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The Plancherel Formula for a Pseudo-Riemannian Symmetric Space

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§1. Introduction

The Plancherel theorem for Riemannian symmetric spaces has been extensively studied and almost completely established by several authors.

Now we believe that it is interesting and important to verify the analogous theorem for pseudo-Riemannian symmetric spaces. (For the definition of a pseudo-Riemannian symmetric space, see [7]). There have been several results in this direction; see [2], [3], [4], [10], [12], [13], [14], [15] and [16].

In this paper we prove the Plancherel formula for the pseudo-Riemannian symmetric space $SU(p, 1)/S(U(1)\times U(p-1, 1))$. Our method, which is mainly due to the theory of Kokaira and Titchmarsh, may be applicable to more general pseudo-Riemannian symmetric spaces.

It is my pleasant duty to express my gratitude to Professor K. Okamoto for his guidance and encouragement.

2. The main result

Let the form $[z, w]=z_1\bar{w}_1+\cdots+z_p\bar{w}_p-z_{p+1}\bar{w}_{p+1}$ be given in the complex $p+1$-dimensional space $\boldsymbol{C}^{p+1}$ $(p\geqq 2)$; let G be the linear group of transformations which have determinant 1 and leave this form invariant.

The mapping

$$\sigma\colon g \longrightarrow J({}^t\bar{g})^{-1}J$$

is an involutive automorphism of G, where $J=\mathrm{diag}(-1, 1,\dots, 1, -1)$. The fixed points of σ constitute the subgroup

$$H=\left\{\begin{pmatrix} e^{i\theta} & 0\cdots 0 \\ 0 & \\ \vdots & * \\ 0 & \end{pmatrix}\in G;\ \theta\in\boldsymbol{R}\right\}.$$

Furthermore there exists a G-invariant indefinite Riemannian metric on G/H and, therefore, G/H is a pseudo-Riemannian symmetric space.

The purpose of this paper is to compute the Plancherel measure for $C_0^\infty(G/H)$. Let $\mathfrak{g}$ be the Lie algebra of G and set

$$\mathfrak{a}=\left\{\begin{pmatrix} & & & t\\ & & 0 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \\ t & & & \end{pmatrix};\ t\in\boldsymbol{R}\right\}.$$

If $\lambda\in\mathfrak{a}^*\backslash 0$, then λ is called a restricted root if $\mathfrak{g}_\lambda=\{X\in\mathfrak{g};\ [H,X]=\lambda(H)X$ for all $H\in\mathfrak{a}\}$ is not $\{0\}$. Let λ_0 be the linear form on $\mathfrak{a}$ defined by

$$\lambda_0:\begin{pmatrix} & & & t\\ & & 0 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 0 & & & \\ t & & & \end{pmatrix}\longrightarrow t.$$

Then the set of the restricted roots is given by $\Lambda=\{\pm\lambda_0,\ \pm 2\lambda_0\}$.

Set

$$K=\left\{\begin{pmatrix} U & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\ 0\cdots 0 & \det U^{-1}\end{pmatrix};\ U\in U(p)\right\},$$

$$A=\left\{a_t=\begin{pmatrix}\mathrm{ch}t & & & \mathrm{sh}t\\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ \mathrm{sh}t & & & \mathrm{ch}t\end{pmatrix};\ t\in\boldsymbol{R}\right\},$$

$$\begin{aligned}
A^+ &= \{a_t;\ t>0\},\\
M &= \text{the centralizer of } A \text{ in } K,\\
N &= \exp(\mathfrak{g}_{\lambda_0}+\mathfrak{g}_{2\lambda_0}),\\
\bar{N} &= \exp(\mathfrak{g}_{-\lambda_0}+\mathfrak{g}_{-2\lambda_0}),\\
m_0 &= \mathrm{diag}(-1,\ -1,\ 1,\dots,\ 1).
\end{aligned}$$

Then the mappings

$$H\times A\times N\ni(h,\ a,\ n)\longrightarrow han\in G,$$

$$K/M\times A^+\ni(kM,\ a)\longrightarrow kaH\in G/H$$

are injective diffeomorphisms; furthermore their images are open and dense.

