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An explicit description of the Teichmüller space as holonomy representations and its applications

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§ 0. Introduction

Let Σ_g be the closed oriented surface of genus $g \geq 2$. We will treat with the space of representations $\mathrm{Hom}(\pi_1(\Sigma_g), PSU(1, 1))^{e=2-2g}/\mathrm{conj}$, where $PSU(1, 1) = \{\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \in PSL(2, \boldsymbol{C})\}$ = the group of orientation preserving isometries on the Poincaré disk and "$e = 2 - 2g$" stands for the connected component (due to W. Goldman [G]) consisting of all representations with the Euler number of the associated oriented $\boldsymbol{RP}^1$-bundles $= 2 - 2g$. The Teichmüller space $\mathscr{T}_g$ of Σ_g is naturally identified with this representation space via holonomy representations after uniformizations. We fix a marking of Σ_g and a pants decomposition $P_1 \cup \cdots \cup P_{2g-2} = \Sigma_g$ associated to it. Hence we have a Fenchel-Nielsen coordinate $\{(l, \tau)\}$ as a parametrization of $\mathscr{T}_g$. Depending on this marking, we describe a global section of the trivial principal $PSU(1, 1)$-bundle

$$\mathrm{Hom}(\pi_1(\Sigma_g), PSU(1, 1))^{e=2-2g} \longrightarrow \mathscr{T}_g = \mathrm{Hom}(\pi_1(\Sigma_g), PSU(1, 1))^{e=2-2g}/\mathrm{conj}.$$

More explicitly we write down the image of a generator system of $\pi_1(\Sigma_g)$ in $PSU(1, 1)$ under a holonomy representation $\rho(l, \tau)$ corresponding to each (l, τ) (Theorem 1). As an application, the length of the closed geodesic in each free homotopy class of (based) closed curves ω in a Riemann surface with Fenchel-Nielsen coordinate (l, τ) is expressed by the trace of $\rho(l, \tau)(\omega)$ (Theorem 3.1). On the other hand the parameters τ of Nielsen twists are described in terms of lengths of closed geodesics (Theorem 3.2). Combining these two, we get a formula of describing the transformation of the mapping class group on the coordinate $\{(l, \tau)\}$ (Theorem 3.3).

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§1. Notation and main theorem

In the following we will give a statement of our main theorem. For fundamental properties on hyperbolic geometry, we refer to Thurston's lecture note [T]. For $\omega \in \pi_1(\Sigma_g)$, let us denote its free homotopy class by $[\omega]$. In a hyperbolic space (Σ_g, h), where h denotes a hyperbolic metric, there exists a unique closed oriented geodesic $\hat{\omega}$ in each free homotopy class $[\omega]$. A marking $(\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g)$ is a system of generators, with $\prod_{i=1}^{g} [\alpha_i, \beta_i] = 1$, of $\pi_1(\Sigma_g)$ modulo base point change by a common path. We will use a marking with the property that, for each $1 \le i \le g$, $\hat{\alpha}_i$ and $\hat{\beta}_i$ intersect at a single point and the ordered set $\langle \hat{\beta}_i, \hat{\alpha}_i \rangle$ of directions coincides with the orientation of Σ_g at this point. Recall that the Teichmüller space $\mathscr{T}_g$ is the set of equivalence classes of these pairs $((\Sigma_g, h), (\alpha_1, \cdots, \beta_g))$; $((\Sigma_g, h), (\alpha_1, \cdots, \beta_g)) \sim ((\Sigma_g, h'), (\alpha'_1, \cdots, \beta'_g))$ if there is an orientation preserving isometry $\varphi\colon (\Sigma_g, h) \to (\Sigma_g, h')$ such that $(\varphi_* \alpha_1, \cdots, \varphi_* \beta_g) = (\alpha'_1, \cdots, \beta'_g)$. To introduce a Fenchel-Nielsen coordinate on $\mathscr{T}_g$, we fix a pants decomposition of Σ_g: Define elements $\gamma_1, \cdots, \gamma_{g-1}$, $\delta_2, \cdots, \delta_{g-1}$ of $\pi_1(\Sigma_g)$ by $\gamma_i = \beta_i \alpha_i^{-1} \beta_i^{-1} \alpha_{i+1}$, $\delta_i = \gamma_i \cdots \gamma_{g-1} \beta_g \alpha_g^{-1} \beta_g^{-1}$.

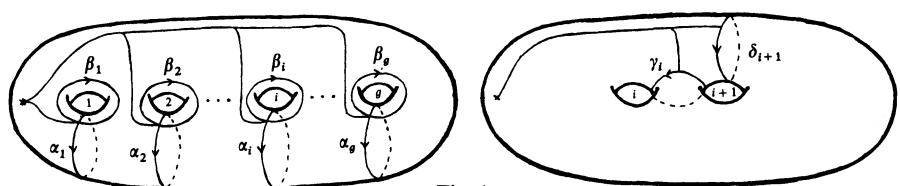

Fig. 1

In (Σ_g, h), $[\alpha_1], \cdots, [\alpha_g]$, $[\gamma_1], \cdots, [\gamma_{g-1}]$, $[\delta_2], \cdots, [\delta_{g-1}]$ are represented by mutually disjoint simple closed curves, hence by mutually disjoint simple closed geodesics. Cutting (Σ_g, h) along these $(3g-3)$ simple closed geodesics, we get a pants decomposition $P_1 \cup \cdots \cup P_{2g-2}$ of (Σ_g, h): $\partial P_1 = \hat{\alpha}_1 \cup \hat{\gamma}_1 \cup \hat{\delta}_2$, $\partial P_2 = \hat{\alpha}_1 \cup \hat{\gamma}_1 \cup \hat{\alpha}_2$, $\partial P_{2i-1} = \hat{\delta}_i \cup \hat{\gamma}_i \cup \hat{\delta}_{i+1}$, $\partial P_{2i} = \hat{\alpha}_i \cup \hat{\gamma}_i \cup \hat{\alpha}_{i+1}$ $(2 \le i \le g-2)$, $\partial P_{2g-3} = \hat{\delta}_{g-1} \cup \hat{\gamma}_{g-1} \cup \hat{\alpha}_g$ and $\partial P_{2g-2} = \hat{\alpha}_{g-1} \cup \hat{\gamma}_{g-1} \cup \hat{\alpha}_g$ (for $g=2$, $\partial P_1 = \partial P_2 = \hat{\alpha}_1 \cup \hat{\gamma}_1 \cup \hat{\alpha}_2$ and define P_1 as right-hand side of $\hat{\alpha}_1$).

