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Regularity criteria for the rational large eddy simulation model

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ABSTRACT. We consider the Rational Large Eddy Simulation (RLES) model introduced by Galdi and Layton (Math. Models Methods Appl. Sci. 10 (2000) 343–350). Various regularity criteria for the strong solution of this model are established here, which improve previous ones.

1. Introduction

The well-known incompressible Navier-Stokes (NS) equations reads:

$$\begin{cases} u_t + u \cdot \nabla u - \frac{1}{Re} \Delta u + \nabla p = 0, & \\ \operatorname{div} u = 0 & \text{in } (0, \infty) \times \mathbf{R}^3, \\ u|_{t=0} = u_0(x) & \text{in } \mathbf{R}^3, \end{cases}$$

where u and p are the velocity and pressure of the fluid, and $Re > 0$ is the Reynolds number. The phenomena of instability of fluid motion at high Reynolds number lead to the study of turbulent flows. The main idea underlying the study of turbulent motion can be traced back to Leonardo da Vinci [3] (at the beginning of the 16th century), who was the first to observe that the motion of vortices trailing a blunt body can be understood as a mean motion plus some turbulent fluctuations. The first mathematical model using this idea was introduced by Reynolds [12]. In fact, Reynolds proposed to consider the velocity as decomposed in

$$u = \bar{u} + u',$$

where $\bar{u}$ is the mean velocity, while u' represents the turbulent fluctuations.

In this paper, we consider the RLES model introduced by Galdi and Layton [5]:

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$$w_t + w \cdot \nabla w + \mathrm{div}(I - \delta^2 \Delta)^{-1}[\nabla w \nabla w] - \frac{1}{Re}\Delta w + \nabla q = 0, \tag{1}$$

$$\mathrm{div}\, w = 0 \qquad \text{in } (0, \infty) \times \mathbf{R}^3, \tag{2}$$

$$w|_{t=0} = w_0 \qquad \text{in } \mathbf{R}^3. \tag{3}$$

Here w and q are the approximations of the averaged flow variables $\bar{u}$ and $\bar{p}$. δ is a positive constant, I is the identity operator, and $[\nabla w \nabla w]_{ij} := \sum_{k=1}^{3} \frac{\partial w_i}{\partial x_k} \frac{\partial w_j}{\partial x_k}$. For simplicity we take $\delta = Re = 1$.

The existence and uniqueness of local strong solutions to the problem (1)–(3) were proved by Berselli-Galdi-Iliescu-Layton [2] when $w_0 \in H^1$. Furthermore, the following results are also proved in [2]:

THEOREM 1. *Let w be a strong solution to* (1)–(3), *and suppose that T^* is the finite maximal existence time, then*

$$\lim_{t \nearrow T^*} \|\nabla w(t)\|_{L^2} = +\infty, \tag{4}$$

$$\int_0^{T^*} \|\nabla w(\tau)\|_{L^r}^s d\tau = \infty, \qquad \textit{for } \frac{2}{s} + \frac{3}{r} = 2,\ 1 \le s < \infty,\ 3/2 < r \le \infty, \tag{5}$$

$$\int_0^{T^*} \|\mathrm{curl}\, w(\tau)\|_{L^r}^s d\tau = \infty, \qquad \textit{for } \frac{2}{s} + \frac{3}{r} = 2,\ 1 < s < \infty,\ 3/2 < r < \infty. \tag{6}$$

Furthermore, there holds the following blow-up estimate

$$\|\nabla w(t)\|_{L^2} \ge \frac{C}{(T^* - t)^{1/4}}, \qquad t < T^*. \tag{7}$$

Before writing down the main result of our paper, let us list some regularity conditions of the strong solution to the Navier-Stokes equations. The first result in this direction is obtained independently by Serrin [13] and Struwe [16] (see also [11]) which states that if weak solution u satisfies

$$u \in L^s(0, T; L^r(\mathbf{R}^3)) \qquad \text{with } \frac{2}{s} + \frac{2}{r} = 1,\ 3 < r \le \infty, \tag{8}$$

then u is smooth in space. After that there are further developments and refinements by Fabes, Jones, and Riviere [4], Giga [7], Sohr and Von Wahl [14], and Galdi and Maremonti [6], which concluded that $u(x,t) \in C^\infty((0, T] \times \mathbf{R}^3)$ with smooth initial data. H. Beirão da Veiga [1] obtained the following regularity criterion

$$\nabla u \in L^s(0, T; L^r(\mathbf{R}^3)) \qquad \text{with } \frac{2}{s} + \frac{3}{r} = 2,\ 3/2 < r \le \infty. \tag{9}$$

