$. We shall denote the image of this homomorphism by $(\mathcal{M}'_1 * \cdots * \mathcal{M}'_n)_{\mathcal{M}_1, \dots, \mathcal{M}_n}$.

Proposition 2. *Let $\mathcal{K}$ be a n-sided ideal of $\mathcal{L}$ which is contained in $\cap_{i=1}^n \mathcal{M}_i$. Then there is a canonical isomorphism*

$$\frac{\mathcal{M}_1}{\mathcal{K}} * \cdots * \frac{\mathcal{M}_n}{\mathcal{K}} \cong \frac{\mathcal{M}_1 * \cdots * \mathcal{M}_n}{\sum_{i=1}^n (\mathcal{M}_1 * \cdots * \mathcal{M}_{i-1} * \mathcal{K} * \mathcal{M}_{i+1} * \cdots * \mathcal{M}_n)_{\mathcal{M}_1, \dots, \mathcal{M}_n}}$$

Proof. The canonical map

$$\Phi : \frac{\mathcal{M}_1}{\mathcal{K}} * \cdots * \frac{\mathcal{M}_n}{\mathcal{K}} \to \frac{\mathcal{M}_1 * \cdots * \mathcal{M}_n}{\sum_{i=1}^n (\mathcal{M}_1 * \cdots * \mathcal{M}_{i-1} * \mathcal{K} * \mathcal{M}_{i+1} * \cdots * \mathcal{M}_n)_{\mathcal{M}_1, \dots, \mathcal{M}_n}}$$

is a well defined homomorphism of Leibniz n-algebras. On the other hand, the canonical map

$$\sigma : \mathcal{M}_1 * \cdots * \mathcal{M}_n \to \frac{\mathcal{M}_1}{\mathcal{K}} * \cdots * \frac{\mathcal{M}_n}{\mathcal{K}}$$

is a homomorphism of Leibniz n-algebras which annihilates

$$\sum_{i=1}^n (\mathcal{M}_1 * \cdots * \mathcal{M}_{i-1} * \mathcal{K} * \mathcal{M}_{i+1} * \cdots * \mathcal{M}_n)_{\mathcal{M}_1, \dots, \mathcal{M}_n}$$

Then σ induces

$$\Sigma : \frac{\mathcal{M}_1 * \cdots * \mathcal{M}_n}{\sum_{i=1}^n (\mathcal{M}_1 * \cdots * \mathcal{M}_{i-1} * \mathcal{K} * \mathcal{M}_{i+1} * \cdots * \mathcal{M}_n)_{\mathcal{M}_1, \dots, \mathcal{M}_n}} \to \frac{\mathcal{M}_1}{\mathcal{K}} * \cdots * \frac{\mathcal{M}_n}{\mathcal{K}}$$

and moreover Σ is inverse of Φ. ∎

Theorem 3. *Let $\mathcal{L}$ be a Leibniz n-algebra, then*

$${}_nHL_1(\mathcal{L}) \cong Ker(\mathcal{L}^{*n} \overset{[-,\dots,-]}{\to} \mathcal{L})$$

Proof. See exact sequence (6). ∎

If $0 \to \mathcal{R} \to \mathcal{F} \to \mathcal{L} \to 0$ is a free presentation of a Leibniz n-algebra $\mathcal{L}$ (always there exist free presentations of a Leibniz n-algebra, see [1]), then having in mind Propositions 1 and 2 we obtain the following isomorphism

$$\mathcal{L}^{*n} \cong \frac{\mathcal{F}}{\mathcal{R}} * \overset{n}{\dots} * \frac{\mathcal{F}}{\mathcal{R}} \cong \frac{\mathcal{F}^{*n}}{\sum_{i=1}^n (\mathcal{F} * \cdots * \mathcal{R} * \dots \mathcal{F})_{\mathcal{F} * \cdots * \mathcal{F}}} \cong \frac{[\mathcal{F}, \overset{n}{\dots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\dots}, \mathcal{F}]} \tag{7}$$

Now we consider a perfect Leibniz n-algebra $\mathcal{L}$, that is $\mathcal{L} = [\mathcal{L}, \overset{n}{\dots}, \mathcal{L}]$, equivalently ${}_nHL_0(\mathcal{L}) = 0$, then exact sequence (6) is the central extension

$$0 \to {}_nHL_1(\mathcal{L}) \to \mathcal{L}^{*n} \overset{[-,\dots,-]}{\longrightarrow} \mathcal{L} \to 0 \tag{8}$$

The following results are devoted to show that exact sequence (8) is the universal central extension of $\mathcal{L}$. Firstly we remember some results about (see [1]).

Definition 5. *A central extension* $(\mathcal{K})$ *of Leibniz n-algebras is an extension of Leibniz n-algebras* $(\mathcal{K}): 0 \to M \to \mathcal{K} \xrightarrow{\pi} \mathcal{L} \to 0$ *for which* $[M, \mathcal{K}^{n-1}] = 0$.

This central extension is called universal if for every central extension $(\mathcal{K}')$: $0 \to M \to \mathcal{K}' \xrightarrow{\pi'} \mathcal{L} \to 0$ *there exists a unique homomorphism* $h : \mathcal{K} \to \mathcal{K}'$ *such that* $\pi' h = \pi$.

Note that a central extension is an abelian extension and that equips M with a structure of trivial $\mathcal{L}$-representation.

