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Matlis duality and the width of a module

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Introduction

The aim of this paper is to study the Matlis duality and some related topics concerning artinian modules over a commutative noetherian ring.

In § 1, we give some results on the Matlis duality. First, we generalize the Matlis duality which is known for noetherian local rings to the case of noetherian semi-local rings. Secondly, we examine the problem on the self-duality. As is well-known, a finite dimensional vector space over a field k (resp. a finite abelian group) is isomorphic to its Matlis dual, i. e. its k-dual (resp. its character group). We determine the class of noetherian rings for which the self-duality holds with respect to the Matlis duality. It turns out that, in case of domains, it characterizes the class of rings of the above-mentioned type.

§ 2 is preparatory. We prove some properties of attached primes, the notion of which has been recently introduced by I. G. Macdonald and R. Y. Sharp.

In § 3, we define *coregular sequences*, the *width* of a module and the *cograde* of a module. These are dual notions to those of regular sequences, the depth of a module and the grade of a module respectively. The first two notions have been already introduced (in different terminologies) by E. Matlis (cf. [5]). We investigate, by using the results of § 2, some properties of these notions. Especially, we characterize the cograde (resp. the width) by the vanishing of Tor modules, and the relationships between the cograde and the grade (resp. the width and the depth) with respect to the Matlis duality are established. Finally, we calculate the width of certain local cohomology modules.

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§1. Matlis duality

1. Throughout this section, we denote by A a commutative noetherian ring with unit. Let E_A be the module $\oplus E_A(A/\mathfrak{m})$, where $\mathfrak{m}$ runs over the set of maximal ideals of A. Then E_A is an injective cogenerator for A; namely, (a) E_A is injective and (b) any A-module can be embedded in a product of E_A (cf. Sharpe, Vámos [7] Chap. 2). Let $\mathscr{C}$ denote the category of A-modules of finite length.

PROPOSITION 1.1. *Suppose that D is a contravariant, left-exact, A-linear* *functor from $\mathscr{C}$ into itself. Then the following statements are equivalent*:

(1) *$DD(M)$ is canonically isomorphic to M for any $M \in \mathscr{C}$.*

(2) *D is exact and $D(A/\mathfrak{m})$ is isomorphic to $A/\mathfrak{m}$ for each maximal ideal $\mathfrak{m}$ of A.*

(3) *D is represented by E_A i.e. $D(M)$ is functorially isomorphic to* $\mathrm{Hom}_A(M, E_A)$ *for each $M \in \mathscr{C}$.*

The arguments in A. Grothendieck [1], §4, are applicable to this case; therefore we omit the proof.

DEFINITION 1.2. *We call a functor D (unique up to a functor isomorphism) satisfying the equivalent conditions of* Prop. 1.1. *the dualizing functor for A.*

For example, if A is an equi-codimensional Gorenstein ring with $\dim(A) = n$, then $D(M) = \mathrm{Ext}_A^n(M, A)$ is the dualizing functor for A. (A is said to be equi-codimensional if $\dim(A_\mathfrak{m})$ is constant for each maximal ideal $\mathfrak{m}$ of A.

2. In this paragraph, we show that the usual Matlis duality for noetherian local rings can be generalized to the case of noetherian semi-local rings. The following lemma is due to R. Y. Sharp (cf. [6]).

LEMMA 1.3. *Let M, N and E be A-modules; suppose that M is finitely generated and E is injective. Then the following (functorial) isomorphism holds*: $M \otimes_A \mathrm{Hom}_A(N, E) \cong \mathrm{Hom}_A(\mathrm{Hom}_A(M, N), E)$.

Let M be an A-module. We call the module $D(M) = \mathrm{Hom}_A(M, E_A)$ the *Matlis dual* of M.

COROLLARY 1.4. *Let M and N be A-modules; suppose that M is finitely generated. Then the following (functorial) isomorphism holds*: $M \otimes_A D(N) \cong D(\mathrm{Hom}_A(M, N))$.

COROLLARY 1.5. *Assumptions being the same as in* Lemma 1.3., *we have* $\mathrm{Tor}_n^A(M, \mathrm{Hom}_A(N, E)) \cong \mathrm{Hom}_A(\mathrm{Ext}_A^n(M, N), E)$ *(functorially). In particular,* $\mathrm{Tor}_n^A(M, D(N)) \cong D(\mathrm{Ext}_A^n(M, N))$ *(functorially).*

PROOF. Let $P. = (P_i)$ be a resolution of M by finite free A-modules. Then $P. \otimes_A \mathrm{Hom}_A(N, E)$ is isomorphic to $\mathrm{Hom}_A(\mathrm{Hom}_A(P., N), E)$ as complexes. Therefore, $\mathrm{Tor}_n^A(M, \mathrm{Hom}_A(N, E)) = H_n(P. \otimes_A \mathrm{Hom}_A(N, E)) \cong H_n(\mathrm{Hom}_A(\mathrm{Hom}_A(P., N), E)) \cong \mathrm{Hom}_A(H^n(\mathrm{Hom}_A(P., N)), E) = \mathrm{Hom}_A(\mathrm{Ext}_A^n(M, N), E)$.

In the rest of this paragraph, we denote by A a noetherian semi-local ring and by $\mathrm{Mod}^F(A)$ (resp. $\mathrm{Mod}^C(A)$) the category of finitely generated (resp. artinian) A-modules.

THEOREM 1.6. *(Matlis duality)* (1) $\mathrm{End}_A(E_A)$ *is isomorphic (as A-algebra) to $\hat{A}$, the completion of A with respect to its Jacobson radical.*

(2) *If M is finitely generated A-module, then $D(M)$ is an artinian A-module.*

(3) *Assume that A is complete. If M is an artinian A-module, then $D(M)$ is finitely generated A-module.*

(4) *If M is finitely generated A-module, then $DD(M)$ is isomorphic to $\hat{M} = M\otimes_A \hat{A}$.*

(5) *If M is an artinian A-module, then $DD(M)$ is isomorphic to M.*

(6) *Assume that A is complete. Then D is an exact, A-linear anti-equivalence from* $\mathrm{Mod}^F(A)$ *to* $\mathrm{Mod}^C(A)$, *namely, D is a contravariant, exact, A-linear functor which is faithful, full and representative.*

(7) $\mathrm{Mod}^C(A)$ *and* $\mathrm{Mod}^F(\hat{A})$ *are equivalent to each other.*

(8) *For any A-module M,* $\mathrm{ann}_A(M)=\mathrm{ann}_A(D(M))$. *In particular,* $\dim_A(M)=\dim_A(D(M))$.

PROOF. (1) Let $\mathfrak{m}_1,\dots,\mathfrak{m}_n$ be the maximal ideals of A. Then $\mathrm{Hom}_A(E_A(A/\mathfrak{m}_i), E_A(A/\mathfrak{m}_j))=0$ for $i\neq j$, and $\mathrm{End}_A(E_A(A/\mathfrak{m}_i))\cong\hat{A}_{\mathfrak{m}_i}$ for any i, where $\hat{A}_{\mathfrak{m}_i}$ stands for the completion of $A_{\mathfrak{m}_i}$ with respect to its maximal ideal (cf. Matlis [4] Th. (3.7)). Hence we have $\mathrm{End}_A(E_A)\cong\prod_{i=1}^n \mathrm{End}_A(E_A(A/\mathfrak{m}_i))=\hat{A}_{\mathfrak{m}_1}\times\cdots\times\hat{A}_{\mathfrak{m}_n}=\hat{A}$.