Each P_j is decomposed into two right-angled hexagons $H_j^{(1)} \cup H_j^{(2)}$ which are isometric to each other (hence denote both by H_j). This decomposition is realized by three geodesic arcs which are connecting two different boundary components of P_j, perpendicular to both. We shall choose a base point $* = *(h, \{P_j\})$ of Σ_g as the end point in $\hat{\alpha}_1$ of the perpendicular geodesic arc between $\hat{\alpha}_1$ and $\hat{\delta}_2$ in P_1. We define the element α_1 of $\pi_1(\Sigma_g, *)$ as $\hat{\alpha}_1$ considered as a closed curve starting from $*$. This defines also the elements

$\beta_1, \alpha_2, \cdots, \beta_g \in \pi_1(\Sigma_g, *)$. (There may be no confusion with the same symbols used in our marking $(\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g)$.)

NOTATION. (Fenchel-Nielsen coordinate $(l, \tau) \in (\boldsymbol{R}_+)^{3g-3} \times \boldsymbol{R}^{3g-3}$ and other dependent parameters)

1. We denote by $\{l_i \in \boldsymbol{R}_+ (i = -\infty, 0, \cdots, 3g-6, \infty)\}$ the lengths L of the geodesic boundaries of the above pants $\{P_j\}$. The index i of l_i is determined as follows: $l_{-\infty} = L(\hat{\alpha}_1)$, $l_0 = L(\hat{\gamma}_1)$, $l_{3i-5} = L(\hat{\delta}_i)$, $l_{3i-4} = L(\hat{\alpha}_i)$, $l_{3i-3} = L(\hat{\gamma}_i)$ $(2 \leq i \leq g-1)$ and $l_\infty = L(\hat{\alpha}_g)$ (hereafter by l_i, we also indicate the corresponding simple closed geodesic). Notice that the isometry class of each hexagon H_j is completely determined by the lengths of its three alternating sides.

2. In each $P_j = H_j^{(1)} \cup H_j^{(2)}$, we shall give a naming which describes the length of each perpendicular geodesic arc connecting two different boundaries of P_j (three alternating sides of H_j other than $l_i/2$'s). We use the symbols s_{3m}, t_{3m}, u_{3n+1}, u_{3n+2}, v_{3n+1} and v_{3n+2}, where $m = -\infty, 0, \cdots, g-2, \infty$ and $n = 0, \cdots, g-3$ (these symbols will also indicate the corresponding sides). s_{3m} (resp. t_{3m}) denotes the length of the opposite side of $l_{3m}/2$ in H_{2i-1} (resp. H_{2i}) for appropriate i. u_{3n+1} (resp. u_{3n+2}, v_{3n+1}, v_{3n+2}) denotes the length of the opposite side of $l_{3n+1}/2$ (resp. $l_{3n+2}/2$, $l_{3n+1}/2$, $l_{3n+2}/2$) in H_{2n+1} (resp. H_{2n+2}, H_{2n+3}, H_{2n+4}). Hence we have the following picture shown in Fig. 2.

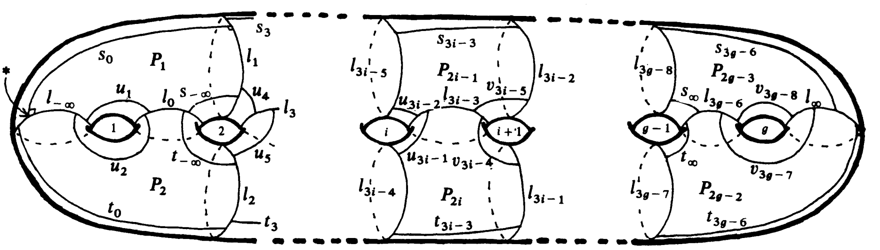

Fig. 2

Here we note that s_k, t_k, u_k and v_k are written as functions of $\{l_i\}$. For example for $1 < i < g$, we have

$$\begin{aligned} & ch(s_{3i-3}) \\ & = (ch(l_{3i-5}/2)\, ch(l_{3i-2}/2) + ch(l_{3i-3}/2))/sh(l_{3i-5}/2)\, sh(l_{3i-2}/2). \end{aligned}$$

3. To determine the parameters of Nielsen twists $\{\tau_i \in \boldsymbol{R} (i = -\infty, 0, \cdots, 3g-6, \infty)\}$ (lengths with signs) along simple closed geodesics $\{l_i\}$, which complete the reconstruction of (Σ_g, h) from $P_1, \cdots, P_{2g-2}$, we have to determine their origin and orientations. The origin $(\tau_{-\infty}, \cdots, \tau_\infty) = (0, \cdots, 0)$ is charac-

terized by the property that $u_1 \cup u_2$, $s_{-\infty} \cup u_4 \cup u_5 \cup t_{-\infty}$, $v_{3i-8} \cup u_{3i-2} \cup u_{3i-1} \cup v_{3i-7}$ $(3 \le i \le g-2)$, $v_{3g-11} \cup s_{\infty} \cup t_{\infty} \cup v_{3g-10}$ and $v_{3g-8} \cup v_{3g-7}$ are closed geodesics and coincide with $\hat{\beta}_1$, $\hat{\beta}_2$, $\hat{\beta}_i$ $(3 \le i \le g-2)$, $\hat{\beta}_{g-1}$ and $\hat{\beta}_g$, respectively (for $g=2$, read as $s_{\infty} \cup t_{\infty} = \hat{\beta}_1$ and $s_{-\infty} \cup t_{-\infty} = \hat{\beta}_2$; for $g=3$, $s_{-\infty} \cup s_{\infty} \cup t_{\infty} \cup t_{-\infty} = \hat{\beta}_2$). The sign of τ_i is determined by the direction of the earthquake (along l_i) between two perpendicular geodesic arcs which were smoothly connected when $\tau_i = 0$, and we fix this as shown in Fig. 3.