For $s \in \boldsymbol{C}$ let X_s be the set of all C^∞-functions ϕ: $G \to \boldsymbol{C}$ such that

$$\phi(gma_t n) = e^{-(is+p)t}\phi(g) \quad (g \in G,\ m \in M,\ a_t \in A,\ n \in N).$$

Let π (resp. π_s) be the left regular representation of G on $C^\infty(G/H)$ (resp. X_s).

Define χ_s: $G \to \boldsymbol{C}$ by

$$\chi_s(g) = \begin{cases} e^{(is-p)t} & (g = ha_t n,\ h \in H,\ a_t \in A,\ n \in N), \\ 0 & (g \notin HAN). \end{cases}$$

Normalize the invariant measure dx on G/H so that

$$\int_{G/H} f(x)\,dx = \int_K dk \int_0^\infty f(ka_t)(\operatorname{ch} t)^{2p-1} \operatorname{sh} t\,dt \quad (f \in C_0(G/H)).$$

Then the operators

$$\mathscr{P}_s\colon X_s \ni \phi \longrightarrow \int_K \chi_s(x^{-1}k)\phi(k)\,dk \in C^\infty(G/H),$$

$$\textstyle\prod_s\colon C_0^\infty(G/H) \ni f \longrightarrow \displaystyle\int_{G/H} \chi_{-s}(x^{-1}g)f(x)\,dx \in X_s$$

commute with the action of G. (The integrals converge for $\operatorname{Im} s < 2-p$, $\operatorname{Im} s > p-2$ respectively; they are analytically continued to the whole complex plane as meromorphic functions).

Normalize the Harr measure $d\bar{n}$ on $\bar{N}$ so that $\int_N \exp\{-2\rho H(\bar{n})\} d\bar{n} = 1$, where $\rho = p\lambda_0$. Let A_s: $X_s \to X_{-s}$ be the interwining operator defined by

$$As\phi(g) = 2^{is}\Gamma(is)^{-1} \int_N \phi(gm_0\bar{n})\,d\bar{n} \quad \text{(see [6, p. 1056])}.$$

Set $c(s) = \int_N \exp\{-(is\lambda_0 + \rho)H(\bar{n})\}\,d\bar{n}$.

For integer $j > -p/2$ we set

$$s(j) = (p+2j)i,$$

$$K_j = \operatorname{Ker} A_{s(j)},$$

$$U_j = \operatorname{Im} A_{s(j)}$$

and

$$\textstyle\prod^j = \{\Gamma(is)^{-1}\prod_{-s}\}|_{s=s(j)}.$$

Then $\operatorname{Im} \prod_-^j = U_j$ (for the proof, see § 5). Hence we can define the operator

$\prod^j: C_0^\infty(G/H) \to X_{s(j)}/K_j$ by the rule

$$\prod_-^j = B_{s(j)} \circ \prod^j ,$$

where the operator $B_{s(j)}: X_{s(j)}/K_j \to U_j$ is defined by

$$B_{s(j)}([\phi]) = A_{s(j)}(\phi) \qquad ([\phi] \in X_{s(j)}/K_j).$$

We define the positive definite inner products on $C_0^\infty(G/H)$, X_ν and $X_{s(j)}/K_j$ by

$$(f, g) = \int_{G/H} f(x)\overline{g(x)}\, dx \qquad (f, g \in C_0^\infty(G/H)),$$

$$(\phi, \psi)_\nu = \int_K \phi(k)\overline{\psi(k)}\, dk \qquad (\phi, \psi \in X_\nu,\ \nu \in R)$$

and

$$([\phi], [\psi])^j = \int_K \phi(k)\overline{(A_{s(j)}\psi)(k)}\, dk \qquad ([\phi], [\psi] \in X_{s(j)}/K_j),$$

respectively. (For the detail, see [6, p. 1057]).