(2) If $A^r\to M\to 0$ is exact, then $0\to D(M)\to D(A^r)=E_A^r$ is exact. Since E_A is artinian, $D(M)$ is also artinian.

(3) If $0\to M\to E_A^r$ is exact, then $A^r=D(E_A^r)\to D(M)\to 0$ is exact, showing that $D(M)$ is finitely generated.

(4) $DD(M)=\mathrm{Hom}_A(\mathrm{Hom}_A(M, E_A), E_A)\cong M\otimes_A \mathrm{Hom}_A(E_A, E_A)\cong M\otimes_A\hat{A}$.

(5) If $0\to M\to E_A^r\to E_A^s$ is exact, then the isomorphism $DD(M)\cong M$ follows from the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & E_A^r & \longrightarrow & E_A^s \\ & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ 0 & \longrightarrow & DD(M) & \longrightarrow & DD(E_A^r) & \longrightarrow & DD(E_A^s). \end{array}$$

(6) It suffices to show that D is faithful and full. Let M, N be finitely generated A-modules. Then $\mathrm{Hom}_A(D(N), D(M))=\mathrm{Hom}_A(\mathrm{Hom}_A(N, E_A), \mathrm{Hom}_A(M, E_A))\cong\mathrm{Hom}_A(\mathrm{Hom}_A(N, E_A)\otimes_A M, E_A)\cong\mathrm{Hom}_A(\mathrm{Hom}_A(M, N), E_A), E_A)\cong$ $\mathrm{Hom}_A(M, N)$ (by (4)).

(7) $\mathrm{Mod}^C(A)$ is equivalent to $\mathrm{Mod}^C(\hat{A})$ and $\mathrm{Mod}^C(\hat{A})$ is anti-equivalent to $\mathrm{Mod}^F(\hat{A})$.

(8) $\mathrm{ann}_A D(M)\supset\mathrm{ann}_A(M)$ is evident. Conversely, let a be an element of $\mathrm{ann}_A D(M)$ and suppose $0\to M\xrightarrow{f} E_A^I$ is exact. Then $f(ax)=(f_i(ax))_{i\in I}=((af_i)$

$(x))_{i\in I}=0$ for any $x\in M$. Hence $ax=0$ for any $x\in M$.

Let R be any (not necessarily commutative) ring. Then it is known that each finitely generated R-module has the projective cover if and only if

(1) $R/Rad(R)$ is semi-simple,

(2) Any idempotent element of $R/Rad(R)$ can be lifted to an idempotent element of R.

When A is a noetherian complete semi-local ring (e.g. an artinian ring), it is easily verified, by Th. 1.6., that for any finitely generated A-module M, the canonical homomorphism $P=D(E_A(D(M)))\to M$ is the projective cover of M.

3. If M is an A-module of finite length, then so is its Matlis dual $D(M)$, and both are of the same length. But, in general, $D(M)$ and M are not necessarily isomorphic. In this paragraph, we determine the class of noetherian rings for which the self-duality holds with respect to Matlis duality.

THEOREM 1.7. *For a noetherian ring A, the following statements are equivalent*:

(a) *Any A-module of finite length is isomorphic to its Matlis dual.*

(b) *A is the product of a finite number of artinian local principal ideal rings and a finite number of Dedekind domains. (In particular, if A is a domain, A is a field or a Dedekind domain.)*

PROOF. (a)$\Rightarrow$(b): We prove this in several steps.

(1) For any ideal $\mathfrak{a}$ of A with $\dim(A/\mathfrak{a})=0$, $A/\mathfrak{a}$ is a Gorenstein ring. In fact, by the primary decomposition and the Chinese Remainder Theorem, we may assume that $\mathfrak{a}$ is $\mathfrak{m}$-primary for some maximal ideal $\mathfrak{m}$ of A. From the hypothesis, $D(A/\mathfrak{a})$ is isomorphic to $A/\mathfrak{a}$. On the other hand,

$$D(A/\mathfrak{a}) = \mathrm{Hom}_A(A/\mathfrak{a}, E_A) = \mathrm{Hom}_A(A/\mathfrak{a}, E_A(A/\mathfrak{m})) \cong E_{A/\mathfrak{a}}(A/\mathfrak{m})$$

Thus, $A/\mathfrak{a}$ is isomorphic to $E_{A/\mathfrak{a}}(A/\mathfrak{m})$, and this implies that $A/\mathfrak{a}$ is a Gorenstein local ring.

(2) $\dim(A)\leqq 1$ and for any maximal ideal $\mathfrak{m}$ of A of height one, $A_\mathfrak{m}$ is a regular local ring (i.e. discrete valuation ring). In fact, let $\mathfrak{m}$ be a maximal ideal of A. From (1), $A/\mathfrak{m}^2$ is a Gorenstein local ring. Therefore,

$$A/\mathfrak{m} \cong \mathrm{Hom}_A(A/\mathfrak{m}, A/\mathfrak{m}^2) \cong (\mathfrak{m}^2 : \mathfrak{m})/\mathfrak{m}^2 \supset \mathfrak{m}/\mathfrak{m}^2$$

Hence, we get $1\geqq \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)\geqq \mathrm{ht}(\mathfrak{m})$. This shows that $\dim(A)\leqq 1$ and for any maximal ideal $\mathfrak{m}$ of height one, $A_\mathfrak{m}$ is a regular local ring.

(3) A is isomorphic to $B\times C$, where B is an artinian ring and C is an equicodimensional regular ring of dimension one, with the proviso that B or C may be a zero-ring. This follows from (2) and the following fact due to S. Kochman:

Any noetherian ring is the direct product of a zero-dimensional ring and a ring in which every maximal ideal has height at least one (cf. [9] 3–3, Ex. 15).

(4) The required statement follows from (3) and the following facts. The artinian ring B is the direct product of a finite number of artinian local rings B_i. Since $B_i/\mathfrak{a}$ is Gorenstein for any ideal $\mathfrak{a}$ of B_i, B_i is a principal ideal ring. (cf. Sharpe, Vámos [7] 6.1. Lemma 6.6.) The regular ring C is the direct product of a finite number of regular domains C_j. Since $\dim(C_j)=1$, C_j is a Dedekind domain.

(b)$\Rightarrow$(a): It suffices to prove the statement when A is an artinian local principal ideal ring or a Dedekind domain. Such a ring A has the following properties:

(1) For any ideal $\mathfrak{a}$ of A, $A/\mathfrak{a}$ is a Gorenstein ring.

(2) Any A-module of finite length is the product of a finite number of cyclic modules. The conclusion follows immediately from these properties.

We give an application of the above theorem. When A is a domain it gives a characterization of Dedekind domains.

