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*Asymptotic Behavior of Solutions for Large $|x|$ of Weakly Coupled Parabolic Systems with Unbounded Coefficients**)

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§ 1. Introduction.

Let E^n be the n-dimensional Euclidean space whose points x is represented by its coordinates $(x_1,\dots, x_n)$. The distance of a point x of E^n to the origin is defined by $|x|=\left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$. Every point in $D\equiv E^n\times(0, T]$ is denoted by (x, t), $x\in E^n$, $t\in(0, T]$ $(T<+\infty)$.

We say that a function $w(x, t)$ belongs to class $E_{\lambda\mu}(D, M, k)$ or shortly $E_{\lambda\mu}$ ($\lambda, \mu>0$ are constants) in D if there exist positive numbers M, k such that

$$|w(x, t)| \leqslant M\exp\{k[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}\}.$$

We say that a function $w(x, t)$ belongs to class $E_{\lambda}(D, M, k)$ or shortly E_{λ} ($\lambda\geqslant 1$ is a constant) in D if there exist positive numbers M, k such that

$$|w(x, t)| \leqslant M\exp\{k[\log(|x|^2+1)+1]^{\lambda}\}.$$

Consider a weakly coupled parabolic system of the form

$$(*)\quad F^p[u^p]\equiv \sum_{i,j=1}^{n} a_{ij}^p(x, t)\frac{\partial^2 u^p}{\partial x_i\partial x_j}+\sum_{i=1}^{n} b_i^p(x, t)\frac{\partial u^p}{\partial x_i}+\sum_{q=1}^{N} c^{pq}(x, t)u^q-\frac{\partial u^p}{\partial t}$$

$$p=1,\dots, N$$

with variable coefficients $a_{ij}^p(=a_{ji}^p)$, b_i^p, c^{pq} defined in $\bar{D}$.

In this paper, we deal with the decay of solutions of

$$(1)\qquad F^p[u^p]=0,\qquad p=1,\dots, N,$$

and the growth of solutions of

$$(2)\qquad F^p[u^p]\leqslant 0,\qquad p=1,\dots, N,$$

for large $|x|$.

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§ 2. The maximum Principle.

The following maximum principle due to Kusano-Kuroda-Chen [4] will be important in the later treatment.

Let R be an unbounded domain in E^n with closure $\bar{R}$ and the boundary ∂R.

LEMMA 1. *Suppose that the coefficients of* (*) *in* $\bar{R}\times[0, T]$ *satisfy the inequalities*

$$(3)\quad \begin{cases} 0\leqslant \sum_{i,j=1}^{n} a_{ij}^{p}(x, t)\xi_i\xi_j \leqslant K_1[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2 \\ \qquad\qquad \text{for any real vector } \xi=(\xi_1,\dots,\xi_n), \quad p=1,\dots,N, \\ |b_i^p(x, t)|\leqslant K_2(|x|^2+1)^{\frac{1}{2}}, \qquad i=1,\dots,n;\ p=1,\dots,N, \\ c^{pq}(x, t)\geqslant 0 \qquad \text{for} \quad p\neq q, \quad p, q=1,\dots,N, \end{cases}$$

$$(4)\quad \sum_{q=1}^{N} c^{pq}(x, t)\leqslant K_3[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}, \qquad p=1,\dots,N,$$

where $K_1>0$, $K_2\geqslant 0$, $K_3>0$, $\mu>0$ *and* λ *are constnats. Let* $\{u^p(x, t)\}$, $p=1,\dots,N$, *be a system of functions satisfying* $F^p[u^p]\geqslant 0$, $p=1,\dots,N$, *in* $R\times(0, T]$ *with the properties*

(i) $u^p(x, t)\leqslant 0$ *for* $(x, t)\in\{\partial R\times[0, t]\}\cup\{R\times(t=0)\}$, $p=1,\dots,N$

(ii) $u^p(x, t)\leqslant M\exp\{k[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu}\}$ *in* $R\times(0, T]$,

for some positive constants M *and* k, $p=1,\dots,N$.

Then $u^p(x, t)\leqslant 0$ *in* $\bar{R}\times[0, T]$, $p=1,\dots,N$.

By the same method, we can prove the following

LEMMA 2. *Assume that the coefficients of* (*) *in* $\bar{R}\times[0, T]$ *satisfy the inequalities*

$$(5)\quad \begin{cases} 0\leqslant \sum_{i,j=1}^{n} a_{ij}^{p}(x, t)\xi_i\xi_j \leqslant K_1[\log(|x|^2+1)+1]^{2-\lambda}(|x|^2+1)|\xi|^2 \\ \qquad\qquad \text{for any real vector } \xi=(\xi_1,\dots,\xi_n), \quad p=1,\dots,N, \\ |b_i^p(x, t)|\leqslant K_2[\log(|x|^2+1)+1](|x|^2+1)^{\frac{1}{2}}, \\ \qquad\qquad\qquad\qquad i=1,\dots,n;\ p=1,\dots,N, \\ c^{pq}(x, t)\geqslant 0 \qquad \text{for} \quad p\neq q, \quad p, q=1,\dots,N, \end{cases}$$

$$(6)\quad \sum_{q=1}^{N} c^{pq}(x, t)\leqslant K_3[\log(|x|^2+1)+1]^{\lambda}, \qquad p=1,\dots,N,$$

where $K_1>0$, $K_2\geqslant 0$, $K_3>0$ *and* $\lambda\geqslant 1$ *are constants. Let* $\{u^p(x,t)\}$, $p=1,\dots,N$, *be a system of functions satisfying* $F^p[u^p]\geqslant 0$, $p=1,\dots,N$, *in* $R\times(0,T]$ *with the properties*

(i) $u^p(x,t)\leqslant 0$ *for* $(x,t)\in\{\partial R\times[0,T]\}\cup\{R\times(t=0)\}$, $p=1,\dots,N$,

(ii) $u^p(x,t)\leqslant M\exp\{k[\log(|x|^2+1)+1]^\lambda\}$ *in* $R\times(0,T]$

for some positive constants M *and* k, $p=1,\dots,N$.

