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Hopf algebra of class functions and inner plethysms

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§ 1. Introduction

This paper is dedicated to Professor Tatsuji Kudo on the occasion of his sixtieth birthday.

The first part of this paper is concerned with a detailed account of Hopf algebra structure of class functions on the symmetric groups and shows how the study incorporates many results in the classical theory of symmetric groups. The second part deals with the operation called inner plethysm. Few calculations have been made for the operation. An attempt is made in this paper to illustrate all necessary procedures for evaluating any inner plethysm, although they may be extremely involved in practice.

In §2 it is shown that the ring $C_{\boldsymbol{Z}}$ of integer-valued class functions on the symmetric groups is a divided polynomial Hopf ring in infinite generators, while the algebra $C_{\boldsymbol{F}}$ over the complex field forms a Hopf polynomial algebra. In §3 the self-duality of $C_{\boldsymbol{F}}$ is established and Newton's formula is obtained in $C_{\boldsymbol{F}}$. A short proof of Frobenius' fundamental theorem is given in §4, by taking advantage of Newton's polynomial established in §3. In §5 a $C_{\boldsymbol{F}}$-version of Liulevicius' self-duality is studied. The structure of the representation ring $R_{\boldsymbol{Z}}$ of symmetric groups is studied in §6. In §7 Atiyah's $\Delta_{n,k}$ is discussed to recover Doubilet's forgotten symmetric functions. The general theory of inner plethysms is given in the final section §8.

§ 2. Hopf algebra of class functions

Let R be a commutative ring with unity and let G be a finite group. By a R-valued class function on G we mean $\zeta : G \to R$ satisfying $\zeta(y^{-1}xy) = \zeta(x)$ for any $x, y \in G$. $C_R(G)$ denotes the R-module of R-valued class functions on G. In the sequel R will be the complex field $\boldsymbol{F}$ or the ring of integers $\boldsymbol{Z}$. For a subgroup H in G, the inclusion map $i : H \to G$ induces the restriction map $i^! = \mathrm{Res}_H^G$: $C_R(G) \to C_R(H)$ and the induction map $i_! = \mathrm{Ind}_H^G : C_R(H) \to C_R(G)$. For $f \in C_R(H)$ and for any $s \in G$,

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$$(\mathrm{Ind}_H^G f)(s) = (1/|H|) \textstyle\sum_{t\in G, t^{-1}st\in H} f(t^{-1}st).$$

Consider a graded connected R-module $C_R=\{C_R(S_n)\,|\,n=0, 1, 2,\ldots\}$ for the symmetric group S_n of degree n. We are going to define a multiplication m: $C_R\otimes C_R\to C_R$ so that C_R forms a graded algebra. Let

$$i_{p,q}\colon\ S_p \times S_q \longrightarrow S_{p+q}$$

be an embedding defined by

$$\begin{aligned} i_{p,q}(\sigma, \tau)(j) &= \sigma(j) && \text{if } 1 \le j \le p, \\ &= p + \tau(j-p) && \text{if } p+1 \le j \le p+q, \end{aligned}$$

for $(\sigma, \tau) \in S_p \times S_q$. If $f_t \in C(S_p)$[1] and $g_s \in C(S_q)$ are characterstic functions of the conjugacy class $\bar{t}$ in S_p and the class $\bar{s}$ in S_q respectively, then the characteristic function h of the conjugacy class $\overline{(t, s)}$ in $S_p \times S_q$ is obtained by

$$h(\sigma, \tau) = f_t(\sigma)\cdot g_s(\tau).$$

Thus there exists the isomorphism

$$\psi_{p,q}\colon C(S_p) \otimes C(S_q) \longrightarrow C(S_p \times S_q).$$

Define $m_{p,q}$: $C(S_p)\otimes C(S_q)\to C(S_{p+q})$ by the composite map $i_{p,q!}\circ\psi_{p,q}=\mathrm{Ind}_{S_p\times S_q}^{S_{p+q}}\circ \psi_{p,q}$.

Given a partition π of n. (In notation, $\pi\vdash n$.) An element σ in S_n is said to have the shape π if the disjoint cycle decomposition of σ produces the partition π. A conjugacy class in S_n is said to have the shape π if its representative has the shape π. Let K_π be the characteristic function of a conjugacy class of the shape π. Then $\{K_\pi | \pi\vdash n\}$ forms a base for $C_R(S_n)$.

For any partition π of n, let π_i be the number of i's in $\pi\,(i=1,\ldots, n)$, i.e., $\pi=\{1^{\pi_1}, 2^{\pi_2},\ldots, n^{\pi_n}\}$, and set $\pi!=\prod_{i=1}^n \pi_i!$ and $|\pi|=\pi!\prod_{i=1}^n i^{\pi_i}$. Then the number of the elements in a conjugacy class of the shape π is $n!/|\pi|$.

For any partitions π of p and σ of q, let $\pi\vee\sigma$ denote the partition of $p+q$ given by the union of π and σ, i.e., $(\pi\vee\sigma)_i=\pi_i+\sigma_i\,(i=1, 2,\ldots)$.

PROPOSITION 2.1. *For any $\pi\vdash p$ and $\sigma\vdash q$, we obtain*

$$K_\pi\cdot K_\sigma(= m_{p,q}(K_\pi\otimes K_\sigma)) = ((\pi \vee \sigma)!/\pi!\sigma!)K_{\pi\vee\sigma}.$$

PROOF. For each $s\in S_{p+q}$, consider

$$\begin{aligned} (K_\pi\cdot K_\sigma)(s) &=(\mathrm{Ind}_{S_p\times S_q}^{S_{p+q}}\, \psi_{p,q}(K_\pi\otimes K_\sigma))(s) \\ &=(1/p!q!) \textstyle\sum_{t\in S_{p+q}, t^{-1}st\in S_p\times S_q} \psi_{p,q}(K_\pi\otimes K_\sigma)(t^{-1}st). \end{aligned}$$

1) If no confusion arises, $C(S_p)$ stands for $C_R(S_p)$.