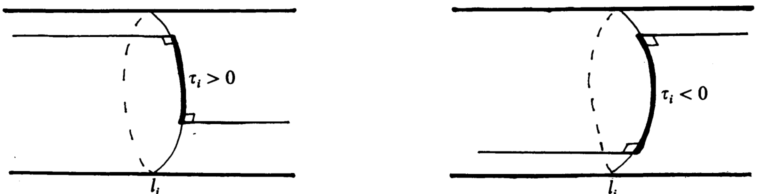

Fig. 3

Note that the above $\{(l, \tau) \in (\boldsymbol{R}_+)^{3g-3} \times \boldsymbol{R}^{3g-3}\}$ gives a global coordinate of $\mathscr{T}_g$.

Next define two typical elements of $PSU(1, 1)$: For $\theta \in \boldsymbol{R}$, let

$$e(\theta) = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix} \in PSU(1, 1)$$

which is the rotation of angle θ around the origin in the Poincaré disk $\mathring{D}(1)$; for $l \in \boldsymbol{R}$, let

$$eh(l) = \begin{bmatrix} ch(l/2) & sh(l/2) \\ sh(l/2) & ch(l/2) \end{bmatrix} \in PSU(1, 1)$$

which, for $l > 0$, is the translation of length l to the positive direction along the real axis in $\mathring{D}(1)$.

Now we can state our main theorem:

THEOREM 1. *Let* $\{(l, \tau) \in (\boldsymbol{R}_+)^{3g-3} \times \boldsymbol{R}^{3g-3}\}$ *be the above Fenchel-Nielsen coordinate of the Teichmüller space* $\mathscr{T}_g$, *which is identified with* $\mathrm{Hom}(\pi_1(\Sigma_g), PSU(1, 1))^{e=2-2g}/\mathrm{conj}$. *Then the following* $\rho(l, \tau) \in \mathrm{Hom}(\pi_1(\Sigma_g, *(l, \tau)), PSU(1, 1))^{e=2-2g}$ *gives a global section of* $\mathrm{Hom}(\pi_1(\Sigma_g), PSU(1, 1))^{e=2-2g} \to \mathrm{Hom}(\pi_1(\Sigma_g), PSU(1, 1))^{e=2-2g}/\mathrm{conj}$:

$$\rho(l, \tau)(\alpha_1) = eh(-l_{-\infty})$$

$$\rho(l, \tau)(\beta_1) = eh(l_{-\infty}/2)e(\pi/2)eh(u_1)e(-\pi/2)eh(\tau_0)e(\pi/2)eh(u_2)$$
$$e(-\pi/2)eh(-l_{-\infty}/2 + \tau_{-\infty})$$

$$\rho(l, \tau)(\alpha_i) = Q_i(l, \tau)e(-\pi/2)eh(l_{3i-4})e(\pi/2)Q_i(l, \tau)^{-1}$$

$$\begin{aligned}\rho(l, \tau)(\beta_i) = {} & Q_i(l, \tau)e(-\pi)eh(v_{3i-7})e(-\pi/2)eh(\tau_{3i-6})e(\pi/2) \\ & eh(v_{3i-8})e(-\pi/2)eh(\tau_{3i-5})e(\pi/2)eh(u_{3i-2})e(-\pi/2)eh(\tau_{3i-3}) \\ & e(\pi/2)eh(u_{3i-1})e(-\pi/2)eh(\tau_{3i-4})e(3\pi/2)Q_i(l, \tau)^{-1} \\ & (1 < i < g)\end{aligned}$$

$$\rho(l, \tau)(\alpha_g) = Q_g(l, \tau)e(-\pi/2)eh(l_\infty)e(\pi/2)Q_g(l, \tau)^{-1}$$

$$\begin{aligned}\rho(l, \tau)(\beta_g) = {} & Q_g(l, \tau)e(-\pi)eh(v_{3g-7})e(-\pi/2)eh(\tau_{3g-6})e(\pi/2) \\ & eh(v_{3g-8})e(-\pi/2)eh(\tau_\infty)e(3\pi/2)Q_g(l, \tau)^{-1}\end{aligned}$$

where for $1 < i \leq g$,

$$\begin{aligned}Q_i(l, \tau) = {} & e(\pi/2)\prod_{j=0}^{i-3}\{eh(s_{3j})e(-\pi/2)eh(\tau_{3j+1})e(\pi/2)\}eh(s_{3i-6}) \\ & e(-\pi/2)eh(l_{3i-5}/2)e(-\pi/2)eh(v_{3i-8})e(-\pi/2)eh(\tau_{3i-6}) \\ & e(\pi/2)eh(v_{3i-7}).\end{aligned}$$

(Here $\prod_{j=0}^{n} A_j$ *means* $A_0 \cdots A_n$. *Also throughout the above statement, we regard, for* $i = 2$ v_{3i-8} *(resp.* v_{3i-7}*) as* $s_{-\infty}$ *(resp.* $t_{-\infty}$*), for* $i = g - 1$ u_{3i-2} *(resp.* u_{3i-1}*) as* s_∞ *(resp.* t_∞*) and for* $i = g$ l_{3i-5} *as* l_∞.*)*

Here we note that the Teichmüller space is contractible and there is no trouble in the identification of different base points.

§2. Proof of Theorem 1

First we must represent the generators $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g \in \pi_1(\Sigma_g, *)$ in terms of the sides of right-angled hexagons $\{H_j\}$. This can be done by means of "piecewise-linear" loops with right-angled bendings, although there are several ways to do.

Second we lift these "piecewise-linear" loops to paths in the universal covering $\mathring{D}(1)$ of $(\Sigma_g, h(l, \tau))$. To this end, we fix a lift of the base point $* = *(l, \tau)$ at the origin of $\mathring{D}(1)$, and a lift of the simple closed geodesic $l_{-\infty}$ at the real axis through the origin, with the direction of $\hat{\alpha}_1$ for $l_{-\infty}$ to be the negative one. Now we can write the lifts of piecewise-linear $\alpha_1, \beta_1, \cdots, \alpha_g, \beta_g$ in $\mathring{D}(1)$ as shown in Fig. 4.