Kozono-Ogawa-Taniuchi [8] refined (8) in the case $s = 2$, $r = \infty$ and (9) in the case $s = 1$, $r = \infty$ by the following condition

$$u \in L^2(0, T; \dot{B}^0_{\infty,\infty}(\mathbf{R}^3)) \tag{10}$$

and

$$\nabla u \in L^1(0, T; \dot{B}^0_{\infty,\infty}(\mathbf{R}^3)) \tag{11}$$

respectively, where $\dot{B}^0_{\infty,\infty}$ denotes the homogeneous Besov spaces.

Kozono-Shimada [9] refined (8) by the following condition

$$u \in L^{2/(1-\alpha)}(0, T; \dot{F}^{-\alpha}_{\infty,\infty}) \qquad \text{for } 0 < \alpha < 1, \tag{12}$$

where $\dot{F}^{-\alpha}_{\infty,\infty}$ denotes the homogeneous Triebel-Lizorkin space. Other regularity criteria for the Navier-Stokes equations can be found in the recent papers [18, 19, 20, 21] by the last author.

The purpose of this paper is to establish regularity criteria for the RLES model in the homogeneous Besov space $\dot{B}^0_{\infty,\infty}$ and homogeneous Triebel-Lizorkin space $\dot{F}^{-\alpha}_{\infty,\infty}$. We now state our main result in this paper.

Theorem 2. *Let* $w_0 \in H^1$ *and* $\operatorname{div} w_0 = 0$ *in* $\mathbf{R}^3$. *Assume that one of the following conditions is satisfied by the solution* $w(x, t)$ *to the RLES model:*

$$w \in L^s(0, T; L^r(\mathbf{R}^3)) \qquad \textit{with } \frac{2}{s} + \frac{2}{r} = 1,\ 3 < r \leq \infty; \tag{13}$$

$$w \in L^2(0, T; \dot{B}^0_{\infty,\infty}(\mathbf{R}^3)); \tag{14}$$

$$\nabla w \in L^1(0, T; \dot{B}^0_{\infty,\infty}(\mathbf{R}^3)); \tag{15}$$

$$\operatorname{curl} w \in L^{2/(2-\alpha)}(0, T; \dot{F}^{-\alpha}_{\infty,\infty}) \qquad \textit{with } 0 < \alpha < 2, \tag{16}$$

then there is no singularity up to T *((20) holds).*

Remark 1. *The criterion* (16) *is interesting, because the vorticity* curl w *attracts attention from engineers.*

2. Proof of Theorem 2

Before going to the proof, let us first recall the definition of the homogeneous Besov space $\dot{B}^0_{\infty,\infty}$ and homogeneous Triebel-Lizorkin space $\dot{F}^{-\alpha}_{\infty,\infty}$.

DEFINITION 1 ([17]). Let $\{\phi_j\}_{j\in\mathbf{Z}}$ be the Littlewood-Paley dyadic decomposition of unity that satisfies $\mathrm{supp}\{\hat{\phi}\} \subset (B_2 \backslash B_{1/2})$, $\hat{\phi}_j(\xi) = \hat{\phi}(2^{-j}\xi)$, and $\sum_{j\in\mathbf{Z}} \hat{\phi}_j(\xi) = 1$ for any $\xi \neq 0$. The homogeneous Besov space $\dot{B}^s_{p,q} := \{f \in \mathscr{S}' : \|f\|_{\dot{B}^s_{p,q}} < \infty\}$ is introduced by the norm

$$\|f\|_{\dot{B}^s_{p,q}} := \left(\sum_{j\in\mathbf{Z}} \|2^{js}\phi_j * f\|^q_{L^p}\right)^{1/q}$$

for $s \in \mathbf{R}$, $1 \leq p, q \leq \infty$. The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q} := \{f \in \mathscr{S}' : \|f\|_{\dot{F}^s_{p,q}} < \infty\}$ is introduced by the norm

$$\|f\|_{\dot{F}^s_{p,q}} := \left\|\left(\sum_{j\in\mathbf{Z}} 2^{jqs}|\phi_j * f|^q\right)^{1/q}\right\|_{L^p}$$

for $s \in \mathbf{R}, 1 \leq p, q \leq \infty$.