It is obvious that if the shape of s is not $\pi \vee \sigma$, then $(K_\pi \cdot K_\sigma)(s)=0$. When s is of the shape $\pi \vee \sigma$, the number of t with the property $\psi_{p,q}(K_\pi \otimes K_\sigma)(t^{-1}st)=1$ is $(p!/|\pi|)(q!/|\sigma|)|\pi \vee \sigma| = p!q!(\pi \vee \sigma)!/\pi!\sigma!$. This completes the proof.

By virtue of Proposition 2.1, it is immediate to see that $K_\sigma \cdot K_\pi = K_\pi \cdot K_\sigma$ and $(K_\pi \cdot K_\sigma)\cdot K_\nu = K_\pi \cdot (K_\sigma \cdot K_\nu)$ for any partitions σ, π and ν. It follows that C_R forms a graded commutative algebra with unit.

PROPOSITION 2.2. *Let C_i denote $K_{\{i\}} \in C_R(S_i)$ where $\{i\}$ is the shape of the i-cycle, and let C_π denote $C_1^{\pi_1} C_2^{\pi_2} \cdots C_n^{\pi_n} \in C_R(S_n)$ for $\pi \vdash n$. Then we obtain*

$$C_\pi = \pi! K_\pi .$$

PROOF. It is evident from Proposition 2.1.

PROPOSITION 2.3. *C_F is a polynomial algebra over the complex field $\boldsymbol{F}$ in an infinite number of variables $C_1, C_2, \ldots, C_n, \ldots$, where the degree of C_n is $2n$. In notation,*

$$C_F = P_F[C_1, C_2, \ldots, C_n, \ldots].$$

PROOF. It is immediate from Proposition 2.2.

We are going to see that unlike C_F, the algebra C_Z is a divided polynomial ring with generators $C_1, C_2, \ldots, C_n, \ldots$. By a divided polynomial ring $D[x]$ with one generator x of an even degree, we mean a graded abelian group $\{\boldsymbol{Z}x_n | n=0, 1, 2, \ldots\}$ with a base $x_0=1$, $x_1=x$, $x_2, \ldots, x_n, \ldots$, such that the multiplication is given by $x_p \cdot x_q = \binom{p+q}{p} x_{p+q}$. Then $x^n = n! x_n$. By abuse of language x is called a generator of the ring $D[x]$.

PROPOSITION 2.4. *The ring C_Z is a divided polynomial ring $D[C_1, C_2, \ldots, C_n, \ldots] = \otimes_{n=1}^{\infty} D[C_n]$.*

PROOF. It is evident from Propositions 2.1. and 2.2.

Let us consider the elements

$$\alpha_n = \textstyle\sum_{\pi \vdash n} (\operatorname{Sgn} \pi) K_\pi, \quad \beta_n = \sum_{\pi \vdash n} K_\pi \quad \text{and} \quad \gamma_n = nC_n$$

of $C_R(S_n)$, where $\operatorname{Sgn} \pi$ denotes ± 1 according as the shape of π is even or odd. Then it is obvious that $C_F = P_F[\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots]$. In a later section we shall show that $C_F = P_F[\alpha_1, \ldots, \alpha_n, \ldots] = P_F[\beta_1, \ldots, \beta_n, \ldots]$ is also true.

Defining $\Delta_{p,q}\colon C_R(S_n) \to C_R(S_p) \otimes C_R(S_q)$ for each p, q with $p+q=n$, by the composition $\psi_{p,q}^{-1} \circ \operatorname{Res}_{S_p \times S_q}^{S_n}$ and setting

$$\Delta_n\colon C_R(S_n) \longrightarrow \textstyle\sum_{p+q=n} C_R(S_p) \otimes C_R(S_q)$$

by

$$\Delta_n(f) = \sum_{p+q=n} \Delta_{p,q}(f)$$

for any $f \in C_R(S_n)$, we obtain a map $\Delta: C_R \to C_R \otimes C_R$. Define a map $\varepsilon: C_R \to R$ by the projection.

PROPOSITION 2.5. *$\Delta_n(K_\pi) = \sum_{\sigma \vee \nu = \pi} K_\sigma \otimes K_\nu$ for each $\pi \vdash n$.*

PROOF. $\mathrm{Res}^{S_n}_{S_p \times S_q} K_\pi$ takes value 1 on conjugacy classes with the shape π in the canonically embedded subgroup $S_p \times S_q$ of S_n and 0 elsewhere. A pair (s, t) in $S_p \times S_q$ with the property that shapes of s, t are σ, ν is embedded by $i_{p,q}$ to an element with shape $\sigma \vee \nu$, and conversely. Hence the proof is complete.

The coassociativity and the counit conditions for a coalgebra are immediate from Proposition 2.5, because

$$(1 \otimes \Delta)\Delta(K_\pi) = \sum_{\rho \vee \rho' \vee \rho'' = \pi} K_\rho \otimes K_{\rho'} \otimes K_{\rho''} = (\Delta \otimes 1)\Delta(K_\pi),$$

$$(1 \otimes \varepsilon)\Delta(K_\pi) = K_\pi \otimes 1, \quad \text{and} \quad (\varepsilon \otimes 1)\Delta(K_\pi) = 1 \otimes K_\pi.$$

It follows that C_R forms a coalgebra with respect to the comultiplication Δ and the counit ε. Then it is straightforward to see that $\Delta(K_\pi \cdot K_\sigma) = \Delta(K_\pi)\Delta(K_\sigma)$ holds true. Thus we have proved

PROPOSITION 2.6. *C_R is a Hopf algebra.*

This fact is known. For example, see Geissinger [3].

THEOREM 2.7. *C_F is a polynomial Hopf algebra in variables $C_1, C_2, \ldots, C_n, \ldots$, or in variables $\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$. C_Z is a divided polynomial Hopf ring $D[C_1, C_2, \ldots, C_n, \ldots]$.*

As a matter of fact, C_F is a polynomial Hopf algebra if F is a field of characteristic 0.