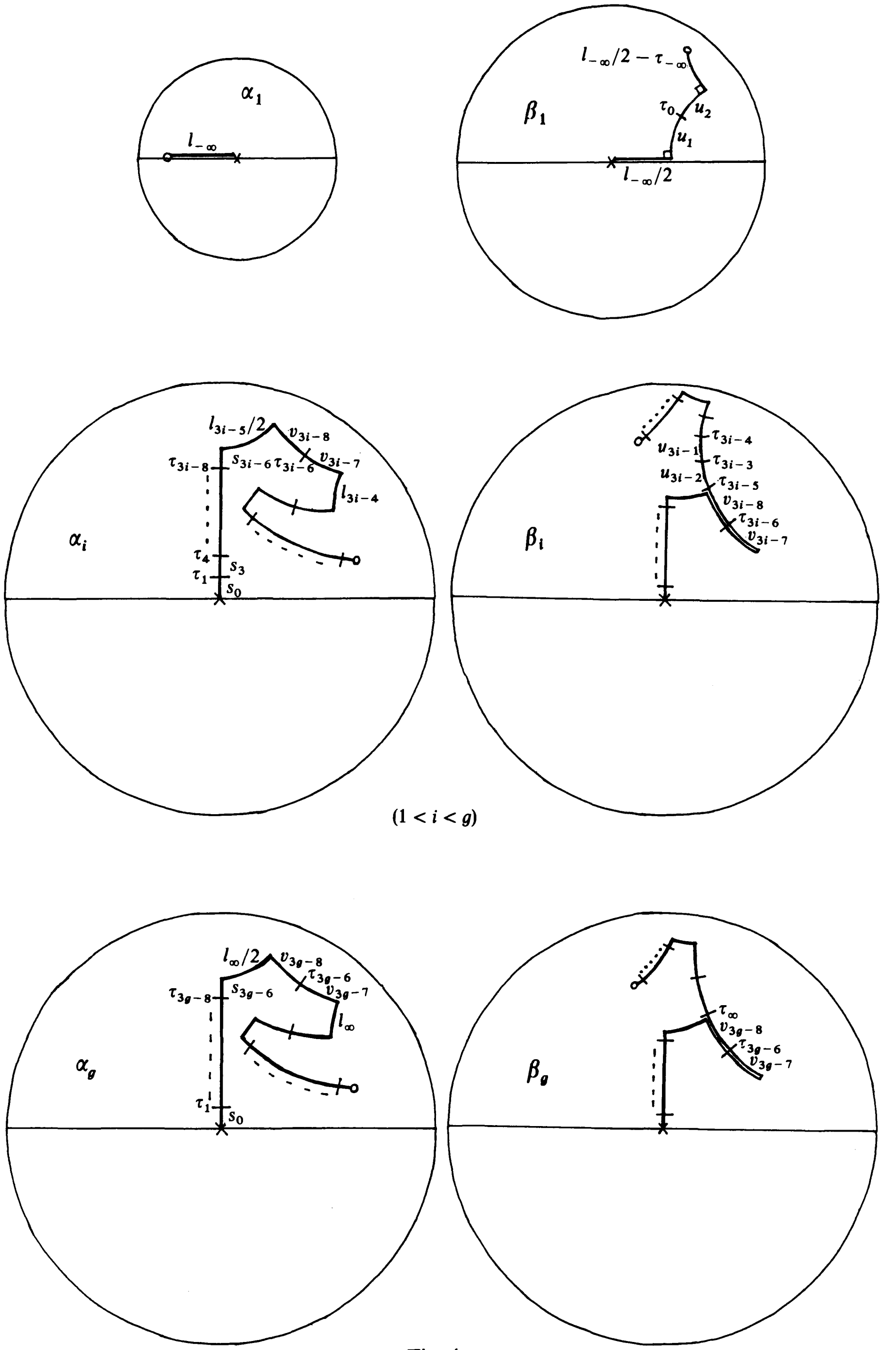
α_1
$l_{-\infty}$
β_1
$l_{-\infty}/2-\tau_{-\infty}$
τ_0
u_2
u_1
$l_{-\infty}/2$
α_i
$l_{3i-5}/2$
v_{3i-8}
τ_{3i-8}
s_{3i-6}
τ_{3i-6}
v_{3i-7}
l_{3i-4}
τ_4
s_3
τ_1
s_0
β_i
u_{3i-1}
τ_{3i-4}
τ_{3i-3}
u_{3i-2}
τ_{3i-5}
v_{3i-8}
τ_{3i-6}
v_{3i-7}
$(1<i<g)$
α_g
$l_\infty/2$
v_{3g-8}
τ_{3g-6}
τ_{3g-8}
s_{3g-6}
v_{3g-7}
l_∞
τ_1
s_0
β_g
τ_∞
v_{3g-8}
τ_{3g-6}
v_{3g-7}

Fig. 4

Here "|" means the possibility of the earthquake τ_i along l_i.

Third we read the holonomy representation $\rho(l, \tau)$ corresponding to $(l, \tau) \in \mathscr{T}_g$ from these pictures. We will demonstrate this procedure only for $\rho(l, \tau)(\beta_1)$, since others can be done in the same way. The first move of β_1 is represented by $eh(l_{-\infty}/2)$. To get the second move (bend by the angle $\pi/2$ and go ahead by the length u_1), we come back to the origin by $eh(l_{-\infty}/2)^{-1}$, rotate by $e(-\pi/2)$, here perform the standard transformation $eh(u_1)$ and return to the original position by $eh(l_{-\infty}/2)e(\pi/2)$. These two processes can be illustrated as in Fig. 5.