THEOREM 1.8. *For a noetherian ring A, the following statements are equivalent*:

(1) *A is an equi-codimensional regular ring of dimension one.*

(2) *For any A-modules M and N of finite length,* $\mathrm{Ext}_A^1(M, N)$, $\mathrm{Ext}_A^1(N, M)$, $\mathrm{Tor}_1^A(M, N)$, $\mathrm{Hom}_A(M, N)$, $\mathrm{Hom}_A(N, M)$ *and* $M\otimes_A N$ *are all isomorphic to one another.*

(3) *For any A-modules M and N of finite length,* $\mathrm{Ext}_A^1(M, N)$ *and* $\mathrm{Hom}_A(M, N)$ *are of the same length.*

(4) *For any A-modules M and N of finite length,* $\mathrm{Tor}_1^A(M, N)$ *and* $M\otimes_A N$ *are of the same length.*

PROOF. (1)$\Rightarrow$(2): First, for any non-negative integer n, we have $\mathrm{Tor}_n^A(M, N)=\mathrm{Tor}_n^A(M, D(N))=D(\mathrm{Ext}_A^n(M, N))=\mathrm{Ext}_A^n(M, N)$. Hence, it suffices to show that $\mathrm{Tor}_1^A(M, N)$ is isomorphic to $M\otimes_A N$. We can assume that A is a Dedekind domain. Since $\mathrm{Tor}_1^A(A/\mathfrak{a}, A/\mathfrak{b})=(\mathfrak{a}\cap\mathfrak{b})/\mathfrak{a}\mathfrak{b}$ for any ideals $\mathfrak{a}, \mathfrak{b}$ of A, we have $\mathrm{Tor}_1^A(A/\mathfrak{m}^m, A/\mathfrak{m}^n)=(\mathfrak{m}^m\cap\mathfrak{m}^n)/\mathfrak{m}^{m+n}=A/\mathfrak{m}^m\otimes_A A/\mathfrak{m}^n$ for any maximal ideal $\mathfrak{m}$ and positive integers m, n. This implies the required fact.

(2)$\Rightarrow$(3) and (3)$\Rightarrow$(4) are clear.

(3)$\Rightarrow$(1): It suffices to show that $A_\mathfrak{m}$ is a regular local ring of dimension one for any maximal ideal $\mathfrak{m}$ of A. Put $k=A/\mathfrak{m}$. Fix a positive integer n. Then, from the exact sequence $0\to\mathfrak{m}^n\to A\to A/\mathfrak{m}^n\to 0$, the following sequence is exact:

$$0 \longrightarrow \mathrm{Hom}_A(A/\mathfrak{m}^n, k) \longrightarrow \mathrm{Hom}_A(A, k) \longrightarrow \mathrm{Hom}_A(\mathfrak{m}^n, k)$$

$$\longrightarrow \mathrm{Ext}_A^1(A/\mathfrak{m}^n, k) \longrightarrow 0$$

Since $\mathrm{Ext}_A^1(A/\mathfrak{m}^n, k)$ and $\mathrm{Hom}_A(A/\mathfrak{m}^n, k)$ are of the same length by the assumption, we get $\dim_k \mathrm{Hom}_A(\mathfrak{m}^n, k)=1$. In particular, $\mathrm{Hom}_A(\mathfrak{m}, k)=k$. If $\dim_k \mathfrak{m}/\mathfrak{m}^2=m$, then $k^m=\mathrm{Hom}_A(\mathfrak{m}/\mathfrak{m}^2, k)\subset\mathrm{Hom}_A(\mathfrak{m}, k)=k$. Hence, $\dim A_{\mathfrak{m}}\leqq m\leqq 1$. Suppose $\dim A_{\mathfrak{m}}=0$. Then $A_{\mathfrak{m}}$ is artinian and $\mathfrak{m}_{\mathfrak{m}}^l=0$ for a sufficiently large l. But this contradicts the fact that $\mathrm{Hom}_{A_{\mathfrak{m}}}(\mathfrak{m}_{\mathfrak{m}}^l, k)=k$. Therefore, $\dim_k(\mathfrak{m}/\mathfrak{m}^2)=\dim(A_{\mathfrak{m}})=1$, namely, $A_{\mathfrak{m}}$ is a discrete valuation ring.

The proof of (4)⇒(1) is similar to that of (3)⇒(1).

§2. Attached primes

In this section, we denote by A a commutative ring with unit. Let M be an A-module.

DEFINITION 2.1. *We say a prime ideal $\mathfrak{p}$ of A is an attached prime of M, if there exists a submodule N of M such that $\mathfrak{p}=\mathrm{ann}_A(M/N)$. We denote by $\mathrm{Att}_A(M)$ the set of attached primes of M.*

When M is *representable* in the sense of [2] (e.g. artinian), our definition of $\mathrm{Att}_A(M)$ coincides with that of Macdonald, Sharp [2], [3].

For example, if $\mathfrak{a}$ is an ideal of A, then $\mathrm{Att}_A(A/\mathfrak{a})$ is the set $V(\mathfrak{a})=\{\mathfrak{p}\in\mathrm{Spec}(A)| \mathfrak{p}\supset\mathfrak{a}\}$.

PROPOSITION 2.2. *Let $\mathfrak{p}$ be a maximal element of the set $\{\mathrm{ann}_A(M/N)|N$ is a submodule of M, $N\neq M\}$. Then $\mathfrak{p}\in\mathrm{Att}_A(M)$.*

PROOF. We prove that $\mathfrak{p}$ is a prime. Let $\mathfrak{p}=\mathrm{ann}_A(M/N_0)$ and suppose $ab\in\mathfrak{p}$, a, $b\notin\mathfrak{p}$. If we put $N_1=N_0+aM$, then $N_1\neq M$. In fact, if $N_0+aM=M$, then $bM=bN_0+baM\subset bN_0+N_0=N_0$. This implies $b\in\mathrm{ann}_A(M/N_0)=\mathfrak{p}$, contrary to the assumption. Since $\mathrm{ann}_A(M/N_0)\subset\mathrm{ann}_A(M/N_1)$, we get $\mathrm{ann}_A(M/N_0)=\mathrm{ann}_A(M/N_1)$ by the maximality of $\mathrm{ann}_A(M/N_0)$. But, in this case, $aM\subset aM+N_0=N_1$. Therefore, we have $a\in\mathrm{ann}_A(M/N_1)=\mathfrak{p}$, a contradiction.

COROLLARY 2.3. *Suppose that A is noetherian. Then $\mathrm{Att}_A(M)=\phi$ if and only if $M=0$.*

DEFINITION 2.4. *We say that an element a of A is M-coregular if $M=aM$. We denote by $W_A(M)$ the set $\{a\in A|M\neq aM\}$.*

PROPOSITION 2.5. *If M is finitely generated (e.g. noetherian), then any M-coregular element is M-regular. If M is artinian, then any M-regular element is M-coregular.*

PROOF. The statements follow from the following more general fact: Let M be an A-module and $f\in\mathrm{End}_A(M)$. (1) If M is finitely generated and f is surjective, then f is an isomorphism. (2) If M is artinian and f is injective, then f is an isomorphism. (cf. [10] Prop. 1.1.)

PROPOSITION 2.6. *We assume that A is noetherian. Then* $W_A(M) = \cup \{\mathfrak{p} | \mathfrak{p} \in \mathrm{Att}_A(M)\}$.

PROOF. Suppose a is an element of $W_A(M)$. Then, by Cor. 2.3., we can take an element $\mathfrak{p}$ of $\mathrm{Att}_A(M/aM)$. Write $\mathfrak{p} = \mathrm{ann}_A(M'/N')$, where $N' = N/aM$ is a submodule of $M' = M/aM$. Since $aM \subset N$ and $M'/N' \cong M/N$, we get $a \in \mathfrak{p} = \mathrm{ann}_A(M/N) \in \mathrm{Att}_A(M)$. Conversely, suppose $a \in \mathfrak{p} \in \mathrm{Att}_A(M)$ and $\mathfrak{p} = \mathrm{ann}_A(M/N)$. Then $aM \subset N \subsetneq M$, showing that $a \in W_A(M)$.