LEMMA 2.8. *$\Delta(\alpha_n) = \sum_{i+j=n} \alpha_i \otimes \alpha_j$, $\Delta(\beta_n) = \sum_{i+j=n} \beta_i \otimes \beta_j$, and*

$$\Delta(\gamma_n) = 1 \otimes \gamma_n + \gamma_n \otimes 1.$$

PROOF.
$$\begin{aligned}\Delta(\alpha_n) &= \textstyle\sum_{\pi \vdash n} (\mathrm{Sgn}\,\pi)\Delta(K_\pi) = \sum_{\pi \vdash n} (\mathrm{Sgn}\,\pi)\left(\sum_{\rho \vee \rho' = \pi} K_\rho \otimes K_{\rho'}\right) \\ &= \textstyle\sum_{i+j=n, \rho \vdash i, \rho' \vdash j} (\mathrm{Sgn}\,(\rho \vee \rho'))\, K_\rho \otimes K_{\rho'} \\ &= \textstyle\sum_{i+j=n} \left(\sum_{\rho \vdash i} (\mathrm{Sgn}\,\rho) K_\rho\right) \otimes \left(\sum_{\rho' \vdash j} (\mathrm{Sgn}\,\rho') K_{\rho'}\right) \\ &= \textstyle\sum_{i+j=n} \alpha_i \otimes \alpha_j.\end{aligned}$$

Similarly, we obtain the last two equalities.

§ 3. Self-duality

By the usual inner product

$$\langle f, g\rangle = (1/n!)\textstyle\sum_{t\in S_n} f(t)\overline{g(t)} \qquad \text{for} \quad f, g \in C_F(S_n),$$

the vector space $C_F(S_n)$ becomes an inner product space over $\boldsymbol{F}$. Then the Frobenius reciprocity theorem states that for any subgroup H in S_n and for $f \in C_F(S_n)$ and $g \in C_F(H)$,

$$\langle \mathrm{Res}_H^{S_n} f, g\rangle = \langle f, \mathrm{Ind}_H^{S_n} g\rangle$$

holds true. If a bilinear form β is defined on C_F by the orthogonal sum such that for $f \in C_F(S_p)$ and $g \in C_F(S_q)$

$$\beta(f, g) = \begin{cases} 0 & \text{if} \quad p \neq q, \\ \langle f, g\rangle & \text{if} \quad p = q, \end{cases}$$

then the graded vector space of finite type C_F becomes an inner product space. It is obvious that β induces a vector space isomorphism $\lambda\colon C_F \to C_F^*$ by the map $\lambda(f) = \beta(f, \ )$ for $f \in C_F$. Since C_F is a Hopf algebra, its dual C_F^* is also a Hopf algebra with multiplication Δ^* and comultiplication m^* if $C_F^* \otimes C_F^*$ is identified with $(C_F \otimes C_F)^*$. It is easy to see that λ is a Hopf algebra isomorphism.

By definition,

$$\langle K_\pi, K_{\pi'}\rangle = (1/n!) \textstyle\sum_{t\in S_n} K_\pi(t) K_{\pi'}(t) = \begin{cases} 0 & \text{if } \pi \neq \pi', \\ 1/|\pi| & \text{if } \pi = \pi'. \end{cases}$$

For a base $\{\gamma_\pi (= \prod_{i=1}^n \gamma_i^{\pi_i}) | \pi \vdash n\}$ for $C_F(S_n)$, we obtain

$$\langle \gamma_\pi, \gamma_{\pi'}\rangle = \begin{cases} 0 & \text{if } \pi \neq \pi', \\ |\pi| & \text{if } \pi = \pi'. \end{cases}$$

It follows that $\{\gamma_\pi\}$ is an orthogonal base. Since

$$\lambda(\gamma_n)(K_\pi) = \langle \gamma_n, K_\pi\rangle = \begin{cases} 0 & \text{if } \pi \neq \{n\}, \\ 1 & \text{if } \pi = \{n\}, \end{cases} \tag{3.1}$$

$\lambda(\gamma_n) = \psi_n$, denoted by Atiyah, maps $K_{\{n\}}$ of the n-cycle into 1 and the other characteristic functions into 0. Thus, we have

PROPOSITION 3.2. *The isomorphism* $\lambda\colon C_F \to C_F^*$ *maps* γ_n *into* ψ_n. *Hence* $C_F^* = P_F[\psi_1, \psi_2, \dots, \psi_n, \dots]$.

THEOREM 3.3. *Let* $\alpha_n=\sum_{\pi\vdash n}(\mathrm{Sgn}\,\pi)K_\pi$ *and let* $\gamma_n = nK_{\{n\}}$. *Then we obtain Newton's formula,*

$$\gamma_n-\alpha_1\gamma_{n-1}+\alpha_2\gamma_{n-2}-\cdots+(-1)^{n-1}\alpha_{n-1}\gamma_1+(-1)^n n\alpha_n = 0. \tag{3.4}$$

PROOF. Denote by $N(\gamma, \alpha)$ the left-hand side of the equation (3.4). If $\lambda(N(\gamma, \alpha))(K_\pi)=\langle N(\gamma, \alpha), K_\pi\rangle=0$ for any $\pi\vdash n$, then we get $N(\gamma, \alpha)=0$. For $i=1,\dots, n$, consider

$$\begin{aligned}\langle(-1)^{n-i}\alpha_{n-i}\gamma_i, K_\pi\rangle &= (-1)^{n-i}\langle\alpha_{n-i}\otimes\gamma_i, \Delta(K_\pi)\rangle \\ &= (-1)^{n-i}\textstyle\sum_{\rho\vee\rho'=\pi}\langle\alpha_{n-i}\otimes\gamma_i, K_\rho\otimes K_{\rho'}\rangle \\ &= (-1)^{n-i}\textstyle\sum_{\rho\vee\rho'=\pi}\langle\alpha_{n-i}, K_\rho\rangle\langle\gamma_i, K_{\rho'}\rangle.\end{aligned}$$