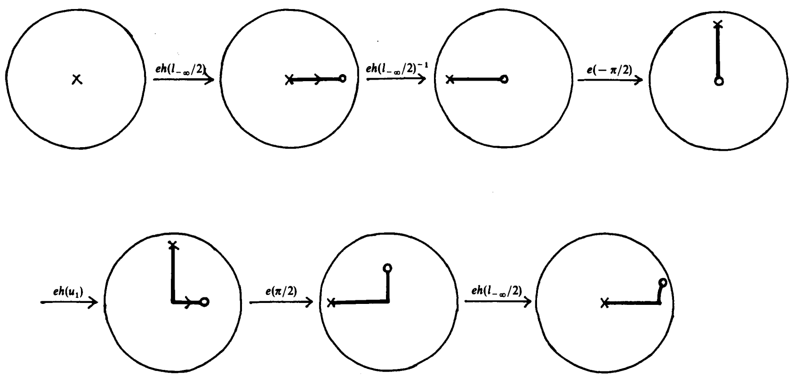

Fig. 5

Continuing this procedure, we get finally the following element (note that $e(-\pi/2)eh(-\tau)e(\pi/2) = e(\pi/2)eh(\tau)e(-\pi/2)$):

$$([3])e(\pi/2)eh(l_{-\infty}/2 - \tau_{-\infty})e(-\pi/2)([3])^{-1}\circ$$

$$\underbrace{\circ([2])e(\pi/2)eh(u_2)e(-\pi/2)([2])^{-1}\circ}_{\longleftarrow\ [3]\ \longrightarrow}$$

$$\underbrace{\circ([1])e(-\pi/2)eh(\tau_0)e(\pi/2)([1])^{-1}\circ}_{\longleftarrow\ [2]\ \longrightarrow}$$

$$\underbrace{\circ([0])e(\pi/2)eh(u_1)e(-\pi/2)([0])^{-1}\circ}_{\longleftarrow\ [1]\ \longrightarrow}$$

$$\underbrace{\circ\, eh(l_{-\infty}/2).}_{\leftarrow\ [0]\ \rightarrow}$$

(here $[0] = eh(l_{-\infty}/2)$, $[1] = [0]e(\pi/2)eh(u_1)$ etc.)

By this element, the origin is sent to the desired point which is determined by the covering transformation corresponding to β_1. Note that any element of $PSU(1, 1)$ is completely determined by its images of the origin and the direction of the real axis from the origin. Hence what is remained for us is to make sure that the direction determined by this element is also the desired one. This can be easily seen by tracing the direction in Fig. 5 and so on. Thus we have $\rho(\beta_1) = eh(l_{-\infty}/2)e(\pi/2)eh(u_1)e(-\pi/2)eh(\tau_0)e(\pi/2)eh(u_2)$ $e(\pi/2)eh(l_{-\infty}/2 - \tau_{-\infty})e(-\pi)$ as desired. □

REMARK. From the above construction, it is easy to deduce the method of writing the images of elements of $\pi_1(\Sigma_g, *)$ under the holonomy representation $\rho(l, \tau)$, only looking at piecewise-linear representatives of these elements. Suppose that a representative has the form shown in Fig. 6. Here $\twoheadrightarrow$ describes a lift of $\hat{\alpha}_1^{-1}$.

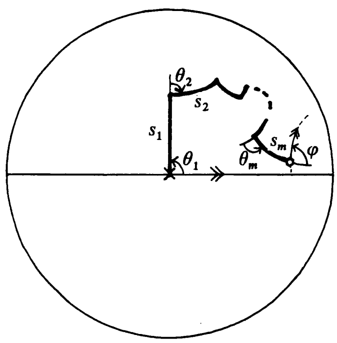

Fig. 6

Then its image is written as

$$e(\theta_1)eh(s_1)e(\theta_2)eh(s_2)\cdots e(\theta_m)eh(s_m)e(\varphi).$$

§3. Applications

3.1. Lengths of closed geodesics

Using the global section constructed in Theorem 1, we can write down the length of the unique closed geodesic $\hat{\omega}$ in each free homotopy class $[\omega]$ of closed curves in $(\Sigma_g, h(l, \tau))$: First connect $[\omega]$ to the base point $* = *(l, \tau)$ by a path and we get a based curve ω of $(\Sigma_g, *(l, \tau))$. We know that the image of ω under $\rho = \rho(l, \tau)$ determines a hyperbolic element of $PSU(1, 1)$ (modulo conjugation) with $|\mathrm{trace}| = 2\,ch(L(\hat{\omega})/2)$ (see [T, §5]). Hence we have the following

THEOREM 3.1. *Take any free homotopy class $[\omega]$ of closed curves ω in Σ_g. Let $h(l, \tau)$ be any hyperbolic metric (modulo $\mathrm{Diff}_+(\Sigma_g)$) on Σ_g. Considering ω as an element of $\pi_1(\Sigma_g, *(l, \tau))$, we fix its word presentation $\omega = \prod_i w_i$ with $w_i = \alpha_1^{\pm 1}, \beta_1^{\pm 1}, \cdots, \alpha_g^{\pm 1}$ or $\beta_g^{\pm 1}$. Then the length $L(l, \tau)(\hat{\omega})$ of the closed geodesic $\hat{\omega}$ in $[\omega]$ with respect to $h(l, \tau)$ is given by*

$$L(l, \tau)(\hat{\omega}) = 2\log\{|\mathrm{tr}(\prod_i \rho(l, \tau)(w_i))| + (|\mathrm{tr}(\prod_i \rho(l, \tau)(w_i))|^2 - 4)^{1/2}\} - 2\log 2,$$

where we know $\rho(l, \tau)(w_i)$ by Theorem 1. □

3.2. Nielsen twists in terms of geodesic-lengths

Here we give a formula of describing the parameters $\{\tau_i\}$ of Nielsen twists in terms of the lengths of closed geodesics in $(\Sigma_g, h(l, \tau))$. Note that the values l_k, s_k, t_k, u_k and v_k can be calculated from $L(\hat{\alpha}_i)$, $L(\hat{\gamma}_i)$ and $L(\hat{\delta}_i)$; in the following we will use these symbols as known numbers. Let $\bar{\alpha}_i = \beta_1^{-1} \cdots \beta_{i-1}^{-1}\alpha_i\beta_{i-1}\cdots\beta_1$ $(\bar{\alpha}_1 = \alpha_1)$ and $\bar{\gamma}_i = \beta_1^{-1}\cdots\beta_i^{-1}\gamma_i\beta_i\cdots\beta_1$.

THEOREM 3.2. *In a hyperbolic surface $(\Sigma_g, h(l, \tau))$, the parameters $\{\tau_i\}$ of Nielsen twists are written in terms of lengths of closed geodesics as follows (we abbreviate $L(\hat{\omega})$ to $L(\omega)$):*

(i) $\tau_{-\infty}$:

$sh(\tau_{-\infty}/2)^2 = \{ch(L(\delta_2^{-1}\bar{\alpha}_2)/2) - ch(l_1/2)ch(l_2/2) -$

$sh(l_1/2)sh(l_2/2)ch(s_0 + t_0)\}/2sh(l_1/2)sh(l_2/2)sh(s_0)sh(t_0)$,

and $\tau_{-\infty} \geq 0$ if and only if $L(\delta_2^{-1}\alpha_1\bar{\gamma}_1^{-1}\alpha_1^{-1}) \geq L(\delta_2^{-1}\bar{\gamma}_1^{-1})$ (for $g = 2$, we regard δ_2 (resp. l_1, l_2) as $\beta_2\alpha_2^{-1}\beta_2^{-1}$ (resp. l_∞, l_∞)).