We recall some results of I. G. Macdonald and R. Y. Sharp. (cf. [2], [3]) We assume that A is noetherian. For an A-module M, we denote by $\mathfrak{N}(M)$ (the *nilradical* of M) the ideal $\sqrt{\mathrm{ann}_A(M)}$ of A. M is said to be *secondary* if the multiplication of M by each element of A is surjective or nilpotent, in other words, $W_A(M) \subset \mathfrak{N}(M)$ (i.e. $W_A(M) = \mathfrak{N}(M)$). In this case, $\mathfrak{p} = \mathfrak{N}(M)$ is a prime ideal of A and M is called $\mathfrak{p}$-*secondary*. The family $\{M_1, \ldots, M_n\}$ of submodules of M is said to be a *secondary representation* of M if (1) $M = M_1 + \cdots + M_n$, (2) each M_i is secondary. If, moreover, the conditions (3) $M_i \not\subset \sum_{j \neq i} M_j$ for each i, (4) $\mathfrak{N}(M_i) \neq \mathfrak{N}(M_j)$ for $i \neq j$ are satisfied, then the representation is called *minimal*. If $M = M_1 + \cdots + M_n$ is the secondary representation of M and $\mathfrak{p}_i = \mathfrak{N}(M_i)$, then $\mathrm{Att}_A(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Any artinian module M has a secondary representation, whence $\mathrm{Att}_A(M)$ is a finite set. For an artinian A-module M, we have $\mathfrak{N}(M) = \cap \{\mathfrak{p} | \mathfrak{p} \in \mathrm{Att}_A(M)\}$ and $\dim_A(M) = \max\{\dim(A/\mathfrak{p}) | \mathfrak{p} \in \mathrm{Att}_A(M)\}$.

PROPOSITION 2.7. *Let A be a noetherian semi-local ring and M a finitely generated A-module. Then* $\mathrm{Ass}_A(M) = \mathrm{Att}_A(D(M))$.

PROOF. In case A is local, this is Th. (2.3) of Sharp [6]. The proof of the semi-local case can be done quite similarly. If A is complete, then the assertion follows easily from the Matlis duality. The general case follows from the next facts: $\mathrm{Ass}_A(M) = \{\mathfrak{P} \cap A | \mathfrak{P} \in \mathrm{Ass}_{\hat{A}}(\hat{M})\}$ and $\mathrm{Att}_A(D(M)) = \{\mathfrak{P} \cap A | \mathfrak{P} \in \mathrm{Att}_{\hat{A}}(D(M))\}$.

For example, if A is a noetherian semi-local ring, then $\mathrm{Att}_A(E_A) = \mathrm{Ass}_A(A)$.

DEFINITION 2.8. *Let A be a commutative ring. For an A-module M, we set* $\mathrm{Cosupp}_A(M) = V(\mathrm{ann}_A(M)) = \{\mathfrak{p} \in \mathrm{Spec}(A) | \mathfrak{p} \supset \mathrm{ann}_A(M)\}$.

PROPOSITION 2.9. (1) $\mathrm{Cosupp}_A(M) = \phi$ *if and only if* $M = 0$. (2) $\mathrm{Supp}_A(M) \subset \mathrm{Cosupp}_A(M)$. *If M is finitely generated, then* $\mathrm{Supp}_A(M) = \mathrm{Cosupp}_A(M)$. (3) $\mathrm{Att}_A(M) \subset \mathrm{Cosupp}_A(M)$. (4) *For any submodule N of M,* $\mathrm{Cosupp}_A(M) = \mathrm{Cosupp}_A(N) \cup \mathrm{Cosupp}_A(M/N)$.

PROOF. (1), (2) and (3) are clear from the definition. (4): Since $\mathrm{ann}_A(M) \subset \mathrm{ann}_A(N) \cap \mathrm{ann}_A(M/N)$, $\mathrm{Cosupp}_A(M) \supset \mathrm{Cosupp}_A(N) \cup \mathrm{Cosupp}_A(M/N)$. Conversely, suppose a is an element of $\mathrm{ann}_A(N) \cap \mathrm{ann}_A(M/N)$. Since $aM \subset N$, $a^2M \subset aN=0$ and this implies $a \in \sqrt{\mathrm{ann}_A(M)}$. Hence $\mathrm{ann}_A(N) \cap \mathrm{ann}_A(M/N) \subset \sqrt{\mathrm{ann}_A(M)}$, showing that $\mathrm{Cosupp}_A(N) \cup \mathrm{Cosupp}_A(M/N) \supset \mathrm{Cosupp}_A(M)$.

PROPOSITION 2.10. *Suppose A is a noetherian ring and M an artinian A-module. Then, any minimal element of* $\mathrm{Cosupp}_A(M)$ *is an element of* $\mathrm{Att}_A(M)$. *Moreover, the set of minimal elements of* $\mathrm{Cosupp}_A(M)$ *coincides with that of* $\mathrm{Att}_A(M)$.

PROOF. Let $\mathfrak{p}$ be a minimal element of $\mathrm{Cosupp}_A(M)$. Since $\mathfrak{p} \supset \mathfrak{N}(M) = \cap \{\mathfrak{q} \mid \mathfrak{q} \in \mathrm{Att}_A(M)\}$ and $\mathrm{Att}_A(M)$ is a finite set, there exists $\mathfrak{q} \in \mathrm{Att}_A(M)$ such that $\mathfrak{p} \supset \mathfrak{q}$. But $\mathfrak{q} \in \mathrm{Cosupp}_A(M)$ from Prop. 2.9. (3). Therefore, $\mathfrak{p}=\mathfrak{q}$ by the minimality of $\mathfrak{p}$.

As for the second assertion, if $\mathfrak{p}$ is a minimal element of $\mathrm{Cosupp}_A(M)$ and $\mathfrak{q} \subset \mathfrak{p}$, $\mathfrak{q} \in \mathrm{Att}_A(M)$, then, similarly as above, we have $\mathfrak{p}=\mathfrak{q}$. Hence $\mathfrak{p}$ is a minimal element of $\mathrm{Att}_A(M)$. Next, suppose $\mathfrak{p}$ is a minimal element of $\mathrm{Att}_A(M)$ and $\mathfrak{q} \subset \mathfrak{p}$, $\mathfrak{q} \in \mathrm{Cosupp}_A(M)$. Take a minimal element $\mathfrak{q}'$ of $\mathrm{Cosupp}_A(M)$ such that $\mathfrak{q}' \subset \mathfrak{q}$. Then, $\mathfrak{q}' \in \mathrm{Att}_A(M)$ from the first assertion. By the minimality of $\mathfrak{p}$, we get $\mathfrak{q}'=\mathfrak{q}=\mathfrak{p}$. This completes the proof.

PROPOSITION 2.11. *Suppose A is a noetherian semi-local ring and M a finitely generated A-module. Then* $\mathrm{Supp}_A(M)=\mathrm{Cosupp}_A(D(M))$.

PROOF. Clear from the fact that $\mathrm{ann}_A(M)=\mathrm{ann}_A(D(M))$.

PROPOSITION 2.12. *Let A be a complete noetherian semi-local ring and M, N A-modules. Suppose that M is finitely generated and N is artinian. Then,*

$$\mathrm{Cosupp}_A(\mathrm{Hom}_A(M, N)) = \mathrm{Cosupp}_A(M) \cap \mathrm{Cosupp}_A(N).$$

PROOF. We know that $D(\mathrm{Hom}_A(M, N))$ is isomorphic to $M \otimes_A D(N)$ (cf. Cor. 1.4.) and that $D(N)$ is finitely generated. Thus, we have $\mathrm{Cosupp}_A(\mathrm{Hom}_A(M, N))= \mathrm{Supp}_A(D(\mathrm{Hom}_A(M, N)) = \mathrm{Supp}_A(M \otimes_A D(N)) = \mathrm{Supp}_A(M) \cap \mathrm{Supp}_A(D(N)) = \mathrm{Cosupp}_A(M) \cap \mathrm{Cosupp}_A(N)$.