Now we can state our result.

THEOREM. *If* $f \in C_0^\infty(G/H)$, *then*

$$(f, f) = \frac{2^{2p}}{2\pi} \int_0^\infty \frac{d\nu}{|c(\nu)|^2} (\prod_\nu f, \prod_\nu f)_\nu$$
$$+ \frac{2^p}{(p-1)!} \sum_j \frac{\{\Gamma(p+j)\}^2}{\Gamma(p+2j)\,(p+2j)!} (\prod^j f, \prod^j f)^j .$$

§3. The reduction of our problem

As is easily seen, we have only to show that the above theorem holds for all f in $C_{00}^\infty(G/H)$ which is defined as the set of all $f \in C_0^\infty(G/H)$ such that $f|_{KU(eH)} \equiv 0$ for an open neighborhood U of e. Let us consider our theorem in more reduced form. To this end, we shall make some preparations.

Let $\hat{K}$ denote the set of equivalence classes of irreducible unitary representations of K. Fix, for $\gamma \in \hat{K}$, $(\pi_\gamma, V_\gamma) \in \gamma$. Put $V_{0,\gamma} = \{v \in V_\gamma;\ \pi_\gamma(m)v = v$ for all $m \in M\}$ and $\hat{K}_0 = \{(\pi_\gamma, V_\gamma) \in \hat{K};\ V_{0,\gamma} \neq 0\}$. Then we can identify $\hat{K}_0$ with the set

$\{(m, n);\ m$ and n are integers, $|n| \leqq m$ and $m - n$ is even$\}$ (see [6], [11]).

Taking account of this identification, we from now on write $\pi_{m,n}$, $V_{m,n}$ and $V_{0,m,n}$ instead of π_γ, V_γ and $V_{0,\gamma}$. Set $d_{m,n}=dim V_{m,n}$ and $\chi_{m,n}(k)=Tr\pi_{m,n}(k)$ $(k\in K)$. Choose $e_{m,n}\in V_{0,m,n}$ such that $\|e_{m,n}\|=1$. Let $C^\infty(G/H)_{m,n}$ (resp. $X_{s,m,n}$) denote the set of all functions in $C^\infty(G/H)$(resp. X_s) which transform according to $\pi_{m,n}$. Let $C_0^\infty(G/H, V_{m,n})$ be the space of all compactly-supported C^∞-functions Φ: $G/H\to V_{m,n}$ such that

$$\Phi(kx) = \pi_{m,n}(k)\Phi(x) \qquad (k\in K,\ x\in G/H).$$

By $C_{00}^\infty(G/H, V_{m,n})$ we shall understand the space of all Φ in $C_0^\infty(G/H, V_{m,n})$ such that $\Phi|_{KU(eH)}\equiv 0$ for an open neighborhood U of e.

For each f in $C_{00}^\infty(G/H)$, set

$$f_{m,n}(x) = d_{m,n}\int_K \chi_{m,n}(k)f(kx)\,dk.$$

Then it is not difficult to see that

$$(f, f) = \sum_{m,n} (f_{m,n}, f_{m,n}),$$

$$(\textstyle\prod_\nu f, \prod_\nu f) = \sum_{m,n} (\prod_\nu f_{m,n}, \prod_\nu f_{m,n})_\nu$$

and

$$(\textstyle\prod^j f, \prod^j f)^j = \sum_{m,n} (\prod^j f_{m,n}, \prod^j f_{m,n})^j.$$

Hence, in order to prove our theorem, we may assume that the function f has the property

$$f(x) = (v, \Phi(x)),$$

where $v\in V_{m,n}$, $\Phi\in C_{00}^\infty(G/H, V_{m,n})$ and (,) is the inner product on $V_{m,n}$.