A basic estimate for product functions reads

LEMMA 1 ([9]). *Let* $1 < p < \infty$, $1 < q < \infty$ *and* $s > 0$, $\alpha > 0$, $\beta > 0$, *and choose* $1 < p_1 < \infty$, $1 < p_2 \leq \infty$ *and* $1 < r_1 \leq \infty$, $1 < r_2 < \infty$ *so that* $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$. *Then for any* $f \in \dot{F}^{s+\alpha}_{p_1,q} \cap \dot{F}^{-\beta}_{r_1,\infty}$ *and* $g \in \dot{F}^{-\alpha}_{p_2,\infty} \cap \dot{F}^{s+\beta}_{r_2,q}$ *we have* $fg \in \dot{F}^s_{p,q}$ *with the estimate*

$$\|fg\|_{\dot{F}^s_{p,q}} \leq C(\|f\|_{\dot{F}^{s+\alpha}_{p_1,q}}\|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\beta}_{r_1,\infty}}\|g\|_{\dot{F}^{s+\beta}_{r_2,q}}). \tag{17}$$

Since it is well-known that (see [2]) there are a $T > 0$ and a unique strong solution w to the problem (1)–(3) in $(0, T]$, in the following calculations, we assume that the solution is sufficiently smooth on $[0, T]$.

Testing (1) by $(I - \Delta)w$ and using (2), we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int w^2 + |\nabla w|^2 dx + \int |\nabla w|^2 + |\Delta w|^2 dx \\ = \int (w \cdot \nabla)w \cdot \Delta w\, dx - \int w\, \mathrm{div}[\nabla w \nabla w] dx =: I(t).\end{aligned} \tag{18}$$

(1) Firstly, let us assume that (13) holds true. In the following calculations, we will use the following Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{L^{2r/(r-2)}} \leq C\|\nabla w\|^{1-3/r}_{L^2}\|\Delta w\|^{3/r}_{L^2} \qquad (r > 3). \tag{19}$$

Using (19), we estimate $I(t)$ as follows.

$$
\begin{aligned}
I(t) &\le C\|w\|_{L^r} \cdot \|\nabla w\|_{L^{2r/(r-2)}} \cdot \|\Delta w\|_{L^2} \\
&\le C\|w\|_{L^r} \cdot \|\nabla w\|_{L^2}^{1-3/r} \cdot \|\Delta w\|_{L^2}^{1+3/r} \\
&\le \frac{1}{2}\|\Delta w\|_{L^2}^2 + C\|w\|_{L^r}^s \|\nabla w\|_{L^2}^2.
\end{aligned}
$$

Inserting the above estimate into (18) and using the Gronwall's inequality lead to

$$
\|w\|_{L^\infty(0,T;H^1)\cap L^2(0,T;H^2)} \le C. \tag{20}
$$

(2) Let us assume that (14) holds true. Noting that

$$
\int (w\cdot\nabla)w\cdot\Delta w\,dx = \sum_{i,k}\int w_i\partial_i w\cdot\partial_k^2 w\,dx = -\sum_{i,k}\int \partial_k w_i\cdot\partial_i w\cdot\partial_k w\,dx, \tag{21}
$$

and

$$
-\int w\,\mathrm{div}[\nabla w\nabla w]dx = \sum_{i,j,k}\int \partial_j w_i\cdot\frac{\partial w_i}{\partial x_k}\cdot\frac{\partial w_j}{\partial x_k}\,dx, \tag{22}
$$

we bound $I(t)$ as follows:

$$
\begin{aligned}
I(t) &\le C\|\nabla w\|_{L^2}\cdot\|\nabla w\|_{L^4}^2 \le C\|w\|_{\dot B^0_{\infty,\infty}}\cdot\|\nabla w\|_{L^2}\cdot\|\Delta w\|_{L^2} \\
&\le \frac{1}{2}\|\Delta w\|_{L^2}^2 + C\|w\|_{\dot B^0_{\infty,\infty}}^2\|\nabla w\|_{L^2}^2.
\end{aligned} \tag{23}
$$

Here we have used the Machihara-Ozawa's inequality [10]:

$$
\|\nabla w\|_{L^4}^2 \le C\|w\|_{\dot B^0_{\infty,\infty}}\cdot\|\Delta w\|_{L^2}. \tag{24}
$$

Inserting (23) into (18), we get (20) due to the Gronwall's inequality.