(ii) τ_{3i-3} $(i = 1, \cdots, g-1)$:

$sh(\tau_{3i-3}/2)^2 = \{ch(L(\delta_{i+1}^{-1}\alpha_{i+1})/2) - ch(l_{3i-2}/2)ch(l_{3i-1}/2) -$

$sh(l_{3i-2}/2)sh(l_{3i-1}/2)ch(v_{3i-5} + v_{3i-4})\}/2sh(l_{3i-2}/2)sh(l_{3i-1}/2)$

$sh(v_{3i-5})sh(v_{3i-4})$,

and $\tau_{3i-3} \geq 0$ if and only if $L(\delta_i\gamma_i\alpha_{i+1}\gamma_i^{-1}) \geq L(\delta_i\alpha_{i+1})$ (we regard, for $i = 1$ v_{3i-5} (resp. v_{3i-4}, δ_i) as $s_{-\infty}$ (resp. $t_{-\infty}, \alpha_1^{-1}$) and for $i = g-1$ δ_{i+1} (resp. l_{3i-2}, l_{3i-1}) as $\beta_g\alpha_g^{-1}\beta_g^{-1}$ (resp. l_∞, l_∞)).

(iii) τ_{3i-5} $(i = 2, \cdots, g-1)$:

$sh(\tau_{3i-5}/2)^2 = \{ch(L(\delta_{i-1}\delta_{i+1})/2) - ch(l_{3i-8}/2)ch(l_{3i-2}/2) -$

$sh(l_{3i-8}/2)sh(l_{3i-2}/2)ch(s_{3i-6} + s_{3i-3})\}/2sh(l_{3i-8}/2)sh(l_{3i-2}/2)$

$sh(s_{3i-6})sh(s_{3i-3})$,

and $\tau_{3i-5} \geq 0$ *if and only if* $L(\delta_{i-1}\gamma_i) \geq L(\delta_{i-1}\delta_i^{-1}\gamma_i\delta_i)$ (*we regard, for* $i = 2$ δ_{i-1} (*resp.* l_{3i-8}) *as* α_1^{-1} (*resp.* $l_{-\infty}$) *and for* $i = g-1$ δ_{i+1} (*resp.* l_{3i-2}) *as* $\beta_g\alpha_g^{-1}\beta_g^{-1}$ (*resp.* l_∞)).

(iv) τ_{3i-4} $(i = 2,\cdots,g-1)$:

$$sh(\tau_{3i-4}/2)^2 = \{ch(L(\bar{\alpha}_{i-1}\bar{\alpha}_{i+1})/2) - ch(l_{3i-7}/2)ch(l_{3i-1}/2) - sh(l_{3i-7}/2)sh(l_{3i-1}/2)ch(t_{3i-6} + t_{3i-3})\}/2sh(l_{3i-7}/2)sh(l_{3i-1}/2) sh(t_{3i-6})sh(t_{3i-3}),$$

and $\tau_{3i-4} \geq 0$ *if and only if* $L(\bar{\alpha}_{i-1}\bar{\alpha}_i\bar{\gamma}_i^{-1}\bar{\alpha}_i^{-1}) \geq L(\bar{\alpha}_{i-1}\bar{\gamma}_i^{-1})$ (*we regard, for* $i = 2$ l_{3i-7} *as* $l_{-\infty}$ *and for* $i = g-1$ l_{3i-1} *as* l_∞).

(v) τ_∞:

$$sh(\tau_\infty/2)^2 = \{ch(L(\delta_{g-1}\beta_g\beta_{g-1}\alpha_{g-1}^{-1}(\beta_g\beta_{g-1})^{-1})/2) - ch(l_{3g-8}/2)ch(l_{3g-7}/2) - sh(l_{3g-8}/2)sh(l_{3g-7}/2)ch(s_{3g-6} + t_{3g-6})\}/ 2sh(l_{3g-8}/2)sh(l_{3g-7}/2)sh(s_{3g-6})sh(t_{3g-6}),$$

and $\tau_\infty \geq 0$ *if and only if* $L(\delta_{g-1}\beta_g\gamma_{g-1}^{-1}\beta_g^{-1}) \geq L(\delta_{g-1}\beta_g\alpha_g\gamma_{g-1}^{-1}\alpha_g^{-1}\beta_g^{-1})$ (*for* $g = 2$, *we regard* δ_{g-1} (*resp.* l_{3g-8}, l_{3g-7}) *as* α_1^{-1} (*resp.* $l_{-\infty}$, $l_{-\infty}$)).

PROOF. We will give a proof only for τ_{3i-5}. Direct calculation shows the following

LEMMA 3.2.1. *Let* $M(x, y) = eh(x)e(\pi/2)eh(y)e(-\pi/2)$ *for* $x, y \in \boldsymbol{R}$. *Then for any* $l, \lambda, s, \sigma, \tau \in \boldsymbol{R}$ *with* $l\cdot\lambda > 0$ *and* $s\cdot\sigma > 0$, *we have*

$$|\mathrm{tr}\{M(l, s)M(\tau, \sigma)M(\lambda, -\sigma)M(-\tau, -s)\}| = 2\{ch(l/2)ch(\lambda/2) + sh(l/2)sh(\lambda/2)(2sh(\tau/2)^2sh(s)sh(\sigma) + ch(s+\sigma))\}. \quad \square$$