If π does not contain i as a member, i.e., $\pi_i=0$, then the last summation is 0 because $\langle\gamma_i, K_{\rho'}\rangle=0$ for any ρ' with $\rho\vee\rho'=\pi$ by (3.1). Assume $\pi_i\neq 0$. Then by removing i from π, we obtain a partition $\pi\wedge\{i\}$ of $n-i$ with $(\pi\wedge\{i\})\vee\{i\}=\pi$, and we get

$$\begin{aligned}&\langle(-1)^{n-i}\alpha_{n-i}\gamma_i, K_\pi\rangle = (-1)^{n-i}\langle\alpha_{n-i}, K_{\pi\wedge\{i\}}\rangle \qquad \text{(by (3.1))} \\ &\quad=(-1)^{n-i}\langle\textstyle\sum_{\pi'\vdash n-i}(\mathrm{Sgn}\,\pi')K_{\pi'}, K_{\pi\wedge\{i\}}\rangle = (-1)^{n-i}(\mathrm{Sgn}\,(\pi\wedge\{i\}))/|\pi\wedge\{i\}|.\end{aligned}$$

Since $\mathrm{Sgn}\,(\pi\wedge\{i\})=(\mathrm{Sgn}\,\pi)(-1)^{i+1}$ and $|\pi\wedge\{i\}|=|\pi|/\pi_i i$, we obtain

$$\langle(-1)^{n-i}\alpha_{n-i}\gamma_i, K_\pi\rangle = (-1)^{n+1}(\mathrm{Sgn}\,\pi)\pi_i i/|\pi|\,.$$

Hence for any $\pi\vdash n$,

$$\begin{aligned}\langle N(\alpha, \gamma), K_\pi\rangle &=\textstyle\sum_{i=1}^n(-1)^{n+1}(\mathrm{Sgn}\,\pi)\pi_i i/|\pi| + (-1)^n n\langle\alpha_n, K_\pi\rangle \\ &=(-1)^{n+1}(\mathrm{Sgn}\,\pi)n/|\pi| + (-1)^n n(\mathrm{Sgn}\,\pi)/|\pi| = 0.\end{aligned}$$

This completes the proof.

Solving the system of linear equations in Theorem 3.3 with respect to $\gamma_1,\dots, \gamma_n$, we obtain $\gamma_n=Q_n(\alpha_1, \alpha_2,\dots, \alpha_n)$, which is the well-known n-th Newton polynomial with coefficients in $\boldsymbol{Z}$. Solving the system with respect to $\alpha_1,\dots, \alpha_n$, we also have $\alpha_n=\bar{Q}(\gamma_1, \gamma_2,\dots, \gamma_n)$ with coefficients in the rationals.

COROLLARY 3.5 (Girard's formula). *Set* $\alpha_\pi=\alpha_1^{\pi_1}\cdots\alpha_n^{\pi_n}$ *for* $\pi\vdash n$. *Then*

$$\gamma_n = (-1)^n n\textstyle\sum_{\pi\vdash n}(-1)^{\pi_1+\cdots+\pi_n}((\pi_1+\cdots+\pi_n-1)!/\pi_1!\cdots\pi_n!)\alpha_\pi.$$

PROOF. It is an immediate consequence of the fact that $\gamma_n=Q_n(\alpha_1,\dots, \alpha_n)$. (See, for example, p. 195 in [9].)

Similarly we can prove

PROPOSITION 3.6. *$\gamma_n+\beta_1\gamma_{n-1}+\cdots+\beta_{n-1}\gamma_1-n\beta_n=0$ holds true. Hence $(-1)^{n-1}\gamma_n=Q_n(\beta_1,\dots,\beta_n)$, $\beta_n=\bar{Q}(\gamma_1,-\gamma_2,\dots,(-1)^{n-1}\gamma_n)$, and*

$$\gamma_n=-n\sum_{\pi\vdash n}(-1)^{\pi_1+\cdots+\pi_n}((\pi_1+\cdots+\pi_n-1)!/\pi_1!\cdots\pi_n!)\beta_\pi, \text{ where } \beta_\pi=\beta_1^{\pi_1}\cdots\beta_n^{\pi_n}.$$

§ 4. Frobenius' fundamental theorem

Let $H_{n,k}=\mathrm{Sym}_k[x_1,x_2,\dots,x_n]$ be the R-module of symmetric functions of degree k in n variables $x_1,x_2,\dots,x_n$ and let $\pi^n_m\colon H_{n,k}\to H_{m,k}$ for non-negative integers n, m with $n\geq m$ be defined by

$$\pi^n_m(f(x_1,\dots,x_n))=f(x_1,\dots,x_m,0,\dots,0).$$

Then $\{H_{n,k};\pi^n_m\}$ forms an inverse system of R-modules. Consider $H_{,k}=\varprojlim_n H_{n,k}$. Then the n-th projection $\pi_{n,k}\colon H_{,k}\to H_{n,k}$ is an isomorphism if $n\geq k$. Let $a_{n,k}$, $h_{n,k}$, and $s_{n,k}$ be the k-th elementary, homogeneous, and power symmetric functions in n variables, whose inverse images under $\pi_{n,k}$ are denoted by a_k, h_k, and s_k, respectively. They are called the k-th elementary, homogeneous, and power symmetric functions in infinite variables $x_1,x_2,\dots,x_n,\dots$. It is obvious that $a_k=(0,\dots,0,a_{k,k},a_{k+1,k},\dots)$, $h_k=(h_{1,k},\dots,h_{k,k},h_{k+1,k},\dots,)$, and $s_k=(s_{1,k},\dots,s_{k,k},s_{k+1,k},\dots)$. The graded R-module $H_R=\{H_{,k}|k=0,1,2,\dots\}$ forms an R-algebra by defining

$$\pi_{n,p+q}(f\cdot g)=\pi_{n,p}(f)\cdot\pi_{n,q}(g)$$

for $f\in H_{,p}$ and $g\in H_{,q}$. It is well known ([3], [4]) that H_R is a polynomial Hopf algebra $P_R[a_1,\dots,a_n,\dots]=P_R[h_1,\dots,h_n,\dots]$ if we define a comultiplication $\Delta(a_n)=\sum_{i+j=n}a_i\otimes a_j$, $\Delta(h_n)=\sum_{i+j=n}h_i\otimes h_j$, and the obvious counit. When $R=\boldsymbol{F}$, then $H_{\boldsymbol{F}}$ is known to form $P_{\boldsymbol{F}}[s_1,\dots,s_n,\dots]$ with $\Delta(s_n)=1\otimes s_n+s_n\otimes 1$.