§4. An application of the classical eigenvalue problem

Let Ω be the Casimir operator for G. Then

$$\pi_s(\Omega) = -(s^2+p^2)/\{4(p+1)\}.$$

If $f\in C^\infty(G/H)_{m,n}$, using the result of [11, Lemma 3.1], we have

$$(\Omega f)(a_t) = \frac{1}{4(p+1)}\Big(\frac{d^2}{dt^2} + 2\{(p-1)\,\mathrm{tht} + \coth 2t\}\frac{d}{dt}$$
$$+ \{m^2 - n^2 + 2(p-1)m\}\,\mathrm{ch}^{-2}t - 4n^2\,\mathrm{sh}^{-2}2t\Big)f(a_t) \qquad (t>0).$$

Set

$$D_1 = -\frac{1}{4}\Big(\frac{d^2}{dt^2} + 2\{(p-1)\,\mathrm{tht} + \coth 2t\}\frac{d}{dt}$$

$$+ \{m^2 - n^2 + 2(p-1)m\}\,\mathrm{ch}^{-2} t - 4n^2\,\mathrm{sh}^{-2} 2t + s^2 + p^2\Big) \quad (t > 0).$$

Putting $z = -\mathrm{sh}^2 t$, we have

$$D_1 = z(1-z)\frac{d^2}{dz^2} + \{1 - (p+1)z\}\frac{d}{dz}$$

$$-\frac{n^2}{4z} - \frac{m(m+2p-2)}{4(1-z)} - \frac{(s^2+p^2)}{4}.$$

Set

$$D_2 = z(1-z)\frac{d^2}{dz^2} + \{\gamma - (\alpha+\beta+1)z\}\frac{d}{dz} - \alpha\beta,$$

where

$$\alpha = \{m - |n| + is + p\}/2,$$

$$\beta = \{m - |n| - is + p\}/2,$$

$$\gamma = 1 - |n|.$$

Then

$$(-z)^{|n|/2}(1-z)^{-m/2}D_1 = D_2(-z)^{|n|/2}(1-z)^{-m/2}.$$

Using the above formula, we have $D_1 f^s_{m,n} = 0$, where

$$f^s_{m,n}(t) = (\mathrm{sh}\,t)^{|n|}(\mathrm{ch}\,t)^m\,\mathrm{F}(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, -\mathrm{sh}^2 t).$$

On the other hand, it is well known that

$$\mathrm{F}(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z)$$

$$= \frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta-\gamma+1)\Gamma(1-\alpha)}(-z)^{\gamma-\alpha-1}\mathrm{F}(\alpha-\gamma+1, \alpha, \alpha-\beta+1, z^{-1})$$

$$+ \frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma+1)\Gamma(1-\beta)}(-z)^{\gamma-\beta-1}\mathrm{F}(\beta-\gamma+1, \beta, \beta-\alpha+1, z^{-1})$$

provided $\alpha - \beta$ is not an integer. Hence if we put

$$c_{m,n}(s) = \frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha-\gamma+1)\Gamma(1-\beta)}$$

and

$$h^s_{m,n}(t) = (\operatorname{sh} t)^{is-m-p}(\operatorname{ch} t)^m \mathrm{F}(\beta - \gamma + 1,\ \beta,\ \beta - \alpha + 1,\ -\operatorname{sh}^{-2} t),$$

then we have

$$f^s_{m,n} = c_{m,n}(s)h^s_{m,n} + c_{m,n}(-s)h^{-s}_{m,n}$$

and

$$D_1 h^s_{m,n} = 0\,.$$

Set

$$r(t) = (\operatorname{ch} t)^{p-1/2}(\operatorname{sh} t)^{1/2} \qquad (t > 0),$$

$$q(t) = \left(p^2 - 2p + \frac{3}{4}\right)\operatorname{th}^2 t - \frac{1}{4}\coth^2 t + 2p - \frac{3}{2}$$
$$- \{m^2 - n^2 + 2(p-1)m\}\operatorname{ch}^{-2} t + 4n^2 \operatorname{sh}^{-2} 2t \qquad (t > 0)$$

and

$$L = q(t) - \frac{d^2}{dt^2}\,.$$

Then

$$(L - (s^2 + p^2))r = 4rD_1 \quad \text{on} \quad C^\infty(0, \infty)\,.$$

Hence

$$(L - (s^2 + p^2))(rf^s_{m,n}) = 0\,,$$
$$(L - (s^2 + p^2))(rh^s_{m,n}) = 0\,.$$