Note that the above equality is not true for a general noetherian ring (even if A is a semi-local domain). In fact, when A is a noetherian semi-local domain and $\mathfrak{m}_1$, $\mathfrak{m}_2$ are two different maximal ideals of A, take $M=A/\mathfrak{m}_1$ and $N=E_A(A/\mathfrak{m}_2)$. Then $\mathrm{Cosupp}_A(\mathrm{Hom}_A(M, N))=\phi$ and $\mathrm{Cosupp}_A(M) \cap \mathrm{Cosupp}_A(N)=\{\mathfrak{m}_1\}$.

PROPOSITION 2.13. *Suppose A is a noetherian ring, M a finitely generated*

A-module and N an artinian A-module. Then, $M\otimes_A N$ is artinian and $\mathrm{Att}_A(M\otimes_A N)=\mathrm{Supp}_A(M)\cap\mathrm{Att}_A(N)$.

PROOF. If $A^n\to M\to 0$ is exact, then $N^n\to M\otimes_A N\to 0$ is exact, showing that $M\otimes_A N$ is artinian and $\mathrm{Att}_A(M\otimes_A N)\subset\mathrm{Att}_A(N^n)=\mathrm{Att}_A(N)$. Next, take any element $\mathfrak{p}$ of $\mathrm{Att}_A(M\otimes_A N)$. Since $\mathfrak{p}\supset\mathfrak{N}(M\otimes_A N)\supset\mathfrak{N}(M)$, $\mathfrak{p}\in V(\mathfrak{N}(M))=\mathrm{Supp}_A(M)$. Conversely, let $\mathfrak{p}$ be an element of $\mathrm{Supp}_A(M)\cap\mathrm{Att}_A(N)$. Then, there exist an exact sequence $N\to L\to 0$ such that L is $\mathfrak{p}$-secondary and a non-zero homomorphism w from M to $A/\mathfrak{p}$ (cf. Bourbaki [8] Chap. 2. §4.4. Prop. 20). If we set $S=w(M)$, then $M\otimes_A N\to S\otimes_A L\to 0$ is exact. On the other hand, since $\mathrm{Ass}_A(S)=\{\mathfrak{p}\}$, we have an epimorphism $S\to A/\mathfrak{p}$. Hence $S\otimes_A L\to L/\mathfrak{p}L\to 0$ is exact and $L\neq\mathfrak{p}L$. In fact, if $L=\mathfrak{p}L$, then there exists an element a of $\mathfrak{p}$ such that $L=aL$. But this implies $a\notin W_A(L)=\mathfrak{p}$, a contradiction. Therefore, $\mathfrak{p}\in\mathrm{Att}_A(L/\mathfrak{p}L)\subset\mathrm{Att}_A(S\otimes_A L)\subset\mathrm{Att}_A(M\otimes_A N)$.

§3. Coregular sequences and the width of modules

Let A be a commutative ring and M an A-module. For an ideal $\mathfrak{a}$ of A, we denote by $M(\mathfrak{a})$ the submodule $\{x\in M\mid \mathfrak{a}x=0\}$ of M, which can be identified with $\mathrm{Hom}_A(A/\mathfrak{a}, M)$. When $\mathfrak{a}$ is generated by $a_1,\dots, a_n$, $a_i\in A$, we also write $M(a_1,\dots, a_n)$ in place of $M(\mathfrak{a})$.

DEFINITION 3.1. *The ordered sequence of elements $(a_1,\dots, a_n)$ of A is said to be an M-coregular sequence if*

(a) $M(a_1,\dots, a_n)\neq 0$;

(b) *a_i is an $M(a_1,\dots, a_{i-1})$-coregular element $(i=1,\dots, n)$.*

DEFINITION 3.2. *Let $\mathfrak{a}$ be an ideal of A. We denote by* $\mathrm{Width}_{\mathfrak{a}}(M)$ *the length of the longest M-coregular sequences in $\mathfrak{a}$. (If such sequences don't exist, then we write* $\mathrm{Width}_{\mathfrak{a}}(M)=\infty$.)

PROPOSITION 3.3. *If M is finitely generated, then* $\mathrm{Width}_{\mathfrak{a}}(M)=0$.

PROOF. Suppose $M=aM$ for some element a of $\mathfrak{a}$. This means that the multiplication of M by a is surjective. Since M is finitely generated, it is necessarily an isomorphism. Thus we get $M(a)=0$, and there are no M-coregular sequences.

In the rest of this section, A stands for a noetherian ring.

The following proposition is due to Matlis (cf. [5] Th. 2). We give here a different proof by using our $W_A(M)$.

PROPOSITION 3.4. *Let $\mathfrak{a}$ be an ideal of A and M an artinian A-module. Then, the following statements are equivalent:*

(1) *There exists an M-coregular element in* $\mathfrak{a}$.

(2) $M \otimes_A A/\mathfrak{a}=0$.

PROOF. (1)$\Rightarrow$(2): Suppose $a \in \mathfrak{a}$ and $M=aM$. Since $M=aM \subset \mathfrak{a}M \subset M$, $M=\mathfrak{a}M$. This implies that $M \otimes_A A/\mathfrak{a}=0$. (2)$\Rightarrow$(1): Suppose $M \neq aM$ for any a of $\mathfrak{a}$. Then, $\mathfrak{a} \subset W_A(M)= \cup\{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Att}_A(M)\}$. Since $\mathrm{Att}_A(M)$ is a finite set, there exists $\mathfrak{p}$ such that $\mathfrak{a} \subset \mathfrak{p}$, $\mathfrak{p} \in \mathrm{Att}_A(M)$. If we put $\mathfrak{p}=\mathrm{ann}_A(M/N)$, then $\mathfrak{a}M \subset \mathfrak{p}M \subset N \subsetneqq M$, contrary to the assumption.

PROPOSITION 3.5. *Let $(A, \mathfrak{m})$ be a noetherian local ring and M a non-zero artinian A-module. Then the following statements are equivalent*:

(1) $\mathrm{Width}_{\mathfrak{m}}(M)=0$.

(2) $\mathfrak{m} \in \mathrm{Att}_A(M)$.

(3) $W_A(M)=\mathfrak{m}$.

PROOF. (1)$\Rightarrow$(2): First, we note that $M \neq \mathfrak{m}M$. In fact, if $M=\mathfrak{m}M$, then there exists an element a of $\mathfrak{m}$ such that $M=aM$. By the assumption, $M(a)=0$ and this contradicts the assumption that M is non-zero. Since there exists an epimorphism $M \to M/\mathfrak{m}M \to A/\mathfrak{m}$, $\mathfrak{m} \in \mathrm{Att}_A(A/\mathfrak{m}) \subset \mathrm{Att}_A(M)$. (2)$\Rightarrow$(3): $W_A(M) = \cup\{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Att}_A(M)\}=\mathfrak{m}$. (3)$\Rightarrow$(1) is clear.

PROPOSITION 3.6. *Let $\mathfrak{a}$ be an ideal of A, M an artinian A-module and $(a_1,\dots, a_n)$, $a_i \in \mathfrak{a}$ an M-coregular sequence. Then,*

(1) $\mathrm{Tor}_i^A(M, A/\mathfrak{a})=0$ *for any* $i<n$.