(3) Let us assume that (15) holds true. Using (21), we bound

$$
\int (w\cdot\nabla)w\cdot\Delta w\,dx = -\sum_{i,k}\int \partial_k w_i\cdot\partial_i w\cdot\partial_k w\,dx
$$

as follows. We decompose $\partial_k w$ as follows

$$
\partial_k w = \sum_{j=-\infty}^{+\infty}\phi_j*\partial_k w = \sum_{j<-N}\phi_j*\partial_k w + \sum_{j=-N}^{N}\phi_j*\partial_k w + \sum_{j>N}\phi_j*\partial_k w,
$$

where N is a positive integer to be chosen later. Plugging this decomposition into $\int (w\cdot\nabla)w\cdot\Delta w\,dx$, we get

$$\begin{aligned}\int (w\cdot\nabla)w\cdot\Delta w\,dx &= -\sum_{j<-N}\sum_{i,k}\int \partial_k w_i\cdot\partial_i w\cdot\phi_j*\partial_k w\,dx\\ &\quad -\sum_{j=-N}^{N}\sum_{i,k}\int \partial_k w_i\cdot\partial_i w\cdot\phi_j*\partial_k w\,dx\\ &\quad -\sum_{j>N}\sum_{i,k}\int \partial_k w_i\cdot\partial_i w\cdot\phi_j*\partial_k w\,dx\\ &=: I_1+I_2+I_3. \end{aligned}\tag{25}$$

Recalling the Bernstein's inequality [17],

$$\|\phi_j*f\|_{L^q}\le C2^{3j(1/p-1/q)}\|\phi_j*f\|_{L^p},\qquad 1\le p\le q\le\infty,\tag{26}$$

with C being a positive constant independent of f and j, we apply Hölder's inequality to deduce that

$$\begin{aligned} I_1 &\le \sum_{i,k}\|\partial_k w_i\|_{L^2}\cdot\|\partial_i w\|_{L^2}\cdot\sum_{j<-N}\|\phi_j*\partial_k w\|_{L^\infty}\\ &\le C\|\nabla w\|_{L^2}^2\cdot\sum_{j<-N}2^{(3/2)j}\|\phi_j*\nabla w\|_{L^2}\\ &\le C2^{-(3/2)N}\|\nabla w\|_{L^2}^3,\\ I_2 &\le \sum_{i,k}\|\partial_k w_i\|_{L^2}\cdot\|\partial_i w\|_{L^2}\cdot\sum_{j=-N}^{N}\|\phi_j*\partial_k w\|_{L^\infty}\\ &\le CN\|\nabla w\|_{L^2}^2\cdot\|\nabla w\|_{\dot B^0_{\infty,\infty}}, \end{aligned}$$

and

$$\begin{aligned} I_3 &\le \sum_{i,k}\|\partial_k w_i\|_{L^6}\cdot\|\partial_i w\|_{L^2}\cdot\sum_{j>N}\|\phi_j*\partial_k w\|_{L^3}\\ &\le C\|\nabla w\|_{L^6}\cdot\|\nabla w\|_{L^2}\cdot\sum_{j>N}2^{j/2}\|\phi_j*\nabla w\|_{L^2}\\ &\le C\|\Delta w\|_{L^2}\cdot\|\nabla w\|_{L^2}\left(\sum_{j>N}2^{-j}\right)^{1/2}\cdot\left(\sum_{j>N}2^{2j}\|\phi_j*\nabla w\|_{L^2}^2\right)^{1/2}\\ &\le C2^{-N/2}\|\nabla w\|_{L^2}\cdot\|\Delta w\|_{L^2}^2. \end{aligned}$$

Now we choose N so that $C2^{-N/2}\|\nabla w\|_{L^2} \le \frac{1}{4}$, to conclude

$$\int (w\cdot\nabla)w\cdot\Delta w\,dx$$

$$\le C\|\nabla w\|_{L^2}^2 + C\|\nabla w\|_{\dot{B}^0_{\infty,\infty}}\cdot\|\nabla w\|_{L^2}^2\log^+\|\Delta w\|_{L^2}^2 + \frac{1}{4}\|\Delta w\|_{L^2}^2. \tag{27}$$

Similarly, using (22), we infer that

$$-\int w\,\mathrm{div}[\nabla w\nabla w]dx \le \text{the right hand side of (27)}. \tag{28}$$

Inserting (27) and (28) into (18), we arrive at (20) by the Gronwall's inequality.