Moreover, if $\operatorname{Im} s > 0$, then

$$r(t)f^s_{m,n}(t) \in L^2(0, 1)\,,$$
$$r(t)h^s_{m,n}(t) \in L^2(1, \infty)\,.$$

The Wronskian of $rf^s_{m,n}$ and $rh^s_{m,n}$ is calculated as follows:

$$W(rf^s_{m,n}, rh^s_{m,n}) = rf^s_{m,n}(rh^s_{m,n})' - (rf^s_{m,n})'rh^s_{m,n}$$
$$= 2is\, c_{m,n}(-s)\,.$$

Now using the theory of Kodaira and Titchmarsh (see [8], [17]), for any f in $C_0^\infty(0, \infty)$ we have

$$f(t)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dv}{c_{m,n}(-v)}\left(h^{v}_{m,n}(t)\int_0^t f(u)f^{v}_{m,n}(u)r^2(u)\,du + f^{v}_{m,n}(t)\int_t^{\infty} f(u)h^{v}_{m,n}(u)r^2(u)du\right)$$

$$- i\sum_j\left\{\operatorname*{Res}_{s=s(j)}\frac{1}{c_{m,n}(-s)}\right\}\left(h^{s(j)}_{m,n}(t)\int_0^t f(u)f^{s(j)}_{m,n}(u)r^2(u)du + f^{s(j)}_{m,n}(t)\int_t^{\infty} f(u)h^{s(j)}_{m,n}(u)r^2(u)du\right)$$

$$= \frac{1}{2\pi}\int_0^{\infty}\frac{dv}{|c_{m,n}(v)|^2}f^{v}_{m,n}(t)\int_0^{\infty} f(u)f^{v}_{m,n}(u)r^2(u)du$$

$$- i\sum_j\left\{\operatorname*{Res}_{s=s(j)}\frac{1}{c_{m,n}(-s)}\right\}\frac{1}{c_{m,n}(s(j))}f^{s(j)}_{m,n}(t)\int_0^{\infty} f(u)f^{s(j)}_{m,n}(u)r^2(u)du\,,$$

where the sum is taken over all integers j; $-p<2j<m-|n|$.

§5. The Plancherel formula for K-finite functions

For $\phi\in X_s$, we define $\phi_{m,n}\in X_{s,m,n}$ by

$$\phi_{m,n}(g) = d_{m,n}\int_K \chi_{m,n}(k)\phi(kg)\,dk\,.$$

Then

$$K_j = \{\phi\in X_{s(j)};\ \phi_{m,n}=0 \quad \text{if}\ \ 2j<m-|n|\},$$

$$U_j = \{\phi\in X_{-s(j)};\ \phi_{m,n}=0 \quad \text{if}\ \ 2j\geqq m-|n|\}.$$

(For the proof, see [6]).

Let $\Psi^s_{m,n}$; $G\to V_{m,n}$ be the function defined by

$$\Psi^s_{m,n}(ka_tn) = e^{(i\bar{s}-p)t}\pi_{m,n}(k)e_{m,n} \quad (k\in K,\ a_t\in A,\ n\in N),$$

and let $\psi^r_{m,n}(g)=(e_{m,n},\ \Psi^s_{m,n}(g))$. Then $D_1(\mathscr{P}_s\psi^s_{m,n}(a_t))=0$. This implies that there exists a complex number $\alpha^s_{m,n}$ such that

$$\mathscr{P}_s\psi^s_{m,n}(a_t) = \alpha^s_{m,n}f^s_{m,n}(t) \qquad (t>0).$$

We shall now compute the constant $\alpha^s_{m,n}$.
If $\operatorname{Im} s$ is a sufficiently large negative value, then