In this section we shall study the fundamental theorem due to Frobenius by bridging between $C_{\boldsymbol{F}}$ and $H_{\boldsymbol{F}}$ rather than between the representation algebras $R_{\boldsymbol{F}}$ and $H_{\boldsymbol{F}}$. By this way our approach will hardly employ representation theoretic arguments.

THEOREM 4.1. *A map $T\colon C_{\boldsymbol{F}}\to H_{\boldsymbol{F}}$ defined by $T(\gamma_m)=s_m$ is a Hopf algebra isomorphism such that $T(\alpha_\pi)=a_\pi(=a_1^{\pi_1}\cdots a_n^{\pi_n})$ and $T(\beta_\pi)=h_\pi(=h_1^{\pi_1}\cdots h_n^{\pi_n})$ for $\pi\vdash n$.*

PROOF. From Theorem 2.7, $C_{\boldsymbol{F}}=P_{\boldsymbol{F}}[\gamma_1,\dots,\gamma_n,\dots]$ with $\Delta(\gamma_n)=1\otimes\gamma_n+\gamma_n\otimes 1$. Hence T is a Hopf algebra isomorphism. Thus $T(\alpha_n)=T(\bar{Q}(\gamma_1,\dots,\gamma_n))=\bar{Q}(T(\gamma_1),\dots,T(\gamma_n))=\bar{Q}(s_1,\dots,s_n)=a_n$ and $T(\alpha_\pi)=a_\pi$, by Corollary 3.5. Similarly, $T(\beta_n)=h_n$ and $T(\beta_\pi)=h_\pi$ by Proposition 3.6. This completes the proof.

COROLLARY 4.2. $C_{\boldsymbol{F}}=P_{\boldsymbol{F}}[\alpha_1,\alpha_2,\dots,\alpha_n,\dots]=P_{\boldsymbol{F}}[\beta_1,\beta_2,\dots,\beta_n,\dots]$.

PROOF. It is evident from Theorem 4.1.

Let $R_F(S_n)$ be the Grothendieck $\boldsymbol{F}$-vector space of isomorphism classes of complex representations of S_n. Then it is well known (for example, see [10]) that the character map χ_n: $R_F(S_n)\to C_F(S_n)$ is an isomorphism.

As in the case of C_F, we define $m_{p,q}$: $R_F(S_p)\otimes R_F(S_q)\to R_F(S_{p+q})$ and Δ_n: $R_F(S_n)\to\sum_{p+q=n}R_F(S_p)\otimes R_F(S_q)$ by $\mathrm{Ind}_{S_p\times S_q}^{S_{p+q}}\circ\psi_{p,q}$ and $\sum_{p+q=n}\psi_{p,q}^{-1}\circ\mathrm{Res}_{S_p\times S_q}^{S_n}$, respectively. Since χ commutes with $\psi_{p,q}$, $\mathrm{Ind}_{S_p\times S_q}^{S_n}$ and $\mathrm{Res}_{S_p\times S_q}^{S_n}$, χ defines a Hopf algebra isomorphism from $R_F=\{R_F(S_n)\}$ to C_F.

For each $\pi\vdash n$, let S_π stand for $\overbrace{S_1\times\cdots\times S_1}^{\pi_1}\times\cdots\times\overbrace{S_n\times\cdots\times S_n}^{\pi_n}=S_1^{\pi_1}\times\cdots\times S_n^{\pi_n}$. Then a trivial representation and a sign representation of S_π are denoted by 1_{S_π} and $\mathrm{Alt}\,S_\pi$ respectively. Let 1_{S_π} and $\mathrm{Alt}\,S_\pi$ represent elements ρ_π and η_π in R_F respectively. If ρ_n and η_n denote $\rho_{\{n\}}$ and $\eta_{\{n\}}$, then by definition $\chi(\rho_n)=\beta_n$ and $\chi(\eta_n)=\alpha_n$.

PROPOSITION 4.3. *χ: $R_F\to C_F$ is a Hopf algebra isomorphism such that $\chi(\rho_\pi)=\beta_\pi$ and $\chi(\eta_\pi)=\alpha_\pi$.*

PROOF. It is easy to check that $\rho_\pi=\rho_1^{\pi_1}\cdots\rho_n^{\pi_n}$ and $\eta_\pi=\eta_1^{\pi_1}\cdots\eta_n^{\pi_n}$ for any partition $\pi\vdash n$. This completes the proof.

Defining F: $R_F\to H_F$ by the composite $T\circ\chi$, we obtain the fundamental theorem:

PROPOSITION 4.4. *The Frobenius isomorphism F: $R_F\to H_F$ maps $\boldsymbol{F}$-basis elements $\rho_\pi=[\mathrm{Ind}_{S_\pi}^{S_n}1_{S_\pi}]$ into h_π and $\eta_\pi=[\mathrm{Ind}_{S_\pi}^{S_n}\mathrm{Alt}\,S_\pi]$ into a_π.*

§5. Liulevicius' self-duality and Atiyah's Δ'

Let $\{V_\pi\}$ be the base consisting of irreducible representations of S_n and let $\langle V_\pi, V_{\pi'}\rangle=\delta_{\pi,\pi'}$. It is well known that the character isomorphism χ: $R_F\to C_F$ preserves inner products. Then an isomorphism μ: $R_F\to R_F^*$ with a commutative diagram

$$\begin{array}{ccc} R_F & \xrightarrow{\chi} & C_F \\ \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\lambda} \\ R_F^* & \xleftarrow{\chi^*} & C_F^* \end{array}$$

is evidently obtained by $\mu([M])([N])=\langle M, N\rangle$ for any representations M and N of symmetric groups. This comes from the verification that $(\chi^*\lambda\chi([M]))([N])=(\lambda(\chi_M))(\chi_N)=\langle\chi_M, \chi_N\rangle=\langle M, N\rangle$. Atiyah [1] denotes by σ_n and λ_n elements in R_F^* satisfying

$$\sigma_n([V_\pi]) = \begin{cases} 1 & \text{if } V_\pi = 1_{S_n}, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \lambda_n([V_\pi]) = \begin{cases} 1 & \text{if } V_\pi = \operatorname{Alt} S_n, \\ 0 & \text{otherwise.} \end{cases}$$