Represent $\delta_{i-1}\delta_{i+1}$, $\delta_{i-1}\gamma_i$ and $\delta_{i-1}\delta_i^{-1}\gamma_i\delta_i$ by the piecewise-linear loops shown in Fig. 7. Here "$\square$" shows the difference of the length caused by the earthquake τ_{3i-5}. Reading the images $\rho(\delta_{i-1}\delta_{i+1})$, $\rho(\delta_{i-1}\gamma_i)$ and $\rho(\delta_{i-1}\delta_i^{-1}\gamma_i\delta_i)$ of the holonomy representation $\rho = \rho(l, \tau)$ as in Theorem 1, we have

$$|\mathrm{tr}\,\rho(\delta_{i-1}\delta_{i+1})| = |\mathrm{tr}\{M(l_{3i-8}, s_{3i-6})M(\tau_{3i-5}, s_{3i-3}) M(l_{3i-2}, -s_{3i-3})M(-\tau_{3i-5}, -s_{3i-6})\}|,$$

$$|\mathrm{tr}\,\rho(\delta_{i-1}\gamma_i)| = |\mathrm{tr}\{M(l_{3i-8}, s_{3i-6})M(\tau_{3i-5} + l_{3i-5}/2, u_{3i-2}) M(l_{3i-3}, -u_{3i-2})M(-\tau_{3i-5} - l_{3i-5}/2, -s_{3i-6})\}|$$

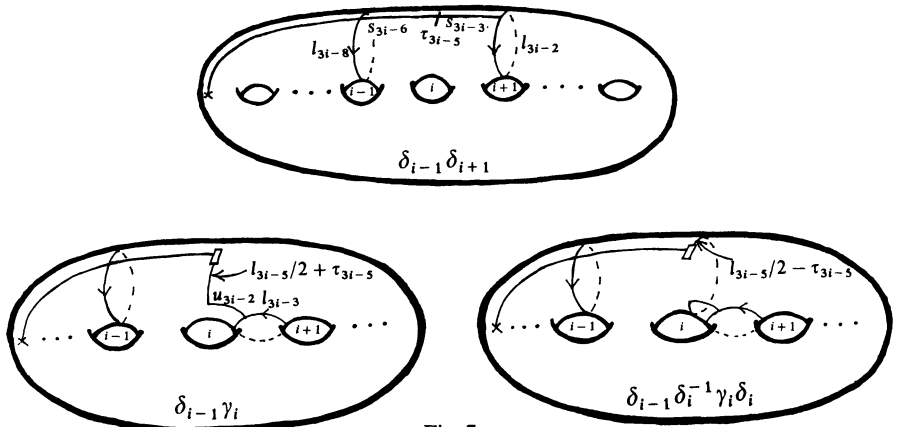

Fig. 7

and $|\mathrm{tr}\,\rho(\delta_{i-1}\delta_i^{-1}\gamma_i\delta_i)| = |\mathrm{tr}\{M(l_{3i-8}, s_{3i-6})M(\tau_{3i-5} - l_{3i-5}/2, u_{3i-2})$

$$M(l_{3i-3}, -u_{3i-2})M(-\tau_{3i-5} + l_{3i-5}/2, -s_{3i-6})\}|.$$

Hence by Lemma 3.2.1, we have the desired formula for $sh(\tau_{3i-5}/2)^2$. Also we deduce that $sh(\tau_{3i-5}/2 + l_{3i-5}/4)^2 \geq sh(\tau_{3i-5}/2 - l_{3i-5}/4)^2$ if and only if $|\mathrm{tr}\,\rho(\delta_{i-1}\gamma_i)| \geq |\mathrm{tr}\,\rho(\delta_{i-1}\delta_i^{-1}\gamma_i\delta_i)|$. $\square$

3.3. Fenchel-Nielsen coordinate and mapping class group

Now we can describe how the mapping class group $\mathfrak{M}_g$ of Σ_g acts on our Fenchel-Nielsen coordinate $\{(l, \tau)\}$. By $[f] \in \mathfrak{M}_g = \pi_0(\mathrm{Diff}_+(\Sigma_g))$, $(l, \tau) = [(\Sigma_g, h), (\alpha_1, \cdots, \beta_g)] \in \mathcal{T}_g$ is mapped to $[(\Sigma_g, h), ((f^{-1})_*\alpha_1, \cdots, (f^{-1})_*\beta_g)] \in \mathcal{T}_g$, which is equal to $[(\Sigma_g, (f^{-1})^*h), (\alpha_1, \cdots, \beta_g)]$. In other words, $[f] \cdot [\rho(l, \tau)] = [\rho(l, \tau) \circ (f^{-1})_*] \in \mathrm{Hom}(\pi_1(\Sigma_g), PSU(1, 1))^{e=2-2g}/\mathrm{conj}$. Because ρ is a global section, there exists (l', τ') with $[f] \cdot [\rho(l, \tau)] = [\rho(l', \tau')]$. This (l', τ') is just our Fenchel-Nielsen coordinate for $[(\Sigma_g, (f^{-1})^*h), (\alpha_1, \cdots, \beta_g)]$. Our aim is to make a formula of calculating this (l', τ') from (l, τ). This is constructed as follows: Starting from (l, τ), we have $\rho(l, \tau)(\alpha_1), \cdots, \rho(l, \tau)(\beta_g)$ by Theorem 1. Given $[f] \in \mathfrak{M}_g$ as an automorphism of $\pi_1(\Sigma_g, *)$, we can calculate $\rho(l, \tau)((f^{-1})_*\alpha_1), \cdots, \rho(l, \tau)((f^{-1})_*\beta_g)$. Now introduce a Fenchel-Nielsen coordinate with respect to the marking $((f^{-1})_*\alpha_1, \cdots, (f^{-1})_*\beta_g)$ just in the same manner as we did in §1. Because $f: (\Sigma_g, h) \to (\Sigma_g, (f^{-1})^*h)$ is an isometry, this Fenchel-Nielsen coordinate is equal to (l', τ'). From $(f^{-1})_*\alpha_1, \cdots, (f^{-1})_*\beta_g$, l' can be calculated by Theorem 3.1 and τ' by Theorem 3.2. Hence we have the following

THEOREM 3.3. *Take any* $[f] \in \mathfrak{M}_g$ *and suppose that the images of the generators* $\alpha_1, \cdots, \beta_g$ *of* $\pi_1(\Sigma_g, *)$ *under* $(f^{-1})_*: \pi_1(\Sigma_g, *) \to \pi_1(\Sigma_g, *)$ *are* *given. Then we can describe the transformation on the Fenchel-Nielsen coordinate* $\{(l, \tau)\}$ *caused by* $[f]$. □