(2) $\mathrm{Tor}_n^A(M, A/\mathfrak{a}) \cong M(a_1,\dots, a_n) \otimes_A A/\mathfrak{a}$.

PROOF. We prove both of (1) and (2) by the induction on n. When $n=0$, there is nothing to prove. Suppose $n>0$. Then, from the exact sequence $0 \to M(a_1) \to M \xrightarrow{a_1} M \to 0$, we get the exact sequence

$$\cdots \xrightarrow{a_1} \mathrm{Tor}_n^A(M, A/\mathfrak{a}) \longrightarrow \mathrm{Tor}_{n-1}^A(M(a_1), A/\mathfrak{a}) \longrightarrow \mathrm{Tor}_{n-1}^A(M, A/\mathfrak{a})$$
$$\xrightarrow{a_1} \mathrm{Tor}_{n-1}^A(M, A/\mathfrak{a}) \longrightarrow \mathrm{Tor}_{n-2}^A(M(a_1), A/\mathfrak{a}) \longrightarrow \cdots.$$

By the induction assumption, we have

$$\mathrm{Tor}_i^A(M(a_1), A/\mathfrak{a})=0 \qquad \text{for any} \quad i<n-1$$
$$\mathrm{Tor}_{n-1}^A(M(a_1), A/\mathfrak{a}) \cong M(a_1)(a_2,\dots, a_n) \otimes_A A/\mathfrak{a}.$$

Since the multiplications by a_1 in the above exact sequence are zero, we get $\mathrm{Tor}_i^A(M, A/\mathfrak{a})=0$ for any $i<n$ and $\mathrm{Tor}_n^A(M, A/\mathfrak{a}) \cong \mathrm{Tor}_{n-1}^A(M(a_1), A/\mathfrak{a}) \cong M(a_1)(a_2, \dots, a_n) \otimes_A A/\mathfrak{a} \cong M(a_1,\dots, a_n) \otimes_A A/\mathfrak{a}$.

COROLLARY 3.7. *Suppose $\mathfrak{a}$ is an ideal of A, M an artinian A-module and* *that* $M(\mathfrak{a})\neq 0$. *Then*, $\mathrm{Width}_{\mathfrak{a}}(M)=\mathrm{Inf}\{n\geqq 0|\mathrm{Tor}_n^A(M, A/\mathfrak{a})\neq 0\}$ *and is finite.*

Note that if A is semi-local, $M\neq 0$ and $\mathfrak{a}$ is contained in the Jacobson radical of A, then the assumption $M(\mathfrak{a})\neq 0$ is always satisfied.

PROPOSITION 3.8. *Let M and N be A-modules and suppose that M is finitely generated and N is artinian. Then the following statements are equivalent:*
(1) *There exists an N-coregular element in* $\mathrm{ann}_A(M)$.
(2) $M\otimes_A N=0$.

PROOF. We put $\mathfrak{a}=\mathrm{ann}_A(M)$. Then,

$$\begin{aligned}\mathrm{Att}_A(M\otimes_A N) &= \mathrm{Supp}_A(M)\cap \mathrm{Att}_A(N) = \mathrm{Supp}_A(A/\mathfrak{a})\cap \mathrm{Att}_A(N)\\ &= \mathrm{Att}_A(A/\mathfrak{a}\otimes_A N).\end{aligned}$$

$$\begin{aligned}\text{Therefore, } M\otimes_A N = 0 &\Longleftrightarrow \mathrm{Att}_A(M\otimes_A N) = \phi\\ &\Longleftrightarrow \mathrm{Att}_A(A/\mathfrak{a}\otimes_A N) = \phi\\ &\Longleftrightarrow A/\mathfrak{a}\otimes_A N = 0,\end{aligned}$$

and the equivalence of (1) and (2) follows immediately from these relations and Prop. 3.4.

THEOREM 3.9. *Let $\mathfrak{a}$ be an ideal of A and N an artinian A-module. Then, the following statements are equivalent:*
(1) $\mathrm{Tor}_i^A(M, N)=0$ *for any finitely generated A-module M such that* $\mathrm{Supp}_A(M)\subset V(\mathfrak{a})$ *and for any $i<n$.*
(2) $\mathrm{Tor}_i^A(A/\mathfrak{a}, N)=0$ *for any $i<n$.*
(3) $\mathrm{Tor}_i^A(M, N)=0$ *for a finitely generated A-module M such that* $\mathrm{Supp}_A(M)=V(\mathfrak{a})$ *and for any $i<n$.*
(4) *There exists an N-coregular sequence $(a_1,\dots, a_n)$ in $\mathfrak{a}$.*

PROOF. (1)⇒(2)⇒(3) are trivial. (3)⇒(4): We prove the statement by the induction on n. When $n=0$, there is nothing to prove. Suppose $n>0$. Then, from Prop. 3.8, there exists an N-coregular element a in $\mathfrak{a}$. Since the sequence $0\to N(a)\to N\xrightarrow{a} N\to 0$ is exact, $\mathrm{Tor}_i^A(M, N(a))=0$ for any $i<n-1$. By the induction assumption, there exists an $N(a)$-coregular sequence $(a_2,\dots, a_n)$ in $\mathfrak{a}$. The sequence $(a, a_2,\dots, a_n)$ is a required N-coregular sequence in $\mathfrak{a}$. (4)⇒(1): Induction on n. When $n=0$, there is nothing to prove. If $n>0$, then $(a_2,\dots, a_n)$ is an $N(a_1)$-coregular sequence in $\mathfrak{a}$. By the induction assumption, $\mathrm{Tor}_i^A(M, N(a_1))=0$ for any $i<n-1$ and any M as in (1). On the other hand, we have the following exact sequence for each $i<n$:

$$\mathrm{Tor}_i^A(M, N) \xrightarrow{a_1} \mathrm{Tor}_i^A(M, N) \longrightarrow \mathrm{Tor}_{i-1}^A(M, N(a_1)) = 0$$

Since $a_1 \in \mathfrak{a} \subset \mathfrak{R}(M)$, $a_1^m M = 0$ for a sufficiently large m. Thus the multiplication of $\mathrm{Tor}_i^A(M, N)$ by a_1^m is surjective and a zero-map. Therefore, we get $\mathrm{Tor}_i^A(M, N) = 0$ for any $i < n$.

DEFINITION 3.10. *Let M and N be A-modules; suppose that M is finitely generated and N is artinian. We denote by* $\mathrm{cograde}_N(M)$ *the length of a maximal N-coregular sequence in* $\mathrm{ann}_A(M)$. *(Note that all such sequences are of the same length.)*

COROLLARY 3.11. *Let the assumptions be the same as* Def. 3.10. *Then,* $\mathrm{cograde}_N(M) = \mathrm{Inf}\{n \geqq 0 | \mathrm{Tor}_n^A(M, N) \neq 0\}$.

Note that $\mathrm{Width}_{\mathfrak{a}}(N) = \mathrm{cograde}_N(A/\mathfrak{a})$.

THEOREM 3.12. *Let A be a noetherian semi-local ring and M, N be finitely generated A-modules. Then we have* $\mathrm{cograde}_{D(N)}(M) = \mathrm{grade}_N(M)$. *In particular,* $\mathrm{cograde}_{E_A}(M) = \mathrm{grade}_A(M)$.