PROPOSITION 5.1. *μ: $R_F \to R_F^*$ is a Hopf algebra isomorphism such that $\mu(\rho_n)=\sigma_n$ and $\mu(\eta_n)=\lambda_n$. Hence $R_F^* = P_F[\sigma_1,\dots,\sigma_n,\dots] = P_F[\lambda_1,\dots,\lambda_n,\dots]$.*

PROOF. $$\mu(\rho_n)([V_\pi]) = \langle 1_{S_n}, V_\pi \rangle = \begin{cases} 1 & \text{if } V_\pi = 1_{S_n}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\mu(\rho_n)=\sigma_n$. Similarly, $\mu(\eta_n)=\lambda_n$. This completes the proof.

Consider a diagram

```tikzcd
R_F \arrow[r, "\chi"] \arrow[d, "\mu"'] \arrow[dr, "F"] & C_F \arrow[d, "T"] \\
R_F^* \arrow[r, "\Delta'"] & H_F
```

where Δ' is Atiyah's isomorphism (Proposition 1.2 and Corollary 1.3 in [1]). Then the diagram commutes, because $\Delta'\mu(\eta_n)=\Delta'(\lambda_n)=a_n$ from Proposition 5.1, (see § 7).

COROLLARY 5.2. *The Frobenius map F is equal to $T\chi=\Delta'\mu$.*

Consider an element $(\alpha_1^n)^*$ in C_F^* which maps α_1^n into 1 and α_π into 0 if $\pi \neq \{1^n\}$. Then we obtain

PROPOSITION 5.3. *λ: $C_F \to C_F^*$ maps β_n into $(\alpha_1^n)^*$.*

PROOF. Observe that $n!\langle \beta_n, \alpha_n \rangle = \sum_{\pi,\pi' \vdash n} n!(\operatorname{Sgn} \pi')\langle K_\pi, K_{\pi'} \rangle = \sum_{\pi \vdash n} n! \cdot (\operatorname{Sgn} \pi)/|\pi| = \sum_{t \in S_n} \operatorname{Sgn} t$. Then we obtain

$$\langle \beta_n, \alpha_n \rangle = 0 \quad \text{if } n \geq 2, \qquad \langle \beta_1, \alpha_1 \rangle = 1.$$

For $\pi \vdash n$, let i be a member of π. Then $\pi = (\pi \wedge \{i\}) \vee \{i\}$ and

$$\begin{aligned} \langle \beta_n, \alpha_\pi \rangle &= \langle \beta_n, \alpha_{\pi \wedge \{i\}} \alpha_i \rangle = \langle \Delta(\beta_n), \alpha_{\pi \wedge \{i\}} \otimes \alpha_i \rangle \\ &= \langle \textstyle\sum_{j=0}^n \beta_{n-j} \otimes \beta_j, \alpha_{\pi \wedge \{i\}} \otimes \alpha_i \rangle = \begin{cases} \langle \beta_{n-i}, \alpha_{\pi \wedge \{i\}} \rangle & \text{if } i = 1, \\ 0 & \text{if } i \geq 2, \end{cases} \end{aligned}$$

by Lemma 2.8 and the above equalities. Therefore we see that

$$\langle \beta_n, \alpha_\pi \rangle = \begin{cases} 1 & \text{if } \pi = \{1^n\}, \\ 0 & \text{otherwise,} \end{cases}$$

by induction on n. This proves the proposition.

PROPOSITION 5.4. *The map* $\ell: C_F \to C_F^*$ *defined by* $\ell(\alpha_n) = (\alpha_1^n)^*$ *is a* C_F-*version of the Liulevicius Hopf algebra isomorphism* ([7]).

PROOF. By Corollary 4.2, $\psi: C_F \to C_F$ defined by $\psi(\alpha_n) = \beta_n$ is an isomorphism. Then $\ell = \lambda \circ \psi$ is an isomorphism. If ℓ is translated via $T: C_F \to H_F$, the Liulevicius isomorphism maps a_n into $(a_1^n)^*$. This completes the proof.

§6. Comment on R_Z

In accordance with Professor Sugawara's suggestion, this section is added to the original draft of the present paper.

By a lattice L in a k dimensional complex vector space V we mean an additive group in V which is generated over $\boldsymbol{Z}$ by a base $\{b_1, b_2, \dots, b_k\}$ for V. Since $\{\rho_\pi | \pi \vdash n\}$ and $\{\eta_\pi | \pi \vdash n\}$ are bases for $R_F(S_n)$ and since $h_\pi = T(\rho_\pi)$ is an integral linear combination of the basis elements $a_\pi = T(\eta_\pi)$ and vice versa, they generate a lattice L_n in $R_F(S_n)$. Then the graded lattice $L = \{L_n\}$ forms a polynomial Hopf ring $P_Z[\rho_1, \rho_2, \dots, \rho_n, \dots] = P_Z[\eta_1, \eta_2, \dots, \eta_n, \dots]$ under operations in R_F. It is also evident that L is a Hopf subring in $R_Z = \{R_Z(S_n)\}$, where $R_Z(S_n)$ is a free abelian group generated by the isomorphism classes of irreducible complex representations of S_n. We are going to show that the inclusion map

$$i: L \longrightarrow R_Z$$

is, in fact, an isomorphism. A bilinear form on R_Z defined by $\langle V_\pi, V_{\pi'} \rangle = \delta_{\pi,\pi'}$ for a base $\{V_\pi | \pi \vdash n\}$ consisting of the irreducible representations of S_n is an inner product on R_Z. Since the group isomorphism $\mu_Z: R_Z \to R_Z^*$ defined by $\mu_Z([M]) = \langle M, \ \rangle$ for any representation M, preserves multiplication and comultiplication by virtue of the Frobenius reciprocity theorem, μ_Z is a Hopf ring isomorphism. Consider a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "i"] \arrow[d, "\mu'_Z"] & R_Z \arrow[d, "\mu_Z"] & \\
 & L^* & R_Z^* \arrow[l, "i^*"'] & 0 \arrow[l]
\end{tikzcd},$$