Then $$u^p(x,t)\leqslant 0 \quad in \quad \bar{R}\times[0,T], \qquad p=1,\dots,N.$$

PROOF: We introduce the auxiliary functions $v(x,t)$ and $w^p(x,t)$, $p=1,\dots,N$, defined by

$$v(x,t)=\exp\{2ke^{bt}[\log(|x|^2+1)+1]^\lambda\}$$

and

$$w^p(x,t)=u^p(x,t)-M\exp\{2ke^{bt}[\log(|x|^2+1)+1]^\lambda-k[\log(B^2+1)+1]^\lambda\},\qquad p=1,\dots,N,$$

where B is a positive number. It is possible to choose the parameter $b>0$ so large that $F^p[v]<0$ in $R\times(0,b^{-1}]$. Now the proof proceeds exactly as in the proof of Lemma 1.

REMARK 1. (i) From the proofs of Lemma 1 and Lemma 2, we see easily that R in those statements can be taken as the whole space E^n. In this case the condition (i) of Lemma 1 and Lemma 2 must be replaced by the following:

$$u^p(x,0)\leqslant 0 \qquad \text{for} \quad x\in E^n, \quad p=1,\dots,N.$$

(ii) Analogues of Lemma 1 and Lemma 2 for a single parabolic inequality have been given by one of the present authors Chen [2] (Theorem 1.1 and Theorem 1.2 respectively).

From Lemma 1, we have the following.

LEMMA 3. *Suppose that the coefficients of* $(*)$ *in* $\bar{D}$ *satisfy the condition* (3) *and* $\sum_{q=1}^{N} c^{pq}(x,t)\leqslant 0$, $p=1,\dots,N$. *Let* $\{u^p(x,t)\}$, $p=1,\dots,N$, *be a usual solution of* $F^p[u^p]=0$, $p=1,\dots,N$, *in* $\bar{D}$ *such that* $u^p(x,t)\in E_{\lambda\mu}$ *and* $|u^p(x,0)|\leqslant M_0$ *in* E^n *for a positive constant* M_0, $p=1,\dots,N$. *Then* $|u^p(x,t)|\leqslant M_0$ *in* $\bar{D}$, $p=1,\dots,N$.

PROOF: Applying Lemma 1 to $v(x,t)=-M_0\pm u^p(x,t)$, we have out lemma directly.

Similarly, we can prove the following.

LEMMA 4. *Suppose that the coefficients of* $(*)$ *in* $\bar{D}$ *satisfy the condition* (5) *and* $\sum_{q=1}^{N} c^{pq}(x,t) \leqslant 0$, $p=1,\dots,N$. *Let* $\{u^p(x,t)\}$, $p=1,\dots,N$, *be a usual solution of* $F^p[u^p]=0$ *in* $\bar{D}$ *such that* $u^p(x,t) \in E_\lambda$ *and* $|u^p(x,0)| \leqslant M_0$ *in* E^n *for a positive constant* M_0, $p=1,\dots,N$. *Then* $|u^p(x,t)| \leqslant M_0$ *in* $\bar{D}$, $p=1,\dots,N$.

§ 3. Exponential Decay of Solutions for large $|x|$.

THEOREM 1. *Suppose that the coefficients of* (1) *in* $\bar{D}$ *satisfy the conditions* (3) *and* (4). *Assume that the constants* K_1, K_2, K_3, λ *and* μ *appeared in* (3), (4) *satisfy*

$$S_1 = 4K_1K_3(\lambda+\mu)^2 - \{K_2 n(\lambda+\mu) + 2[(1-\mu)(\lambda+\mu)+\lambda]K_1\}^2 > 0, \quad \text{if } \lambda \geqslant 0,\ \mu \in (0,1];$$

$$S_1' = 4K_1K_3(\lambda+\mu)^2 - [K_2 n(\lambda+\mu) + 2K_1\lambda]^2 > 0, \quad \text{if } \lambda \geqslant 0,\ \mu > 1;$$

$$S_2 = 4K_1K_3(\lambda^2+\mu^2) - (\mu-\lambda)^2[K_2 n + 2(\mu+\lambda-1)K_1]^2 > 0, \quad \text{if } \lambda < 0,\ \mu \in (0,1];$$

$$S_2' = 4K_1K_3(\lambda^2+\mu^2) - (\mu-\lambda)^2[K_2 n + 2(1-\lambda)K_1]^2 > 0, \quad \text{if } \lambda < 0,\ \mu > 1.$$