$$
\begin{aligned}
&e^{-(is-p)t}\mathscr{P}_s\psi^s_{m,n}(a_t)\\
&= e^{-(is-p)t}\int_K \chi_s(a_t^{-1}k)(e_{m,n}, \pi_{m,n}(k)e_{m,n})\,dk\\
&= e^{-(is-p)t}\int_K \chi_s(a_t k)(\pi_{m,n}(m_0)e_{m,n}, \pi_{m,n}(k)e_{m,n})\,dk\\
&= e^{-(is-p)t}\int_{\bar N} \chi_s(a_t\bar n)(\pi_{m,n}(m_0)e_{m,n}, \pi_{m,n}(k(\bar n))e_{m,n})e^{-(is\lambda_0+\rho)H(\bar n)}\,d\bar n\\
&= \int_{\bar N} \chi_s(a_t\bar n a_t^{-1})(\pi_{m,n}(m_0)e_{m,n}, \pi_{m,n}(k(\bar n))e_{m,n})e^{-(is\lambda_0+\rho)H(\bar n)}\,d\bar n\\
&\quad\longrightarrow \int_{\bar N} e^{-(is\lambda_0+\rho)H(\bar n)}(\pi_{m,n}(m_0)e_{m,n}, \pi_{m,n}(k(\bar n))e_{m,n})\,d\bar n\\
&\quad= \Big(e_{m,n}, \pi_{m,n}(m_0)\int_{\bar N} e^{(i\bar s\lambda_0-\rho)H(\bar n)}\pi_{m,n}(k(\bar n))e_{m,n}\,d\bar n\Big) \quad (t \longrightarrow +\infty).
\end{aligned}
$$

Moreover the result of [6] says that

$$
\begin{aligned}
&\pi_{m,n}(m_0)\int_{\bar N} e^{(i\bar s\lambda_0-\rho)H(\bar n)}\pi_{m,n}(k(\bar n))e_{m,n}\,d\bar n\\
&= c(-\bar s)\prod_{k=1}^{(m-n)/2}\frac{2k-2+p+i\bar s}{2k-2+p-i\bar s}\prod_{k=1}^{(m+n)/2}\frac{2k-2+p+i\bar s}{2k-2+p-i\bar s}\,e_{m,n}.
\end{aligned}
$$

Therefore, if $\operatorname{Im} s$ is a sufficiently large negative value, we have

$$
\begin{aligned}
&e^{-(is-p)t}\mathscr{P}_s\psi^s_{m,n}(a_t)\\
&\longrightarrow c(s)\prod_{k=1}^{(m-n)/2}\frac{2k-2+p-is}{2k-2+p+is}\prod_{k=1}^{(m+n)/2}\frac{2k-2+p-is}{2k-2+p+is}\\
&\qquad\qquad (t \longrightarrow +\infty).
\end{aligned}
$$

On the other hand, if $\operatorname{Im} s<0$, then

$$
\begin{aligned}
e^{-(is-p)t}f^s_{m,n}(t) &= e^{-(is-p)t}(c_{m,n}(s)h^s_{m,n}(t)+c_{m,n}(-s)h^{-s}_{m,n}(t))\\
&\longrightarrow 2^{p-is}c_{m,n}(s) \qquad (t \longrightarrow +\infty).
\end{aligned}
$$

Consequently,

$$
\alpha^s_{m,n}=2^{is-p}c(s)\frac{1}{c_{m,n}(s)}\prod_{k=1}^{(m-n)/2}\frac{2k-2+p+is}{2k-2+p+is}\prod_{k=1}^{(m+n)/2}\frac{2k-2+p-is}{2k-2+p+is}.
$$

By the principle of analytic continuation, the above formula is valid for all s in

C. Using this formula, we obtain

$$\operatorname{Ker}(\Gamma(is)^{-1}\mathscr{P}_s)|_{s=s(j)} = K_j .$$

Hence the operator $\mathscr{P}^j\colon X_{s(j)}/K_j \to C^\infty(G/H)$ is defined by

$$\mathscr{P}^j([\phi]) = (\Gamma(is)^{-1}\mathscr{P}_s)|_{s=s(j)}(\phi) \qquad ([\phi]\in X_{s(j)}/K_j).$$