PROOF. From Cor. 1.5, $\mathrm{Tor}_n^A(M, D(N))$ is isomorphic to $D(\mathrm{Ext}_A^n(M, N))$. Therefore, $\mathrm{cograde}_{D(N)}(M) = \mathrm{Inf}\{n \geqq 0 | \mathrm{Tor}_n^A(M, D(N)) \neq 0\} = \mathrm{Inf}\{n \geqq 0 | \mathrm{Ext}_A^n(M, N) \neq 0\} = \mathrm{grade}_N(M)$.

COROLLARY 3.13. $\mathrm{Width}_{\mathfrak{a}} D(M) = \mathrm{Depth}_{\mathfrak{a}}(M)$.

For example, if $(A, \mathfrak{m})$ is a noetherian local ring, then $\mathrm{Width}\, E_A(A/\mathfrak{m}) = \mathrm{Depth}(A)$.

As is seen from Th. 3.9, $\mathrm{cograde}_N(M)$ (resp. $\mathrm{Width}_{\mathfrak{a}}(N)$) does not depend on M (resp. $\mathfrak{a}$), but on $\mathrm{Supp}_A(M)$ (resp. $V(\mathfrak{a})$).

PROPOSITION 3.14. *Let A be a noetherian semi-local ring and M an artinian A-module. Then, $M \otimes_A \hat{A}$ is isomorphic to M, and M is also artinian as an $\hat{A}$-module. Moreover,* $\mathrm{Width}_{\hat{A}}(M)$ *is equal to* $\mathrm{Width}_A(M)$. *(Here,* $\mathrm{Width}_A(M)$ *means* $\mathrm{Width}_{Rad(A)}(M)$.*)*

PROOF. The first assertion is obvious. As for the second one, since $\mathrm{Tor}_n^A(A/\mathfrak{a}, M) \otimes_A \hat{A} \cong \mathrm{Tor}_n^{\hat{A}}(\hat{A}/\hat{\mathfrak{a}}, M \otimes_A \hat{A}) \cong \mathrm{Tor}_n^{\hat{A}}(\hat{A}/\hat{\mathfrak{a}}, M)$, where $\mathfrak{a} = Rad(A)$, and $\hat{A}$ is faithfully flat over A, the assertion follows from Cor. 3.7.

PROPOSITION 3.15. *Let $\mathfrak{a}$ be an ideal of A and M an artinian A-module. If $a \in \mathfrak{a}$ is an M-coregular element and $M(a) \neq 0$, then* $\mathrm{Width}_{\mathfrak{a}}(M) = \mathrm{Width}_{\mathfrak{a}}(M(a)) + 1$.

PROOF. The exact sequence $0 \to M(a) \to M \xrightarrow{a} M \to 0$ leads to the following exact sequence:

$$\cdots \longrightarrow \mathrm{Tor}_{n+1}^A(M, A/\mathfrak{a}) \xrightarrow{a} \mathrm{Tor}_{n+1}^A(M, A/\mathfrak{a}) \longrightarrow \mathrm{Tor}_n^A(M(a), A/\mathfrak{a})$$

$$\longrightarrow \mathrm{Tor}_n^A(M, A/\mathfrak{a}) \xrightarrow{a} \cdots.$$

In the above sequence, the multiplications by a are all zero-maps. The assertion follows from this and Cor. 3.7.

PROPOSITION 3.16. *Let $\mathfrak{a}$ be an ideal of A; let M', M and M'' be artinian A-modules. Suppose that $M'(\mathfrak{a})$, $M(\mathfrak{a})$ and $M''(\mathfrak{a})$ are non-zero. If $0 \to M' \to M \to M'' \to 0$ is exact, then the following statements hold:*

(1) *If* $\mathrm{Width}_{\mathfrak{a}}(M) < \mathrm{Width}_{\mathfrak{a}}(M')$, *then* $\mathrm{Width}_{\mathfrak{a}}(M) = \mathrm{Width}_{\mathfrak{a}}(M'')$.
If $\mathrm{Width}_{\mathfrak{a}}(M) > \mathrm{Width}_{\mathfrak{a}}(M')$, *then* $\mathrm{Width}_{\mathfrak{a}}(M'') = \mathrm{Width}_{\mathfrak{a}}(M') + 1$.
If $\mathrm{Width}_{\mathfrak{a}}(M) = \mathrm{Width}_{\mathfrak{a}}(M')$, *then* $\mathrm{Width}_{\mathfrak{a}}(M'') \geqq \mathrm{Width}_{\mathfrak{a}}(M)$.

(2) *If* $\mathrm{Width}_{\mathfrak{a}}(M) < \mathrm{Width}_{\mathfrak{a}}(M'')$, *then* $\mathrm{Width}_{\mathfrak{a}}(M) = \mathrm{Width}_{\mathfrak{a}}(M')$.
If $\mathrm{Width}_{\mathfrak{a}}(M) > \mathrm{Width}_{\mathfrak{a}}(M'')$, *then* $\mathrm{Width}_{\mathfrak{a}}(M'') = \mathrm{Width}_{\mathfrak{a}}(M') + 1$.
If $\mathrm{Width}_{\mathfrak{a}}(M) = \mathrm{Width}_{\mathfrak{a}}(M'')$, *then* $\mathrm{Width}_{\mathfrak{a}}(M) \leqq \mathrm{Width}_{\mathfrak{a}}(M') + 1$.

(3) $\mathrm{Width}_{\mathfrak{a}}(M) \geqq \min(\mathrm{Width}_{\mathfrak{a}}(M'), \mathrm{Width}_{\mathfrak{a}}(M''))$.

PROOF. (1) Suppose $\mathrm{Width}_{\mathfrak{a}}(M) > 0$ and $\mathrm{Width}_{\mathfrak{a}}(M') > 0$. Then, $\mathfrak{a}$ is contained neither in $W_A(M)$ nor in $W_A(M')$. Therefore, $\mathfrak{a}$ is not contained in $W_A(M) \cup W_A(M')$ (cf. Prop. 2.6. and [9] 2–2, Th. 81). Hence, we can take an element a of $\mathfrak{a}$ which is M-coregular and M'-coregular. Then, from Prop. 3.15., $\mathrm{Width}_{\mathfrak{a}}(M'(a)) = \mathrm{Width}_{\mathfrak{a}}(M') - 1$, $\mathrm{Width}_{\mathfrak{a}}(M(a)) = \mathrm{Width}_{\mathfrak{a}}(M) - 1$, $\mathrm{Width}_{\mathfrak{a}}(M''(a)) = \mathrm{Width}_{\mathfrak{a}}(M'') - 1$ and $0 \to M'(a) \to M(a) \to M''(a) \to 0$ is exact by the snake lemma. Hence, by the induction, we can assume that $\mathrm{Width}_{\mathfrak{a}}(M) = 0$ or $\mathrm{Width}_{\mathfrak{a}}(M') = 0$.

Case 1) $\mathrm{Width}_{\mathfrak{a}}(M) = 0$: It suffices to show the first statement. By the assumption, $\mathfrak{a}$ is contained in $W_A(M)$. Therefore, there exists a prime ideal $\mathfrak{p} \in \mathrm{Att}_A(M)$ such that $\mathfrak{a} \subset \mathfrak{p}$. Suppose $\mathfrak{p} \in \mathrm{Att}_A(M')$, then $\mathfrak{a} \subset W_A(M')$. Hence, we have $\mathrm{Width}_{\mathfrak{a}}(M') = 0$, a contradiction. Thus we get $\mathfrak{p} \in \mathrm{Att}_A(M'')$. This implies $\mathfrak{a} \subset W_A(M'')$, namely, $\mathrm{Width}_{\mathfrak{a}}(M'') = 0$.