Let $\{u^p(x,t)\}$, $p=1,\dots,N$, *be a usual solution of* (1) *in* $\bar{D}$ *such that* $u^p(x,t) \in E_{\lambda\mu}$ *and*

$$|u^p(x,0)| \leqslant M_0 \exp\{-k_0[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\}$$

in E^n *for some positive constants* M_0 *and* k_0, $p=1,\dots,N$. *Put*

$$T_0 = \begin{cases} \min\left(T, \dfrac{1}{\sqrt{S_1}}\tan^{-1}\dfrac{\sqrt{S_1}}{K_2 n(\lambda+\mu+2[(1-\mu)(\lambda+\mu)+\lambda]K_1+K_3k_0^{-1}}\right) \\ \qquad\qquad \text{if } \lambda \geqslant 0, \quad \mu \in (0,1]; \\ \min\left(T, \dfrac{1}{\sqrt{S_1'}}\tan^{-1}\dfrac{\sqrt{S_1'}}{K_2 n(\lambda+\mu)+2K_1\lambda+K_3k_0^{-1}}\right), \quad \text{if } \lambda \geqslant 0,\ \mu > 1, \end{cases}$$

or

$$T_0 = \begin{cases} \min\left(T, \dfrac{1}{\sqrt{S_2}}\tan^{-1}\dfrac{\sqrt{S_2}}{K_2 n(\mu-\lambda)+2K_1(\mu-\lambda)(\mu+\lambda-1)+K_3k_0^{-1}}\right) \\ \qquad\qquad \text{if } \lambda < 0, \quad \mu \in (0,1]; \\ \min\left(T, \dfrac{1}{\sqrt{S_2'}}\tan^{-1}\dfrac{\sqrt{S_2'}}{K_2 n(\mu-\lambda)+2K_1(1-\lambda)(\mu-\lambda)+K_3k_0^{-1}}\right), \\ \qquad\qquad \text{if } \lambda < 0, \quad \mu > 1. \end{cases}$$

Then $\lim_{|x|\to\infty} u^p(x,t) = 0$ *for any* $t \in [0, T'] \subset [0, T_0)$, $p = 1,\dots, N$.

PROOF: We only consider the case $\lambda \geqslant 0$, $\mu \in (0, 1]$. Put

$$u^p(x,t) = w^p(x,t) H_{k_0}(x,t), \qquad p = 1,\dots, N,$$

where $H_{k_0}(x,t) \equiv H = \exp\{-k_0[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu \rho^{b(k_0)t}\}$, $b(k_0) = -\{4k_0K_1(\lambda+\mu)^2 + 2nK_2(\lambda+\mu) + 4[(1-\mu)(\lambda+\mu)+\lambda]K_1 + K_3 k_0^{-1}\rho\} \times (\log\rho)^{-1}$, and ρ is a number greater than 1. Then it is obvious that $F^p[H] \leqslant 0$ in $E^n \times [0, T_{k_0}]$, $p=1,\dots, N$, where $T_{k_0} = \min(T, |b(k_0)|^{-1})$. We see that

$$\sum_{i,j=1}^n a_{ij}^p(x,t)\frac{\partial^2 w^p}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i^{*p}(x,t)\frac{\partial w^p}{\partial x_i} + \frac{F^p[H]}{H} w^p - \frac{\partial w^p}{\partial t} = 0$$

in $E^n \times [0, T_{k_0}]$, $p=1,\dots, N$. Further in $E^n \times [0, T_{k_0}]$ we have $|b_i^{*p}(x,t)| \leqslant K_2'(|x|^2+1)^{\frac{1}{2}}$ for a positive constant K_2' which is independent of t and clearly $|w^p(x,0)| \leqslant M_0$ for $x \in E^n$, $p=1,\dots, N$. Hence we conclude from Lemma 3 that $|w^p(x,t)| \leqslant M_0$, $p=1,\dots, N$. Therefore it holds that in $E^n \times [0, T_{k_0}]$

$$|u^p(x,t)| \leqslant M_0 \exp\{-k_0[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu \rho^{b(k_0)t}\}, \qquad p = 1,\dots, N.$$

If $T_{k_0} < T$, then we consider $u^p(x, T_{k_0})$ to be the initial data of $u^p(x,t)$ in $E^n \times (T_{k_0}, T)$, $p = 1,\dots, N$, and repeat the above procedure. Since

$$|u^p(x, T_{k_0})| \leqslant M_0 \exp\{-k_0\rho^{-1}[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\},$$

we get

$$|u^p(x,t)| \leqslant M_0 \exp\{-k_0\rho^{-1}[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu \rho^{b(k_0\rho^{-1})t}\}$$

in $E^n \times [T_{k_0}, T_{k_0} + T_{k_1}]$, where

$$T_{k_1} = \min(T - T_{k_0}, |b(k_0\rho^{-1})|^{-1}), \qquad p = 1,\dots, N.$$

In general, if $T_{k_0} + \cdots + T_{k_m} < T$, then by the argument used above, we can conclude that in $E^n \times [T_{k_0} + \cdots + T_{k_m}, T_{k_0} + \cdots + T_{k_m} + T_{k_{m+1}}]$

$$|u^p(x,t)| \leqslant M_0 \exp\{-k_0\rho^{-(m+1)}[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu \rho^{b(k_0\rho^{-(m+1)})t}\}$$

where

$$T_{k_{m+1}} = \min(T - (T_{k_0} + \cdots + T_{k_m}), \quad |b(k_0\rho^{-(m+1)})|^{-1}) > 0,$$
$$p = 1,\dots, N.$$

