HIROSHIMA MATH. J.
9 (1979), 321–322

Non-Existence of Torsion Free Flat Connections on Reductive Homogeneous Spaces

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(Received December 10, 1978)

1. Introduction

In [2], K. Nomizu studied invariant connections on homogeneous spaces. In particular, he proved that there exists one and only one canonical invariant affine connection which is torsion free (cf. [2, Theorem 10.1]). Now we are interested in an invariant affine connection which is not necessarily canonical. In this paper we shall prove that there exist no invariant torsion free flat affine connections on reductive homogeneous spaces of semisimple Lie groups. This generalizes the result of Matsushima-Okamoto [3].

2. Proof of the Theorem

A homogeneous space G/H of a Lie group G is called reductive if the following condition is satisfied: in the Lie algebra $\mathfrak{g}$ of G there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ (direct sum of vector spaces) and $\mathrm{Ad}(h)\mathfrak{m}=\mathfrak{m}$ for all $h\in H$, where $\mathfrak{h}$ is the subalgebra of corresponding to the identity component of H. From now on, we shall consider a fixed decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ of the Lie algebra satisfying the above condition. Since $[\mathfrak{h}, \mathfrak{m}]\subset\mathfrak{m}$, we shall regard ad W as an endomorphism of $\mathfrak{m}$ for each $W\in\mathfrak{h}$. We denote by $X_{\mathfrak{h}}$ and $X_{\mathfrak{m}}$, the $\mathfrak{h}$-component of X and the $\mathfrak{m}$-component of X ($X\in\mathfrak{g}$), respectively. An affine connection on a reductive homogeneous space G/H is called G-invariant if every element of G acts on G/H as an affine transformation.

THEOREM. *Let G/H be a reductive homogeneous space of a real semisimple Lie group. Then G/H has no G-invariant torsion free flat affine connections.*

PROOF. Assume that there exists a G-invariant torsion free flat affine connection. Let α be the corresponding connection function, i.e. the bilinear function on $\mathfrak{m}\times\mathfrak{m}$ with values in $\mathfrak{m}$ which is invariant by $\mathrm{Ad}(h)$ for all $h\in H$ (see, [2, Theorem 8.1]). We write $\alpha(X)Y$ for $\alpha(X, Y)$. It is the immediate consequence of [2, Theorem 8.1, (9.1), (9.6)] that

(1) $$[\mathrm{ad}\, W, \alpha(X)] = \alpha(\mathrm{ad}\, W(X)) \quad \text{for} \quad W\in\mathfrak{h} \quad \text{and} \quad X\in\mathfrak{m},$$

(2) $$\alpha(X)Y-\alpha(Y)X = [X, Y]_{\mathfrak{m}} \quad \text{for} \quad X, Y\in\mathfrak{m},$$

$$(3) \qquad [\alpha(X), \alpha(Y)] = \alpha([X, Y]_{\mathfrak{m}}) + \mathrm{ad}\,[X, Y]_{\mathfrak{h}} \qquad \text{for} \quad X, Y \in \mathfrak{m}.$$

Put $P(X) = \mathrm{ad}\, X_{\mathfrak{h}} + \alpha(X_{\mathfrak{m}})$ for $X \in \mathfrak{g}$. Then it is easy to check by (1) and (3) that P is a representation of $\mathfrak{g}$ on $\mathfrak{m}$. We denote by $C^q(\mathfrak{g}, \mathfrak{m})$, $H^q(\mathfrak{g}, P)$ the space of q-dimensional $\mathfrak{m}$-cochains, the q-dimensional cohomology group associated with the representation P, respectively (see, [1, Chapter IV]). We define $c \in C^1(\mathfrak{g}, \mathfrak{m})$ by $c(X) = X_{\mathfrak{m}}$ $(X \in \mathfrak{g})$. Then we have from (2) that $dc(X, Y) = -c([X, Y]) + P(X)c(Y) - P(Y)c(X) = 0$. Now G is semisimple. This implies $H^1(\mathfrak{g}, P) = 0$ (cf. [1, Theorem 25.1]). Therefore there exists an element Y in $C^0(\mathfrak{g}, \mathfrak{m})$ $(=\mathfrak{m})$ such that $dY = c$. For any $X \in \mathfrak{m}$, we have $X = c(X) = dY(X) = \alpha(X)Y$. Hence it follows from (2) that

$$P(Y)X = \alpha(Y)X = X + [Y, X]_{\mathfrak{m}} \qquad \text{for all} \quad X \in \mathfrak{m}.$$

Since $[Y, \mathfrak{h}] \subset \mathfrak{m}$, this implies $\mathrm{tr}_{\mathfrak{m}}\,[Y, \cdot\,]_{\mathfrak{m}} = \mathrm{tr}_{\mathfrak{g}}\,\mathrm{ad}\, Y = 0$. So it shows that $\mathrm{tr}_{\mathfrak{m}}\, P(Y) = \dim \mathfrak{m}$. On the other hand we have that $\mathrm{tr}_{\mathfrak{m}}\, P(Y) = 0$, because $\mathfrak{g}$ is semisimple. This is absurd. Q. E. D.

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