where $\mu'_Z = \mu_Z | L$. Since the ranks of free groups $R_Z(S_n)$ and L_n are both the number of the partitions of n for each n, Coker i is a torsion group and hence i^* is a monomorphism. Note that $\mu'_Z(\rho_n) = (\eta_1^n)^*$ maps η_1^n into 1 and η_π into 0 if $\pi \neq \{1^n\}$. To see it, we observe that

$$\mu'_Z(\rho_n)(\eta_\pi) = \langle \beta_n, \alpha_\pi \rangle = \begin{cases} 1 & \text{if } \pi = \{1^n\}, \\ 0 & \text{otherwise.} \end{cases}$$

If μ'_Z is proved to be epic, i as well as μ'_Z are isomorphisms because of the commutativity of the diagram shown above.

Let $\mathscr{Q}(L)$ be the cokernel of $I(m)\colon I(L)\otimes I(L)\to I(L)$, where $I(m)$ is the restriction of the multiplication m in L to the augmentation ideal $I(L)=\{L_n|n\geq 1\}$. It is well known ([8]) that μ'_Z is epic iff $\mathscr{Q}(\mu'_Z)\colon \mathscr{Q}(L)\to\mathscr{Q}(L^*)$ is epic. It is evident that $\mathscr{Q}(L_n)$ is a free group whose generator is represented by an indecomposable element ρ_n for each n. If $v_n=Q(\eta_1, \eta_2,\dots, \eta_n)$ which is the n-th Newton polynomial in $\eta_1, \eta_2,\dots, \eta_n$, then v_n is primitive in L_n because $\chi(v_n)=\gamma_n$ and $\Delta(\gamma_n)=1\otimes\gamma_n+\gamma_n\otimes 1$. Since any primitive element in $C_F(S_n)$ is a scalar multiple of $K_{\{n\}}$ and since $nK_{\{n\}}=\gamma_n=(-1)^n\alpha_1^n+\cdots$ by Girard's formula, the subgroup $\mathscr{P}(L_n)$, consisting of primitive elements in L_n, is a free group generated by v_n and is a direct summand of L_n. Consider an exact sequence

$$0 \longrightarrow \mathscr{P}(L_n) \xrightarrow{j_n} L_n \xrightarrow{\tilde{\Delta}} \textstyle\sum_{p=1}^{n-1} L_p\otimes L_{n-p}$$

where $\tilde{\Delta}(x)=\Delta(x)-1\otimes x-x\otimes 1$ for any $x\in L_n$. Since $j=\{j_n\}$ is split, we obtain an exact sequence

$$I(L^*)\otimes I(L^*) \xrightarrow{\tilde{\Delta}^*} I(L^*) \xrightarrow{j^*} \mathscr{P}(L)^* \longrightarrow 0.$$

It follows that $\mathscr{Q}(L^*)=\mathscr{P}(L)^*$ where $\mathscr{P}(L)^*=\{\mathscr{P}(L_n)^*\}$.

Consider a commutative diagram

$$\begin{array}{ccccc} L_n & \xrightarrow{p_n} & \mathscr{Q}(L_n) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \mu_{Z,n}} & & \downarrow{\scriptstyle \mathscr{Q}(\mu_{Z,n})} & & \\ L_n & \xrightarrow{j_n^*} & \mathscr{Q}(L_n^*) & \longrightarrow & 0, \end{array}$$

where $p_n(\rho_n)$ is the generator of $\mathscr{Q}(L_n)$ and $\mu_{Z,n}(\rho_n)=(\eta_1^n)^*$. However, $j_n^*((\eta_1^n)^*)(v_n)=(-1)^n$. Hence $\mathscr{Q}(\mu_Z)\colon \mathscr{Q}(L)\to\mathscr{Q}(L^*)$ is epic.

This proves the following

THEOREM 6.1. $R_Z = P_Z[\rho_1, \rho_2,\dots, \rho_n,\dots] = P_Z[\eta_1, \eta_2,\dots, \eta_n,\dots]$.

It should be mentioned that the proof employed for Theorem 6.1 is a representation theoretic version of Liulevicius' argument in [7], although the entire content in the preceding five sections does not depend upon his paper.

§ 7. Atiyah's Δ' and Doubilet's forgotten symmetric functions

Let E be an n dimensional complex vector space with a base $\{e_1,\cdots, e_n\}$ and let $E^{\otimes k}$ be the k-th tensor product of E. By letting S_k act on $E^{\otimes k}$ in an obvious way, $E^{\otimes k}$ becomes an S_k-module. Then there exists the well known decomposition isomorphism

$$\zeta\colon \sum_{\pi\vdash k} \hom_{S_k}(V_\pi, E^{\otimes k})\otimes V_\pi \longrightarrow E^{\otimes k}$$

defined by $\zeta(f\otimes x)=f(x)$ for $f\in\hom_{S_k}(V_\pi, E^{\otimes k})$ and $x\in V_\pi$, where $\{V_\pi|\pi\vdash k\}$ is the complete set of irreducible S_k-modules. Let $T\colon E\to E$ be a linear map defined by $T(e_i)=x_ie_i$ for each i. Then $T^{\otimes k}\colon E^{\otimes k}\to E^{\otimes k}$ is an S_k-map and hence induces a linear map $\pi(T)\colon \hom_{S_k}(V_\pi, E^{\otimes k})\to\hom_{S_k}(V_\pi, E^{\otimes k})$. It is easy to see that Trace $(\pi(T))$ is symmetric in $x_1,\dots, x_n$ with integer coefficients. Define

$$\Delta_{n,k} = \sum [V_\pi]\otimes_{\boldsymbol{Z}} \mathrm{Trace}(\pi(T))\in R_{\boldsymbol{Z}}(S_k)\otimes H_{n,k},$$