Now we suppose

$$G(\rho) = \sum_{m=0}^{\infty} |b(k_0\rho^{-m})|^{-1} = \log\rho \sum_{m=0}^{\infty} \{4k_0K_1(\lambda+\mu)^2\rho^{-m} + 2K_2n(\lambda+\mu)$$
$$+4[(1-\mu)(\lambda+\mu)+\lambda]K_1 + K_3k_0^{-1}\rho^{m+1}\}^{-1}.$$

For brevity we put $f=4k_0K_1(\lambda+\mu)^2$, $g=2K_2n(\lambda+\mu)+4[(1-\mu)(\lambda+\mu)+\lambda]K_1$, $h=K_3k_0^{-1}$. Then

$$G(\rho) = \log\rho \sum_{m=0}^{\infty} \frac{1}{f\rho^{-m-1}+g+h\rho^m}.$$

The function $(f\rho^{-m-1}+g+h\rho^s)^{-1}$ of $s\in(-\infty,\infty)$ has its maximum at $s=s_0=\frac{1}{2}\log_\rho \frac{f}{h\rho}$.

First suppose that $f>h$. Then we can find $\rho_0\ (>1)$ such that $\rho_0>\rho>1$ implies $\frac{f}{h\rho}>1$ and $4fh\rho-g^2>0$, that is $s_0>0$,. Let r be the nonnegative integer such that $r<s_0\leqslant r+1$. Then

$$G(\rho) \geqslant \log\rho \int_1^r \frac{ds}{f^{-s}+g+h^{s+1}} + \log\rho \int_{r+1}^{\infty} \frac{ds}{f^{-s}+g+h^{s+1}} = \frac{2}{\sqrt{4hf\rho-g^2}} \times$$

$$\tan^{-1}\frac{\sqrt{4hf\rho-g^2}[4hf\rho-g^2+(2h\rho^{r+1}+g)(2h\rho+g)+2h\rho(\rho^r-1)(2h\rho^{r+2}+g)]}{(2h\rho^{r+2}+g)[4hf\rho-g^2+(2h\rho^{r+1}+g)(2h\rho+g)]-(4hf\rho-g^2)2h\rho(\rho^r-1)}$$

$$\overset{df}{=} T^*(\rho).$$

In the case when $f\leqslant h$, we see that $f\leqslant h\rho$, $s_0\leqslant 0$ and that

$$G(\rho) \geqslant \log\rho \int_1^{\infty} \frac{ds}{f\rho^{-s}+g+h^{s+1}} = \frac{2}{\sqrt{4hf\rho-g^2}} \tan^{-1}\frac{\sqrt{4hf\rho-g^2}}{2h\rho+g} \overset{df}{=} T^{**}(\rho).$$

$T^*(\rho)$, $T^{**}(\rho)$ are all continuous in $[1,\infty)$. Putting

$$\tilde{T}(\rho) = \begin{cases} T^*(\rho), & (f>h) \\ T^{**}(\rho), & (f\leqslant h), \end{cases}$$

we see easily from the continuity of $\tilde{T}(\rho)$ in $[1,\infty)$ that there exist a positive integer L and a positive number $\rho\ (>1)$ such that

$$T' \leqslant \sum_{m=0}^{L} |b(k_0\rho^{-m})|^{-1}.$$

Therefore, for $k' = \max_{0\leqslant m\leqslant L} (k_0\rho^{-m+b(k_0\rho^{-m})t})$, we have

$$|u^p(x,t)| \leqslant M_0 \exp\{-k'[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\}, \qquad p=1,\dots,N,$$

at every point $(x,t)\in E^n\times[0,T']$, which proves the theorem.

Similarly, we can prove the following.

THEOREM 2. *Suppose that the coefficients of* (1) *in* $\bar{D}$ *satisfy the conditions* (5) *and* (6). *Assume that the constants* K_1, K_2, K_3 *and* λ *appeared in* (5), (6) *satisfy*

$$S = \lambda^2[4K_1K_3-(nK_2+2K_1)^2] > 0.$$

Let $\{u^p(x, t)\}$, $p=1,\dots, N$, *be a usual solution of* (1) *in* $\bar{D}$ *such tthat* $u^p(x, t)\in E_\lambda$ *and* $|u^p(x, 0)|\leqslant M_0 \exp\{-k_0[\log(|x|^2+1)+1]^\lambda\}$ *in* E^n *for some positive constants* M_0 *and* k_0, $p=1,\dots, N$. *Put*

$$T_0 = \min\left(T, \frac{1}{\sqrt{S}}\tan^{-1}\frac{\sqrt{S}}{nK_2\lambda+2K_1\lambda+K_3k_0^{-1}}\right).$$

Then for any $t\in[0, T')\subset[0, T_0)$ *there exists a positive constant* k' *such that*

$$|u^p(x, t)| \leqslant M_0\exp\{-k'[\log(|x|^2+1)+1]^\lambda\} \quad \textit{for any } x\in E^n,\ p=1,\dots, N.$$

§ 4. Unbounded growth of solutions for large $|x|$.

From Lemma 1, we have the following.

LEMMA 5. *Assume that the coefficients of* (2) *in* $\bar{D}$ *satisfy the condition* (3) and

$$\text{(7)}\qquad k_3[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu \leqslant \sum_{q=1}^{N} c^{pq}(x, t)$$
$$\leqslant K_3[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu, \qquad p=1,\dots, N,$$

where $k_3>0$, $K_3>0$, $\mu>0$ *and* λ *are constants. Let* $\{u^p(x, t)\}$, $p=1,\dots, N$, *be a usual solution of* (2) *in* $\bar{D}$ *with the properties*:

(i) $u^p(x, t) \geqslant -M\exp\{k[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\}$, $p=1,\dots, N$, *in* D *for some positive constants* M *and* k,

(ii) $u^p(x, 0) \geqslant M_0$ *in* E^n *for a positive constant* M_0, $p=1,\dots, N$.