Now let $f(x)=(e_{m,n}, \Phi(x))$ $(\Phi\in C_0^\infty(G/H, V_{m,n}))$. Then it is clear that $\prod_s f(g)=\prod_s f(e)\psi^s_{m,n}(g)$. Moreover

$$\begin{aligned}
\prod\nolimits_s f(e) &= \int_{G/H} \chi_{-s}(x^{-1})(e_{m,n}, \Phi(x))\,dx \\
&= \int_0^\infty du\, r^2(u)\int_K \chi_{-s}(a_u^{-1}k^{-1})(e_{m,n}, \Phi(ka_u))\,dk \\
&= \int_0^\infty du\, r^2(u)\int_K \chi_{-s}(a_u^{-1}k)(\pi_{m,n}(k)e_{m,n}, \Phi(a_u))\,dk \\
&= \int_0^\infty du\, r^2(u) f(a_u)\int_K \chi_{-s}(a_u^{-1}k)(\pi_{m,n}(k)e_{m,n}, e_{m,n})\,dk \\
&= \int_0^\infty du\, r^2(u) f(a_u)\overline{\mathscr{P}_{\bar{s}}\psi^{\bar{s}}_{m,n}(a_u)} \\
&= a^{-s}_{m,n}\int_0^\infty f(a_u) f^s_{m,n}(u) r^2(u)\,du .
\end{aligned}$$

Thus

$$\prod\nolimits_s f(g) = \left(\alpha^{-s}_{m,n}\int_0^\infty f(a_u) f^s_{m,n}(u) r^2(u)\,du\right)\psi^s_{m,n}(g).$$

In particular we have that $\operatorname{Im}\prod_-^j = U_j$.

We can then prove the following

PROPOSITION. *Let* $f(x)=(v, \Phi(x))$ $(v\in V_{m,n}, \Phi\in C_{00}^\infty(G/H, V_{m,n}))$. *Then*

$$\begin{aligned}
f(x) &= \frac{2^{2p}}{2\pi}\int_0^\infty \frac{dv}{|c(v)|^2}\mathscr{P}_v\prod\nolimits_v f(x) \\
&\quad + \frac{2^p}{(p-1)!}\sum_j \frac{\{\Gamma(p+j)\}^2}{\Gamma(p+2j)\,(p+2j)!}\mathscr{P}^j\prod\nolimits^j f(x) \quad (x\in KA^+(eH)).
\end{aligned}$$

PROOF. We need only verify that

$$f(a_t) = \frac{2^{2p}}{2\pi}\int_0^\infty \frac{dv}{|c(v)|^2}\mathscr{P}_v\prod\nolimits_v f(a_t)$$

$$+\frac{2^p}{(p-1)!}\sum_j\frac{\{\Gamma(p+j)\}^2}{\Gamma(p+2j)\,(p+2j)!}\mathscr{P}^j\prod^j f(a_t)\quad (t>0)\,.$$

Furthermore we may assume that $v=e_{m,n}$. Then, taking account of the above results, we have

$$(\mathscr{P}_s\prod_s f)(a_t)=\alpha^s_{m,n}\alpha^{-s}_{m,n}f^s_{m,n}(t)\int_0^\infty f(a_u)f^s_{m,n}(u)r^2(u)\,du\,.$$

If ν is a positive real number,

$$\alpha^\nu_{m,n}\alpha^{-\nu}_{m,n}=|\alpha^\nu_{m,\nu}|^2=2^{-2p}|c(\nu)|^2|c_{m,n}(\nu)|^{-2}\,.$$

Hence it follows that

$$2^{2p}\frac{1}{|c(\nu)|^2}\mathscr{P}_\nu\prod_\nu f(a_t)=\frac{1}{|c_{m,n}(\nu)|^2}f^\nu_{m,n}(t)\int_0^\infty f(a_u)f^\nu_{m,n}(u)r^2(u)\,du\,.$$