(4) Finally, let us assume that (16) holds true. Applying curl to (1), we find that

$$\partial_t \,\mathrm{curl}\, w + w\cdot\nabla\,\mathrm{curl}\, w - \Delta\,\mathrm{curl}\, w + \mathrm{curl}\,\mathrm{div}(I-\Delta)^{-1}[\nabla w\nabla w]$$

$$= (\mathrm{curl}\, w\cdot\nabla)w. \tag{29}$$

Testing (29) by $(I-\Delta)\,\mathrm{curl}\, w$, we see that

$$\frac{1}{2}\frac{d}{dt}\int |\mathrm{curl}\, w|^2 + |\nabla\,\mathrm{curl}\, w|^2 dx + \int |\nabla\,\mathrm{curl}\, w|^2 + |\Delta\,\mathrm{curl}\, w|^2 dx$$

$$= -\int (\mathrm{curl}\, w\cdot\nabla)w\cdot\Delta\,\mathrm{curl}\, w\,dx + \int (w\cdot\nabla)\,\mathrm{curl}\, w\cdot\Delta\,\mathrm{curl}\, w\,dx$$

$$-\int \mathrm{curl}\cdot\mathrm{div}\cdot(I-\Delta)^{-1}[\nabla w\nabla w]\cdot(I-\Delta)\,\mathrm{curl}\, w\,dx$$

$$+\int (\mathrm{curl}\, w\cdot\nabla)w\cdot\mathrm{curl}\, w\,dx$$

$$=: J_1 + J_2 + J_3 + J_4 \tag{30}$$

Using Lemma 1 and the interpolation inequality, we bound J_1 as follows.

$$J_1 = \int (\mathrm{curl}\, w\cdot\nabla w)\cdot\Lambda^2\,\mathrm{curl}\, w\,dx \qquad (\Lambda := (-\Delta)^{1/2})$$

$$= \int \Lambda^{1-\alpha}(\mathrm{curl}\, w\cdot\nabla w)\cdot\Lambda^{1+\alpha}\,\mathrm{curl}\, w\,dx$$

$$\le \|\Lambda^{1-\alpha}(\mathrm{curl}\, w\cdot\nabla w)\|_{L^2}\cdot\|\Lambda^{1+\alpha}\,\mathrm{curl}\, w\|_{L^2}$$

$$\le C\|\mathrm{curl}\, w\cdot\nabla w\|_{\dot{F}^{1-\alpha}_{2,2}}\cdot\|\Lambda^{1+\alpha}\,\mathrm{curl}\, w\|_{L^2}$$

$$\le C(\|\mathrm{curl}\, w\|_{\dot{F}^{-\alpha}_{\infty,\infty}}\|\nabla w\|_{\dot{F}^1_{2,2}} + \|\nabla w\|_{\dot{F}^{-\alpha}_{\infty,\infty}}\|\mathrm{curl}\, w\|_{\dot{F}^1_{2,2}})\|\Lambda^{1+\alpha}\,\mathrm{curl}\, w\|_{L^2}$$

$$\leq C\|\mathrm{curl}\ w\|_{\dot{F}^{-\alpha}_{\infty,\infty}} \cdot \|\nabla\ \mathrm{curl}\ w\|_{L^2} \cdot \|\Lambda^{1+\alpha}\ \mathrm{curl}\ w\|_{L^2}$$

$$\leq C\|\mathrm{curl}\ w\|_{\dot{F}^{-\alpha}_{\infty,\infty}} \cdot \|\nabla\ \mathrm{curl}\ w\|_{L^2}^{2-\alpha} \cdot \|\Delta\ \mathrm{curl}\ w\|_{L^2}^{\alpha}$$

$$\leq \frac{1}{16}\|\Delta\ \mathrm{curl}\ w\|_{L^2}^2 + C\|\mathrm{curl}\ w\|_{\dot{F}^{-\alpha}_{\infty,\infty}}^{2/(2-\alpha)}\|\nabla\ \mathrm{curl}\ w\|_{L^2}^2. \tag{31}$$

Here, we have used the following inequalities [15]:

$$\|\Lambda^{1-\alpha}f\|_{L^2} \leq C\|f\|_{\dot{F}^{1-\alpha}_{2,2}}, \qquad \|\nabla f\|_{\dot{F}^{-\alpha}_{\infty,\infty}} \leq C\|\mathrm{curl}\ f\|_{\dot{F}^{-\alpha}_{\infty,\infty}},$$

and

$$\|\nabla f\|_{\dot{F}^{1}_{2,2}} \leq C\|\mathrm{curl}\ f\|_{\dot{F}^{1}_{2,2}}.$$