Case 2) $\mathrm{Width}_{\mathfrak{a}}(M') = 0$: We have only to show the second statement. Firstly, we have the exact sequence:

$$\cdots \longrightarrow \mathrm{Tor}_1^A(M'', A/\mathfrak{a}) \longrightarrow M' \otimes_A A/\mathfrak{a} \longrightarrow M \otimes_A A/\mathfrak{a} \longrightarrow M'' \otimes_A A/\mathfrak{a} \longrightarrow 0.$$

By assumptions, $M' \otimes_A A/\mathfrak{a} \neq 0$ and $M \otimes_A A/\mathfrak{a} = 0$. Therefore $\mathrm{Tor}_1^A(M'', A/\mathfrak{a}) \neq 0$ and $M'' \otimes_A A/\mathfrak{a} = 0$. Thus we have $\mathrm{Width}_{\mathfrak{a}}(M'') = 1$.

The assertions (2) and (3) follow immediately from (1).

THEOREM 3.17. *Suppose A is a noetherian local ring and M is an artinian A-module. Then for each $\mathfrak{p} \in \mathrm{Att}_A(M)$, we have* $\mathrm{Width}_{\mathfrak{a}}(M) \leqq \dim(A/\mathfrak{p}) \leqq \dim_A(M)$.

PROOF. The last inequality follows from $\mathfrak{N}(M) = \cap\{\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Att}_A(M)\}$. Next, let $\mathfrak{p}$ be an element of $\mathrm{Att}_A(M)$. Since $\mathrm{Att}_A(M) = \{\mathfrak{P} \cap A \mid \mathfrak{P} \in \mathrm{Att}_{\hat{A}}(M)\}$, we can take $\mathfrak{P} \in \mathrm{Att}_{\hat{A}}(M)$ such that $\mathfrak{p}=\mathfrak{P} \cap A$. Then $\dim_{\hat{A}}(\hat{A}/\mathfrak{P}) \leqq \dim_{\hat{A}}(\hat{A}/\mathfrak{p}\hat{A})=\dim_A(A/\mathfrak{p})$ and $\mathrm{Width}_A(M)=\mathrm{Width}_{\hat{A}}(M)$. Hence we can assume that A is complete. We prove the first inequality by the induction on $\mathrm{Width}_A(M)$. If $\mathrm{Width}_A(M)=0$, then the assertion is obvious. Suppose $\mathrm{Width}_A(M)>0$ and take an M-coregular element a in $\mathfrak{m}$. Since $\mathfrak{p} \in \mathrm{Att}_A(M)$, a is not contained in $\mathfrak{p}$. Let $\mathfrak{q}$ be a minimal prime ideal over $(\mathfrak{p}, a)$.

LEMMA 3.18. *Let A be a noetherian complete semi-local ring, M an artinian A-module and $a \in A$ an M-coregular element. Let $\mathfrak{p} \in \mathrm{Att}_A(M)$ and $\mathfrak{q}$ a minimal prime over $(\mathfrak{p}, a)$. Then $\mathfrak{q} \in \mathrm{Att}_A(M(a))$.*

PROOF. We can take an exact sequence $0 \to M' \to M \to M'' \to 0$ such that $\mathrm{Att}_A(M'')=\{\mathfrak{p}\}$ and $\mathrm{Att}_A(M')=\mathrm{Att}_A(M)-\{\mathfrak{p}\}$. In fact, if $M=M_1+\cdots+M_n$ is a minimal secondary representation of M and M_1 is $\mathfrak{p}$-secondary, then put $M'=M_2+\cdots+M_n$ and $M''=M/M'$. Since a is M-coregular, a is also M'-coregular. (Because $\mathrm{Att}_A(M)=\mathrm{Att}_A(M') \cup \mathrm{Att}_A(M'')$, $W_A(M)=W_A(M') \cup W_A(M'')$. Since a is not in $W_A(M)$, a is not in $W_A(M')$.) Then by the snake lemma, the sequence $0 \to M'(a) \to M(a) \to M''(a) \to 0$ is exact. We show that $\mathfrak{q}$ is a minimal element of $\mathrm{Cosupp}_A(M''(a))=\mathrm{Cosupp}_A(\mathrm{Hom}_A(A/aA, M''))=V(a) \cap \mathrm{Cosupp}_A(M'')$ (cf. Prop. 2.12.). In fact, since $a \in \mathfrak{q}$, $\mathfrak{p} \subset \mathfrak{q}$ and $\mathrm{Att}_A(M'')=\{\mathfrak{p}\}$, $\mathfrak{q}$ is an element of $\mathrm{Cosupp}_A(M''(a))$. On the other hand, if $\mathfrak{q}' \subset \mathfrak{q}$ and $\mathfrak{q}'$ is an element of $\mathrm{Cosupp}_A(M''(a))$, then $\mathfrak{q}' \supset (\mathfrak{p}, a)$. Thus $\mathfrak{q}=\mathfrak{q}'$ by the minimality of $\mathfrak{q}$. Therefore, by Prop. 2.10, $\mathfrak{q} \in \mathrm{Att}_A(M''(a)) \subset \mathrm{Att}_A(M(a))$.

By the above lemma, $\mathfrak{q}$ is an element of $\mathrm{Att}_A(M''(a))$. Then, by the induction assumption, $\mathrm{Width}_{\mathfrak{a}}(M)-1=\mathrm{Width}_{\mathfrak{a}} M(a) \leqq \dim(A/\mathfrak{q}) \leqq \dim(A/\mathfrak{p})-1$. Therefore, $\mathrm{Width}_{\mathfrak{a}}(M) \leqq \dim(A/\mathfrak{p})$ and this completes the proof of Th. 3.17.

PROPOSITION 3.19. *Let $(A, \mathfrak{m})$ be a noetherian local ring and M a Cohen-Macaulay A-module with $\dim_A(M)=n$. Then $\mathrm{Width}_{\mathfrak{m}}(H^n_{\mathfrak{m}}(M))=n$.*

PROOF. We prove the statement by the induction on n. If $n=0$, then $H^0_{\mathfrak{m}}(M)=M$ is of finite length. Thus $\mathrm{Width}_{\mathfrak{m}}(H^0_{\mathfrak{m}}(M))=0$. Suppose $n>0$ and a is an M-coregular element in $\mathfrak{m}$. Then M/aM is Cohen-Macaulay and $\dim(M/aM)=n-1$. From the exact sequence $0 \to M \xrightarrow{a} M \to M/aM \to 0$, we have the following exact sequence:

$$0 = H^{n-1}_{\mathfrak{m}}(M) \longrightarrow H^{n-1}_{\mathfrak{m}}(M/aM) \longrightarrow H^n_{\mathfrak{m}}(M) \xrightarrow{a} H^n_{\mathfrak{m}}(M) \longrightarrow H^n_{\mathfrak{m}}(M/aM)$$
$$= 0.$$

This implies that a is $H^n_{\mathfrak{m}}(M)$-coregular and $H^{n-1}_{\mathfrak{m}}(M/aM) \cong H^n_{\mathfrak{m}}(M)(a)$. By the induction assumption and Prop. 3.15., we get $\mathrm{Width}_{\mathfrak{m}}(H^n_{\mathfrak{m}}(M))=\mathrm{Width}_{\mathfrak{m}}(H^n_{\mathfrak{m}}(M)(a))+1=\mathrm{Width}_{\mathfrak{m}}(H^{n-1}_{\mathfrak{m}}(M/aM))+1=(n-1)+1=n$.

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