Then it holds that

$$u^p(x, t) \geqslant M_0\exp\{k_0[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu t\},$$

$p=1,\dots, N$, *in* $\bar{D}$ *for a positive constant* k_0.

PROOF: We employ the method as described in [2]. We only prove the case $\lambda\geqslant 0$, $\mu\in(0, 1]$, because the other cases: $\lambda\geqslant 0, \mu\in[1, \infty)$; $\lambda<0, \mu\in(0, 1]$; $\lambda<0, \mu\in[1, \infty)$ can be discussed similarly. Take k_0 as such as

$$0 < k_0 \leqslant \frac{k_3}{\{4[(1-\mu)(\lambda+\mu)+\lambda]K_1+2K_2n(\lambda+\mu)\}T+1}.$$

Put

$$v(x, t) = M_0 \exp\{k_0[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu t\}.$$

Then, from (3) and (7) we see easily that

$$\frac{F^p[v]}{v} \geqslant [\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu$$

$$\times\{k_0[-4[(1-\mu)(\lambda+\mu)+\lambda]TK_1-2K_2n(\lambda+\mu)T-1]+k_3\} \geqslant 0$$

in D. Putting $w^p(x, t)=v(x, t)-u^p(x, t)$, $p=1,\dots, N$, and applying Lemma 1 to $w^p(x, t)$, we have $w^p(x, t)\leqslant 0$ in $\bar{D}$, that is, $u^p(x, t)\geqslant v(x, t)$ in $\bar{D}$, $p=1,\dots, N$, which proves the Lemma.

By the same method, we can prove

LEMMA 6. *Assume that the coefficients of* (2) *in* $\bar{D}$ *satisfy the condition* (5) *and*

$$(8) \qquad k_3[\log(|x|^2+1)+1]^\lambda \leqslant \sum_{q=1}^N c^{pq}(x, t) \leqslant K_3[\log(|x|^2+1)+1]^\lambda,$$

$p=1,\dots, N$, *where* $k_3>0$, $K_3>0$ *and* $\lambda\geqslant 1$ *are constants. Let* $\{u^p(x, t)\}$, $p=1,\dots, N$, *be a usual solution of* (2) *in* $\bar{D}$ *with the properties*:

(i) $u^p(x, t) \geqslant -M\exp\{k[\log(|x|^2+1)+1]^\lambda\}, \qquad p=1,\dots, N,$

in D *for some positive constants* M *and* k,

(ii) $u^p(x, 0) \geqslant M_0$ *in* E^n *for a positive constant* M_0, $\qquad p=1,,\dots, N.$

Then $u^p(x, t)\geqslant M_0\exp\{k_0[\log(|x|^2+1)+1]^\lambda t\}$ *in* $\bar{D}$ *for a positive constant* k_0, $p=1,\dots, N$.

THEOREM 3. *Suppose that the coefficients of* (2) *in* $\bar{D}$ *satisfy the condition* (7) *and the inequalities*

$$k_1[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2 \leqslant \sum_{i,j=1}^n a_{ij}^p(x, t)\xi_i\xi_j$$

$$\leqslant K_1[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2, \qquad \text{for all } \xi\in E^n,\ p=1,\dots, N,$$

$$|b_i^p(x, t)| \leqslant K_2(|x|^2+1)^{\frac{1}{2}}, \qquad i = 1,\dots, n;\ p = 1,\dots, N,$$

$$c^{pq}(x, t) \geqslant 0 \qquad \text{for} \quad p \neq q,\ p,\ q = 1,\dots, N,$$

where $k_1>0$, $K_1>0$, $K_2\geqslant 0$, $\mu>0$ *and* λ *are constants. Let* $\{u^p(x,t)\}$, $p=1,\dots,N$, *be a usual solution of* (2) *in* $\bar{D}$ *with the property* (i) *mentioned in* Lemma 5 *and such that*

$$u^p(x,0)\geqslant M_0\exp\{-k_0[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\} \quad in \quad E^n,$$

$p=1,\dots,N$, *for some positive constants* M_0 *and* k_0. *Assume that if* $\lambda\geqslant 0$, *then*

$$-2K_2n(\lambda+\mu)+k_3k_0^{-1}>0, \tag{9}$$

$$H_1=4k_1k_3\mu^2-K_2^2n^2(\lambda+\mu)^2>0, \tag{10}$$

or if $\lambda<0$, *then*

$$-2(\mu-\lambda)K_2n+k_3k_0^{-1}>0,$$

$$H_2=-4k_1k_3\lambda\mu-K_2^2n^2(\mu-\lambda)^2>0.$$

Put

$$T_0^*=\frac{1}{\sqrt{H_1}}\tan^{-1}\frac{\sqrt{H_1}}{-K_2n(\lambda+\mu)+k_3k_0^{-1}}<T, \qquad if \ \lambda\geqslant 0,$$

$$T_0^*=\frac{1}{\sqrt{H_2}}\tan^{-1}\frac{\sqrt{H_2}}{-K_2n(\mu-\lambda)+k_3k_0^{-1}}<T, \qquad if \ \lambda<0.$$

Then there exists a positive constant M^* *such that* $u^p(x,T^*)\geqslant M^*$. *Further if* $t\in(T_0^*,T)$, *then there exists a positive constant* k^* *such that*

$$u^p(x,t)\geqslant M^*\exp\{k^*(t-T_0^*)[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\}$$

for any $x\in E^n$, $p=1,\dots,N$.