Theorem 4. *1. If* $(\mathcal{K}): 0 \to \mathrm{M} \to \mathcal{K} \xrightarrow{\pi} \mathcal{L} \to 0$ *is a central extension with* $\mathcal{K}$ *a perfect Leibniz n-algebra and every central extension of* $\mathcal{K}$ *splits, then* $(\mathcal{K})$ *is universal.*

2. A Leibniz n-algebra $\mathcal{L}$ *admits a universal central extension if and only if* $\mathcal{L}$ *is perfect.*

3. The kernel of the universal central extension is canonically isomorphic to ${}_nHL_1(\mathcal{L}, K)$.

Lemma 4. *Let* $\varphi : \mathcal{L} \to \mathcal{M}$ *be a surjective homomorphism of Leibniz n-algebras. Then the canonical homomorphism* $\varphi * \overset{n}{\ldots} * \varphi : \mathcal{L}^{*n} \to \mathcal{M}^{*n}$ *is surjective and its kernel is the n-sided ideal*

$$Im(Ker(\varphi) * \mathcal{L} * \cdots * \mathcal{L} + \cdots + \mathcal{L} * \cdots * \mathcal{L} * Ker(\varphi) \to \mathcal{L} * \mathcal{L} * \cdots * \mathcal{L})$$

Lemma 5. *Let* $0 \to \mathcal{N} \to \mathcal{H} \xrightarrow{\pi} \mathcal{L} \to 0$ *be a central extension of Leibniz n-algebras, being* $\mathcal{H}$ *a perfect Leibniz n-algebra. Let* $0 \to \mathcal{M} \to \mathcal{K} \xrightarrow{\sigma} \mathcal{L} \to 0$ *be another central extension of Leibniz n-algebras. If there exists a homomorphism of Leibniz n-algebras* $\phi : \mathcal{H} \to \mathcal{K}$ *such that* $\sigma\phi = \pi$, *then* ϕ *is unique.*

Proof. Let $\phi, \psi : \mathcal{H} \to \mathcal{K}$ be two homomorphisms of Leibniz n-algebras such that $\sigma\phi = \pi$ and $\sigma\psi = \pi$. Then for any $h \in \mathcal{H}$ there exists $m \in \mathcal{M}$ such that $\phi(h) = \psi(h) + m$. From here, ϕ and ψ coincide on commutators $[h_1, \ldots, h_n] \in \mathcal{H}$ thanks to centrality of $\mathcal{M}$ on $\mathcal{K}$. Since $\mathcal{H}$ is a perfect Leibniz n-algebra, it is spanned by commutators, so $\phi = \psi$.

Theorem 5. *Let* $\mathcal{L}$ *be a perfect Leibniz n-algebra, then*

$$0 \to {}_nHL_1(\mathcal{L}) \to \mathcal{L}^{*n} \xrightarrow{[-,\ldots,-]} \mathcal{L} \to 0 \tag{9}$$

is the universal central extension of $\mathcal{L}$.

Proof. Let $(\mathcal{H}) : 0 \to \mathcal{M} \to \mathcal{K} \xrightarrow{\sigma} \mathcal{L} \to 0$ be an arbitrary central extension of $\mathcal{L}$. The homomorphism of Leibniz n-algebras $\tau : \mathcal{K}^{*n} \to \mathcal{K}, \tau(x_1 * \cdots * x_n) = [x_1, \ldots, x_n]$, can be factored throughout the homomorphism $\sigma * \overset{n}{\ldots} * \sigma : \mathcal{H}^{*n} \to \mathcal{L}^{*n}$ by lemma 4 and centrality of $\mathcal{M} = Ker(\sigma)$. This provides a homomorphism $\phi : \mathcal{L}^{*n} \to \mathcal{H}$ such that $\sigma.\phi(l_1 * \cdots * l_n) = [l_1, \ldots, l_n]$, for all $l_1, \ldots, l_n \in \mathcal{L}$.

On the other hand, $\mathcal{L}$ perfect implies that $\mathcal{L}^{*n}$ is perfect since $[\mathcal{L}^{*n}, \ldots, \mathcal{L}^{*n}] = [\mathcal{L}, \ldots, \mathcal{L}] * \cdots * [\mathcal{L}, \ldots, \mathcal{L}]$. Now lemma 5 ends the proof. ∎

Having in mind formula (7), then we can write the universal central extension of a perfect Leibniz n-algebra $\mathcal{L}$ as follows

$$0 \to {}_nHL_1(\mathcal{L}) \to \frac{[\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}]}{[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]} \overset{[-,\ldots,-]}{\to} \mathcal{L} \to 0$$

From here we can deduce that ${}_nHL_1(\mathcal{L}) \cong (\mathcal{R} \cap [\mathcal{F}, \overset{n}{\ldots}, \mathcal{F}])/[\mathcal{R}, \mathcal{F}, \overset{n-1}{\ldots}, \mathcal{F}]$, being $0 \to \mathcal{R} \to \mathcal{F} \to \mathcal{L} \to 0$ a free presentation of a Leibniz n-algebra $\mathcal{L}$. This result was obtained in [1] using other techniques.

In the universal central extension (9), ${}_nHL_1(\mathcal{L})$ can be considered as a trivial representation of $\mathcal{L}$. By Theorem 3 in [1] (Theorem of Universal Coefficient) we have that

$${}_nHL^1(\mathcal{L}, {}_nHL_1(\mathcal{L})) \cong Hom({}_nHL_1(\mathcal{L}), {}_nHL_1(\mathcal{L}))$$

But it is well-known the bijection (see [2])

$${}_nHL^1(\mathcal{L}, {}_nHL_1(\mathcal{L})) \cong Ext(\mathcal{L}, {}_nHL_1(\mathcal{L}))$$

One can see that the universal central extension corresponds to the element

$$Id_{{}_nHL_1(\mathcal{L})} \in Hom({}_nHL_1(\mathcal{L}), {}_nHL_1(\mathcal{L}))$$

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