It is well-known that any $[f] \in \mathfrak{M}_g$ can be given as a sequence of words of Dehn twists D_ω along $\hat{\omega} = \hat{\alpha}_i, \hat{\beta}_i$ or $\hat{\gamma}_i$ [L]. So it is enough to know about the transformations caused by these Dehn twists. For the sake of completeness, we will make a list of the images of $\alpha_1, \cdots, \beta_g$ under D_ω along $\hat{\omega} = \hat{\alpha}_i, \hat{\beta}_i$ or $\hat{\gamma}_i$. To determine D_ω, we will adopt a rule that the right-hand side of the oriented $\hat{\omega}$ is moved to the direction of $\hat{\omega}$. Then we have

(1) $(D_{\alpha_i})_* : \beta_i \longmapsto \beta_i \alpha_i^{-1}$ (other α_j, β_j's are fixed),

(2) $(D_{\beta_i})_* : \alpha_i \longmapsto \alpha_i \beta_i$,

(3) $(D_{\gamma_i})_* : \beta_i \longmapsto \gamma_i \beta_i,\ \beta_{i+1} \longmapsto \beta_{i+1} \gamma_i^{-1},\ \alpha_{i+1} \longmapsto \gamma_i \alpha_{i+1} \gamma_i^{-1}$.

EXAMPLE. Let $g \geq 3$ and $[f] = [D_{\beta_1}] \in \mathfrak{M}_g$. Then by $(D_{\beta_1}^{-1})_* : \pi_1(\Sigma_g, *(l, \tau)) \to \pi_1(\Sigma_g, *(l, \tau))$, $\alpha_1 \mapsto \alpha_1 \beta_1^{-1}$, $\gamma_1 \mapsto \beta_1 \gamma_1$ and other α_j, γ_j, δ_j's are unchanged. Hence in $(l', \tau') = [D_{\beta_1}] \cdot (l, \tau)$, $l'_{-\infty}$ is calculated as

$$\begin{aligned} ch(l'_{-\infty}/2) &= |\operatorname{tr} \rho(l, \tau)((D_{\beta_1}^{-1})_* \alpha_1)|/2 \\ &= |\operatorname{tr}\{M(l_{-\infty} + \tau_{-\infty}, u_2) M(\tau_0, u_1)\}|/2 \\ &= ch((l_{-\infty} + \tau_{-\infty})/2) ch(\tau_0/2) ch((u_1 + u_2)/2) + sh((l_{-\infty} + \tau_{-\infty})/2) sh(\tau_0/2) \\ &\quad ch((u_1 - u_2)/2), \end{aligned}$$

where u_1 is given by $ch(u_1) = \{ch(l_{-\infty}/2) ch(l_0/2) + ch(l_1/2)\}/sh(l_{-\infty}/2) sh(l_0/2)$ and u_2 by $ch(u_2) = \{ch(l_{-\infty}/2) ch(l_0/2) + ch(l_2/2)\}/sh(l_{-\infty}/2) sh(l_0/2)$. Similarly l'_0 is calculated as

$$\begin{aligned} ch(l'_0/2) &= |\operatorname{tr} \rho(l, \tau)((D_{\beta_1}^{-1})_* \gamma_1)|/2 \\ &= \{ch(\tau_0/2) ch(\tau_{-\infty}/2) ch((u_1 + u_2)/2) + sh(\tau_0/2) sh(\tau_{-\infty}/2) ch((u_1 - u_2)/2)\} \\ &\quad ch(l_0/2) + \{sh(\tau_0/2) ch(\tau_{-\infty}/2) ch((u_1 + u_2)/2) + ch(\tau_0/2) sh(\tau_{-\infty}/2) \\ &\quad ch((u_1 - u_2)/2)\} sh(l_0/2). \end{aligned}$$

Also for $j \neq -\infty, 0$, we have $l'_j = l_j$.

We now determine new Nielsen twists, for example, $\tau'_{-\infty}$ by Theorem 3.2. $|\tau'_{-\infty}|$ is calculated as

$$\begin{aligned} sh(\tau'_{-\infty}/2)^2 = \{&|\operatorname{tr} \rho(l, \tau)((D_{\beta_1}^{-1})_* \delta_2^{-1} \bar{\alpha}_2)|/2 - ch(l'_1/2) ch(l'_2/2) \\ &- sh(l'_1/2) sh(l'_2/2) ch(s'_0 + t'_0)\}/2 sh(l'_1/2) sh(l'_2/2) sh(s'_0) sh(t'_0), \end{aligned}$$

where s'_0 is given by $ch(s'_0) = \{ch(l'_{-\infty}/2) ch(l'_1/2) + ch(l'_0/2)\}/sh(l'_{-\infty}/2) sh(l'_1/2)$ and t'_0 by $ch(t'_0) = \{ch(l'_{-\infty}/2) ch(l'_2/2) + ch(l'_0/2)\}/sh(l'_{-\infty}/2) sh(l'_2/2)$. We know that $(D_{\beta_1}^{-1})_* \delta_2^{-1}\bar{\alpha}_2 = \delta_2^{-1}\bar{\alpha}_2$, $l_1' = l_1$ and $l_2' = l_2$. Hence we have

$$sh(\tau'_{-\infty}/2)^2$$
$$= \{2sh(\tau_{-\infty}/2)^2 sh(s_0)sh(t_0) + ch(s_0 + t_0) - ch(s_0' + t_0')\}/2sh(s_0')sh(t_0').$$

Also we have $\tau'_{-\infty} \geq 0$ if and only if $|\mathrm{tr}\,\rho(l, \tau)((D_{\beta_1}^{-1})_* \delta_2^{-1}\alpha_1\bar{\gamma}_1^{-1}\alpha_1^{-1})| \geq |\mathrm{tr}\,\rho(l, \tau)((D_{\beta_1}^{-1})_* \delta_2^{-1}\bar{\gamma}_1^{-1})|$, which is rewritten as $|\mathrm{tr}\,\rho(l, \tau)(\delta_2^{-1}\alpha_1\beta_1^{-1}\bar{\gamma}_1^{-1}\alpha_1^{-1})| \geq |\mathrm{tr}\,\rho(l, \tau)(\delta_2^{-1}\bar{\gamma}_1^{-1}\beta_1^{-1})|$.

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