On the other hand if $\phi\in X_{s,m,n}$, then

$$A_s(\phi)=2^{is}\Gamma(is)^{-1}c(s)\prod_{k=1}^{(m-n)/2}\frac{2k-2+p-is}{2k-2+p+is}\prod_{k=1}^{(m+n)/2}\frac{2k-2+p-is}{2k-2+p+is}\,\phi$$

(see [6]).
This implies that

$$\frac{2^p}{(p-1)!}\frac{\{\Gamma(p+j)\}^2}{\Gamma(p+2j)\,(p+2j)!}\mathscr{P}^j\prod^j f(a_t)$$

$$=\begin{cases}-i\left\{\operatorname*{Res}_{s=s(j)}\dfrac{1}{c_{m,n}(-s)}\right\}\dfrac{1}{c_{m,n}(s(j))} & \\ \quad\times f^{s(j)}_{m,n}(t)\displaystyle\int_0^\infty f(a_u)f^{s(j)}_{m,n}(u)r^2(u)\,du & (2j<m-|n|)\,,\\ 0 & (2j\geqq m-|n|)\,.\end{cases}$$

Therefore, owing to the formula derived in §4, we obtain the desirable result as follows:

$$f(a_t)=\frac{2^{2p}}{2\pi}\int_0^\infty\frac{d\nu}{|c(\nu)|^2}\mathscr{P}_\nu\prod_\nu f(a_t)$$
$$+\frac{2^p}{(p-1)!}\sum_j\frac{\{\Gamma(p+j)\}^2}{\Gamma(p+2j)\,(p+2j)!}\mathscr{P}^j\prod^j f(a_t)\,.$$

§6. The proof of our theorem

We are now in a position to prove our main theorem. We easily see that, if $\phi\in X_s$ and $f\in C_0^\infty(G/H)$, then

$$\int_{G/H} \mathscr{P}_s \phi(x) \overline{f(x)}\, dx = \int_K \phi(k) \overline{\prod_{\bar{s}} f(k)}\, dk .$$

Therefore, from the definition of the inner products, we have

$$\int_{G/H} \mathscr{P}_\nu \prod_\nu f(x) \overline{f(x)}\, dx = (\prod_\nu f, \prod_\nu f)_\nu \qquad (\nu \in R, f \in C_0^\infty(G/H))$$

and

$$\int_{G/H} \mathscr{P}^j \prod^j f(x) \overline{f(x)}\, dx = (\prod^j f, \prod^j f)^j \qquad (f \in C_0^\infty(G/H)).$$

Now let $f(x) = (v, \Phi(x))$ $(v \in V_{m,n}, \ \Phi \in C_{00}^\infty(G/H, \ V_{m,n}))$. Then, according to Proposition in § 5, we find that

$$\begin{aligned}(f, f) &= \frac{2^{2p}}{2\pi} \int_{G/H} \left(\int_0^\infty \frac{d\nu}{|c(\nu)|^2} \mathscr{P}_\nu \prod_\nu f(x) \right) \overline{f(x)}\, dx \\ &\quad + \frac{2^p}{(p-1)!} \int_{G/H} \left(\sum_j \frac{\{\Gamma(p+j)\}^2}{\Gamma(p+2j)\,(p+2j)!} \mathscr{P}^j \prod^j f(x) \right) \overline{f(x)}\, dx \\ &= \frac{2^{2p}}{2\pi} \int_0^\infty \frac{d\nu}{|c(\nu)|^2} (\prod_\nu f, \prod_\nu f)_\nu \\ &\quad + \frac{2^p}{(p-1)!} \sum_j \frac{\{\Gamma(p+j)\}^2}{\Gamma(p+2j)\,(p+2j)!} (\prod^j f, \prod^j f)^j .\end{aligned}$$

The proof of our theorem is now complete.

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