By integration by parts, we rewrite J_2 as

$$J_2 = \sum_{i,k}\int w_i\partial_i\ \mathrm{curl}\ w \cdot \partial_k^2\ \mathrm{curl}\ w\ dx = -\sum_{i,k}\int w_i\ \mathrm{curl}\ w \cdot \partial_i\partial_k^2\ \mathrm{curl}\ w\ dx$$

$$= \sum_{i,k}\int \partial_k w_i \cdot \mathrm{curl}\ w \cdot \partial_i\partial_k\ \mathrm{curl}\ w\ dx$$

$$= \sum_{i,k}\int \Lambda^{1-\alpha}(\partial_k w_i \cdot \mathrm{curl}\ w) \cdot \partial_i\partial_k(-\Delta)^{-1} \cdot \Lambda^{1+\alpha}\ \mathrm{curl}\ w\ dx$$

$$\leq \sum_{i,k}\|\Lambda^{1-\alpha}(\partial_k w_i \cdot \mathrm{curl}\ w)\|_{L^2} \cdot \|\partial_i\partial_k(-\Delta)^{-1} \cdot \Lambda^{1+\alpha}\ \mathrm{curl}\ w\|_{L^2}$$

$$\leq C\sum_{i,k}\|(\partial_k w_i \cdot \mathrm{curl}\ w)\|_{\dot{F}^{1-\alpha}_{2,2}} \cdot \|\Lambda^{1+\alpha}\ \mathrm{curl}\ w\|_{L^2}$$

and we obtain, in the same way as that of J_1,

$$J_2 \leq \text{the right hand side of (31)}. \tag{32}$$

By integration by parts, we bound J_3 as follows.

$$J_3 = -\int[\nabla w \nabla w] \cdot \nabla\ \mathrm{curl}^2\ w\ dx$$

$$= -\int \Lambda^{1-\alpha}[\nabla w \nabla w] \cdot \Lambda^{\alpha-1}\nabla\Delta w\ dx$$

$$= \int \Lambda^{1-\alpha}[\nabla w \nabla w] \cdot \Lambda^{1+\alpha}\nabla w\ dx$$

and we get, in the same way as that of J_1,

$$J_3 \leq \text{the right hand side of (31).} \tag{33}$$

Finally, we bound J_4 as follows.

$$\begin{aligned} J_4 &= \int \Lambda^{1-\alpha}(\mathrm{curl}\, w \cdot \nabla w) \cdot \Lambda^{\alpha-1}\, \mathrm{curl}\, w\, dx \\ &\leq \|\Lambda^{1-\alpha}(\mathrm{curl}\, w \nabla w)\|_{L^2} \cdot \|\Lambda^{\alpha-1}\, \mathrm{curl}\, w\|_{L^2} \\ &\leq C\|\Lambda^{1-\alpha}(\mathrm{curl}\, w \cdot \nabla w)\|_{L^2} \cdot \|\Lambda^{\alpha} w\|_{L^2} \end{aligned}$$

and we get, in the same way as that of J_1,

$$\begin{aligned} J_4 &\leq C\|\mathrm{curl}\, w\|_{\dot{F}^{-\alpha}_{\infty,\infty}} \cdot \|\nabla w\|_{\dot{F}^{1}_{2,2}} \cdot \|\Lambda^{\alpha} w\|_{L^2} \\ &\leq C\|\mathrm{curl}\, w\|_{\dot{F}^{-\alpha}_{\infty,\infty}} \cdot \|\Delta w\|_{L^2} \cdot \|\Lambda^{\alpha} w\|_{L^2} \\ &\leq C\|\mathrm{curl}\, w\|_{\dot{F}^{\alpha}_{\infty,\infty}} (\|w\|^2_{L^2} + \|\Delta w\|^2_{L^2}) \end{aligned} \tag{34}$$

On the other hand, from (18), (21) and (22), we find that in the same way as that of J_4,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int |w|^2 + |\nabla w|^2 dx + \int |\nabla w|^2 + |\Delta w|^2 dx \\ \leq \text{the right hand side of (34).} \end{aligned} \tag{35}$$

Combining (30), (31), (32), (33), (34) and (35) and using the Gronwall's inequality, we arrive at

$$\|w\|_{L^\infty(0,T;H^2)\cap L^2(0,T;H^3)} \leq C,$$

This completes the proof. $\square$

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