Proof: We only prove the case $\lambda\geqslant 0$, $\mu\in(0,1]$, because other cases $\lambda\geqslant 0$, $\mu\in[1,\infty)$; $\lambda<0$, $\mu\in(0,1]$ and $\lambda<0$, $\mu\in[1,\infty)$ can be discussed analogously. Now we use the idea of [3] and put

$$v(x,t)=M_0\exp\Big\{-k_0[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\rho^{-r_0t}$$

$$-\frac{2(\lambda+\mu)(n+2\lambda)K_1k_0}{r_0\log\rho}(1-\rho^{-r_0t})-\frac{2\mu^2k_1k_0^2}{r_0\log\rho}(1-\rho^{-2r_0t})\Big\},$$

where $r_0=[4\mu^2k_1k_0\rho^{-1}-2(\lambda+\mu)nK_2+k_3k_0^{-1}](\log\rho)^{-1}$ and $\rho>1$ is a number. From (9) we see $r_0>0$. Since $\lambda\geqslant 0$, $\mu\in(0,1]$, it is easy to see that

$$\frac{F^p[v]}{v}\geqslant k_0\rho^{-r_0t}[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu$$

$$\times [4k_1 k_0 \mu^2 \rho^{-r_0 t} - 2(\lambda+\mu)K_2 n + k_3(k_0\rho^{-r_0 t})^{-1} - r_0 \log \rho].$$

If $0 \leqslant t < r_0^{-1}$, then

$$4k_1 k_0 \mu^2 \rho^{-r_0 t} - 2(\lambda+\mu)nK_2 + k_3(k_0\rho^{-r_0 t})^{-1} - r_0 \log \rho \geqslant 0.$$

Hence it follows that $F^p[v] \geqslant 0$ provided that $0 \leqslant t \leqslant r_0^{-1}$. In the following we assume $r_0^{-1} < T$. Putting $w^p(x, t) = v(x, t) - u^p(x, t)$, $p = 1, \dots, N$, we see easily $w^p(x, 0) \leqslant 0$, $F^p[w^p] \geqslant 0$ and $w^p(x, t) \leqslant M' \exp\{k[\log(|x|^2+1)+1]^\lambda (|x|^2+1)^\mu\}$ in $E^n \times [0, r_0^{-1}]$ for a suitable positive constant M', $p = 1, \dots, N$. Therefore Lemma 1 implies $w^p(x, t) \leqslant 0$, that is, $u^p(x, t) \geqslant v(x, t)$ in $E^n \times [0, r_0^{-1}]$, $p = 1, \dots, N$. Hence

$$(11) \qquad u^p(x, r_0^{-1}) \geqslant v(x, r_0^{-1}) = M_0 \exp\Big\{ -k_0 \rho^{-1} [\log(|x|^2+1)+1]^\lambda (|x|^2+1)^\mu - \frac{2(\lambda+\mu)(n+2\lambda)K_1 k_0}{r_0 \log \rho}(1-\rho^{-1}) - \frac{2\mu^2 k_1 k_0^2}{r_0 \log \rho}(1-\rho^{-2}) \Big\}, \quad p = 1, \dots, N.$$

If $r_0^{-1} < T$, then we consider $t = r_0^{-1}$ to be the initial time and (11) to be the initial data of $u^p(x, t)$. Repeating the above procedure, we obtain

$$u^p(x, t) \geqslant M_1 \exp\Big\{ -k_0 \rho^{-1} [\log(|x|^2+1)+1]^\lambda (|x|^2+1)^\mu \rho^{-r_1(t-r_0^{-1})} - \frac{2(\lambda+\mu)(n+2\lambda)K_1 k_0 \rho^{-1}}{r_1 \log \rho}(1-\rho^{-r_1(t-r_0^{-1})}) - \frac{2\mu^2 k_1 k_0^2 \rho^{-2}}{r_1 \log \rho}(1-\rho^{-2r_1(t-r_0^{-1})}) \Big\}, \qquad p = 1, \dots, N,$$

in $E^n \times [r_0^{-1}, r_0^{-1} + r_1^{-1}]$, where

$$r_1 = (4\mu^2 k_1 k_0 \rho^{-2} - 2(\lambda+\mu)K_2 n + k_3 k_0^{-1})(\log \rho)^{-1},$$

$$M_1 = M_0 \exp\Big\{ -\frac{2(\lambda+\mu)(n+2\lambda)K_1 k_0}{r_0 \log \rho}(1-\rho^{-1}) - \frac{2\mu^2 k_1 k_0^2}{r_0 \log \rho}(1-\rho^{-2}) \Big\}$$

provided that $r_0^{-1} + r_1^{-1} < T$. Hence

$$u^p(x, r_0^{-1} + r_1^{-1}) \geqslant M_0 \exp\Big\{ \frac{-2(\lambda+\mu)(n+2\lambda)K_1 k_0}{\log \rho}(1-\rho^{-1})(r_0^{-1} + \rho^{-1} r_1^{-1}) - \frac{2\mu^2 k_1 k_0^2}{\log \rho}(1-\rho^{-2})(r_0^{-1} + \rho^{-2} r_1^{-1}) \Big\} \times$$

$$\exp\{ -k_0 \rho^{-2} [\log(|x|^2+1)+1]^\lambda (|x|^2+1)^\mu \},$$

$p=1,\ldots, N$.

In general, if $r_0^{-1}+\cdots+r_j^{-1}<T$, then it holds that

$$(12)\qquad u^p(x, r_0^{-1}+\cdots+r_j^{-1}) \geqslant M_0 \exp\Big\{-\frac{2(\lambda+\mu)(n+2\lambda)K_1k_0}{\log\rho}(1-\rho^{-1})$$

$$\times(r_0^{-1}+\rho^{-1}r_1^{-1}+\cdots+\rho^{-j}r_j^{-1})$$

$$-\frac{2\mu^2k_1k_0^2}{\log\rho}(1-\rho^{-2})(r_0^{-1}+\rho^{-2}r_1^{-1}+\cdots+\rho^{-2j}r_j^{-1})\Big\}$$

$$\times\exp\{-k_0\rho^{-j-1}[\log(|x|^2+1)+1]^\lambda(|x|^2+1)^\mu\},\qquad p=1,\ldots,N,$$

where $r_j=(4\mu^2k_1k_0\rho^{-j-1}-2(\lambda+\mu)K_2n+k_3k_0^{-1}\rho^j)(\log\rho)^{-1}$.

Now suppose

$$G(\rho)=\sum_{j=0}^{\infty} r_j^{-1} < T.$$

First we estimate the sum $G(\rho)$ from above and below. For brevity we put $f=4\mu^2k_1k_0$, $g=-2(\lambda+\mu)K_2n$, $h=k_3k_0^{-1}$. Then

$$G(\rho)=\log\rho\sum_{j=0}^{\infty}(f\rho^{-j-1}+g+h\rho^j)^{-1}.$$

The function $(f\rho^{-s-1}+g+h\rho^s)^{-1}$ of $s\in(-\infty,\infty)$ has its maximum at $s=s_0=\frac{1}{2}\log_\rho\frac{f}{h\rho}$. From (10) we see that

$$(13)\qquad 4hf-g^2=4h_1>0.$$

There are two cases: (i) $f>h$ and (ii) $f\leqslant h$.

In case (i), we can find a number ρ_0 (>1) such that $\rho_0>\rho>1$ implies $f>h\rho$ and such that $4fh\rho^{-1}-g^2>0$. For such a number ρ it is evident that $s_0>0$. Let d be the non-negative integer such that $d<s_0\leqslant d+1$. Then

$$G(\rho)\geqslant\log\rho\left[\int_s^d(f\rho^{-s-1}+g+h\rho^s)^{-1}ds+\int_{d+1}^{\infty}(f\rho^{-s-1}+g+h\rho^s)^{-1}ds\right]$$

$$=\frac{2}{\sqrt{4fh\rho^{-1}-g^2}}\times$$

$$\tan^{-1}\frac{\sqrt{4fh\rho^{-1}-g^2}\,[4fh\rho^{-1}-g^2+(2h\rho^d+g)(2h+g)+2h(\rho^d-1)(2h\rho^{d+1}+g)]}{(2h\rho^{d+1}+g)[4fh\rho^{-1}-g^2+(2h\rho^d+g)(2h+g)]-(4fh\rho^{-1}-g^2)2h(\rho^d-1)}$$

$$\overset{df}{=}T_1(\rho).$$

It is easy to see that

$$G(\rho) \leqslant T_1(\rho)+r_{d+1}^{-1}+r_d^{-1}$$
$$= T_1(\rho)+\log\rho[f\rho^{-d-1}+g+h\rho^d)^{-1}+(f\rho^{-d-2}+g+h\rho^{d+1})^{-1}],$$
$$(1<\rho<\rho_0).$$

In the case (ii), it is obvious that $s_0<0$ for any $\rho>1$. As in the case (i), there is a ρ_0 (>1) such that $4fh\rho^{-1}-g^2>0$ for any ρ satisfying $\rho_0>\rho>1$. So for such a ρ we get

$$G(\rho) \geqslant \log\rho\int_0^\infty (f\rho^{-s-1}+g+h\rho^s)^{-1}\,ds$$
$$= \frac{2}{\sqrt{4fh\rho^{-1}-g^2}}\tan^{-1}\frac{\sqrt{4fh\rho^{-1}-g^2}}{2h+g} \overset{df}{=} T_2(\rho).$$

We see easily that

$$G(\rho) \leqslant T_2(\rho)+(f\rho^{-1}+g+h)^{-1}\log\rho, \qquad (1<\rho<\rho_0).$$

Therefore, in both cases (i) and (ii), from the assumption (9), we have

$$\lim_{\rho\to 1} G(\rho) = \frac{2}{\sqrt{4fh-g^2}}\tan^{-1}\frac{\sqrt{4fh-g^2}}{2h+g} = T_0^*. \tag{14}$$

It is easy to see from (9) that

$$\sum_{j=0}^\infty \rho^{-j}r_j^{-1} = \log\rho\sum_{j=0}^\infty \frac{\rho^{-j}}{4\mu^2k_1k_0\rho^{-j-1}-2(\lambda+\mu)K_2n+k_3k_0^{-1}\rho^j} \tag{15}$$
$$\leqslant \log\rho\sum_{j=0}^\infty \rho^{-j}\frac{1}{-2(\lambda+\mu)K_2n+k_3k_0^{-1}}$$
$$= \frac{1}{-2(\lambda+\mu)K_2n+k_3k_0^{-1}}\,\frac{\log\rho}{1-\rho^{-1}}.$$

By the same reasoning as above, it follows that

$$\sum_{j=0}^\infty \rho^{-2j}r_j^{-1} \leqslant \frac{1}{-2(\lambda+\mu)K_2n+k_3k_0^{-1}}\,\frac{\log\rho}{1-\rho^{-2}}. \tag{16}$$

From (14), for any given positive number ε, we can find ρ_0 (>1) such that if $\rho_0>\rho>1$, then $u^p(x, T_0^*)>u^p(x, G(\rho))-\frac{1}{2}\varepsilon$, $p=1,\dots, N$. On the other hand, there exists a positive integer N_0 such that $L\geqslant N_0$ implies $u^p(x, G(\rho))> u^p(x, \sum_{j=0}^L r_j^{-1})-\frac{1}{2}\varepsilon$, $p=1,\dots, N$. Therefore it holds that $u^p(x, T_0)>u^p(x, \sum_{j=0}^L r_j^{-1})-\varepsilon$, $p=1,\dots, N$. From (12), (15) and (16), we get

$$u^p(x, T_0^*) > M_0 \exp\left[-\frac{2(\lambda+\mu)(n+2\lambda)K_1 k_0 + 2\mu^2 k_1 k_0^2}{-2(\lambda+\mu)K_2 n + k_3 k_0^{-1}} \right]$$

$$\times \exp\{-k_0 \rho^{-L-1}[\log(|x|^2+1)+1]^\lambda (|x|^2+1)^\mu\} - \varepsilon \qquad p=1,\ldots, N.$$

We fix $x \in E^n$ arbitrarily. Letting L tend to infinity and ε to zero, we have

$$u^p(x, T_0^*) \geqslant M_0 \exp\left\{\frac{-2(\lambda+\mu)(n+2\lambda)K_1 k_0 - 2\mu^2 k_1 k_0^2}{-2(\lambda+\mu)K_2 n + k_3 k_0^{-1}}\right\} \overset{df}{=} M^*,$$

$$p = 1,\ldots, N.$$

For this M^*, it suffices from Lemma 5, to show the existence of a positive constant k^* such that

$$u^p(x, t) \geqslant M^* \exp\{k^*(t-T_0^*)[\log(|x|^2+1)+1]^\lambda (|x|^2+1)^\mu\}$$

for $(x, t) \in E^n \times (T_0^*, T)$, $p=1,\ldots, N$.

By the same method, we can prove

THEOREM 4. *Suppose that the coefficients of* (2) *in* $\bar{D}$ *satisfy the condition* (8) *and the inequalities*

$$k_1[\log(|x|^2+1)+1]^{2-\lambda}(|x|^2+1)|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}^p(x, t)\xi_i \xi_j$$

$$\leqslant K_1[\log(|x|^2+1)+1]^{2-\lambda}(|x|^2+1)|\xi|^2, \qquad \textit{for} \quad \xi \in E^n,\ p = 1,\ldots, N,$$

$$|b_i^p(x, t)| \leqslant K_2[\log(|x|^2+1)+1](|x|^2+1)^{\frac{1}{2}}, \qquad i = 1,\ldots, n;\ p = 1,\ldots, N,$$

$$c^{pq}(x, t) \geqslant 0 \qquad \textit{for} \quad p \neq q,\ p, q = 1,\ldots, N,$$

where $k_1 > 0$, $K_1 > 0$, $K_2 \geqslant 0$ *and* $\lambda \geqslant 1$ *are constants. Let* $\{u^p(x, t)\}$, $p=1,\ldots, N$, *be a usual solution of* (2) *with the property* (i) *mentioned in* Lemma 6 *and such that*

$$u^p(x, 0) \geqslant M_0 \exp\{-k_0[\log(|x|^2+1)+1]^\lambda\} \qquad \textit{in} \quad E^n,\ p = 1,\ldots, N,$$

for some positive constants M_0 *and* k_0. *Assume that the inequalities* $-2n(K_1+K_2)\lambda + k_3 k_0^{-1} > 0$ *and* $4k_1 k_3 - (K_1+K_2)^2 n^2 > 0$ *hold. Put*

$$T^* = \frac{1}{\lambda\sqrt{4k_1 k_3 - (K_1+K_2)^2 n^2}} \tan^{-1} \frac{\lambda\sqrt{4k_1 k_3 - (K_1+K_2)^2 n^2}}{-n(K_1+K_2)\lambda + k_3 k_0^{-1}} < T.$$

Then there exists a positive constant M^* *such that* $u^p(x, T_0^*) \geqslant M^*$. *Further if* $t \in (T_0^*, T)$, *then there exists a positive constant* k^* *such that*

$$u^p(x, t) \geqslant M^* \exp\{k^*(t-T_0^*)[\log(|x|^2+1)+1]^\lambda\}$$

for $x \in E^n$, $p=1,\ldots, N$.

REMARK 2. In the case $\lambda=0$, $N=1$, Theorem 1 coincides with a result stated in [1].

REMARK 3. In the case $\lambda=0$, $N=1$, Theorem [3] is a special case of our Theorem 3.

REMARK 4. If $N=1$, then Theorem 4.1, 4.2, 4.5, 4.6 of [2] are special cases of our Theorem 1, 3, 2, 4 respectively.

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