and define a homomorphism

$$\Delta'_{n,k}\colon R^*_{\boldsymbol{Z}}(S_k) = \hom_{\boldsymbol{Z}}(R_{\boldsymbol{Z}}(S_k), \boldsymbol{Z}) \longrightarrow H_{n,k}$$

by $\Delta'_{n,k}(\xi)=\sum_{\pi\vdash k}\xi(V_\pi)\,\mathrm{Trace}(\pi(T))\in H_{n,k}$ for $\xi\in R^*_{\boldsymbol{Z}}(S_k)$. Then we have $\Delta'_{\ ,k}\colon R^*_{\boldsymbol{Z}}(S_k)\to H_{\ ,k}$, and hence Atiyah's homomorphism

$$\Delta'\colon R^*_{\boldsymbol{Z}} \longrightarrow H$$

is defined. By the definition of $\Delta'_{n,k}$ it is immediate to see that $\Delta'(\sigma_k)=h_k$ and $\Delta'(\lambda_k)=a_k$, because $\hom_{S_k}(1_{S_k}, E^{\otimes k})$ is the k-th symmetric power $\sigma^k(E)$ and $\hom_{S_k}(\mathrm{Alt}\,S_k, E^{\otimes k})$ is the k-th exterior power $\lambda^k(E)$. Atiyah (Proposition 1.2 in [1]) shows that Δ' is a ring isomorphism.

Atiyah (Corollary 1.4 in [1]) shows that when $\Delta_{n,k}=\sum_i a_i\otimes b_i$ for $n\geq k$, then $\{a_i\}$ and $\{b_i\}$ are "dual bases" to each other. The following proposition states how the a_i determines the b_i and vice-versa.

PROPOSITION 7.1. *Given bases $\{a_i\}$ for $R_{\boldsymbol{Z}}(S_k)$ and $\{b_i\}$ for $H_{\ ,k}$. Then $\Delta_{\ ,k}=\sum_i a_i\otimes b_i$ if and only if $\langle a_i, F^{-1}(b_j)\rangle=\delta_{ij}$, where F is the Frobenius map and δ_{ij} denotes the Kronecker delta.*

PROOF. Let $F(c_j)=b_j$ and $\Delta_{\ ,k}=\sum_i a_i\otimes b'_i$. Then we obtain

$$\begin{aligned} F(c_j) &= \Delta'\mu(c_j) && \text{from Corollary 5.2}\\ &= \textstyle\sum_i \mu(c_j)(a_i)b'_i && \text{by definition of } \Delta'\\ &= \textstyle\sum_i \langle c_j, a_i\rangle b'_i = \sum_i \langle a_i, F^{-1}(b_j)\rangle b'_i. \end{aligned}$$

Thus, $b'_i=b_i$ if and only if $\langle a_i, F^{-1}(b_j)\rangle=\delta_{ij}$. This completes the proof.

Corresponding to a base $\{a_\pi|\pi\vdash k\}$ for $H_{\ ,k}$ there exists a base $\{d_\pi|\pi\vdash k\}$ for $R_{\boldsymbol{Z}}(S_k)$ such that $\Delta_{\ ,k}=\sum d_\pi\otimes a_\pi$. Then, by proposition 7.1

$$\langle d_\pi, F^{-1}(a_{\pi'})\rangle = \langle d_\pi, \eta_{\pi'}\rangle = \delta_{\pi\pi'}.$$

Since $\{\eta_\pi|\pi\vdash k\}$ is a base for $R_{\boldsymbol{Z}}(S_k)$, we obtain

$$\Delta_{\ ,k} = \sum \eta_\pi\otimes F(d_\pi)$$

by repeated use of the proposition.

DEFINITION 7.2. A base $\{F(d_\pi)|\pi\vdash k\}$ for $H_{,k}$ is called the *Doubilet forgotten symmetric functions* ([2]).

In the rest of the section we shall determine the d_π so that the Doubilet functions will be recovered. Note that $d_{\{k\}}$ is determined by Atiyah (Proposition 1.9 in [1]).

THEOREM 7.3. *Let $\Delta_{,k}=\sum_\pi d_\pi\otimes a_\pi=\sum_\pi \eta_\pi\otimes F(d_\pi)$, where a_π is a monomial of elementary symmetric functions. Then for $\pi\vdash k$, we have*

$$d_\pi = (1/\pi!)\textstyle\sum_{\sigma\vdash k}(q_\sigma/|\sigma|)Q_1(\eta_1)^{\sigma_1}\cdots Q_k(\eta_1,\dots,\eta_k)^{\sigma_k},$$

where $Q_i(a_1,\dots,a_i)$ is the i-th Newton polynomial for s_i and

$$q_\sigma = (\partial/\partial a_1)^{\pi_1}\cdots(\partial/\partial a_k)^{\pi_k}s_\sigma \quad (s_\sigma=s_1^{\sigma_1}\cdots s_k^{\sigma_k} = Q_1(a_1)^{\sigma_1}\cdots Q_k(a_1,\dots,a_k)^{\sigma_k}).$$

PROOF. $\gamma_\sigma=|\sigma|K_\sigma$ by Proposition 2.1, and we get

$$\langle\chi^{-1}(K_\sigma), F^{-1}(T(\gamma_{\sigma'}))\rangle = \langle K_\sigma, \gamma_{\sigma'}\rangle = \delta_{\sigma\sigma'} \quad \text{(by Corollary 5.2 and (3.1))}.$$

Therefore by Proposition 7.1 and Theorem 4.1,

$$\Delta_{,k} = \textstyle\sum_{\sigma\vdash k}\chi^{-1}(K_\sigma)\otimes T(\gamma_\sigma) = \sum_{\sigma\vdash k}\chi^{-1}(K_\sigma)\otimes s_\sigma.$$

Since s_σ is a polynomial of degree k in variables $a_1,\dots,a_k$, the coefficient of the monomial $a_\pi=a_1^{\pi_1}\cdots a_k^{\pi_k}$ in s_σ is equal to $q_\sigma/\pi!$, where q_σ is the one given in the theorem. Therefore, by rewriting $\Delta_{,k}$ in terms of a_π, we obtain

$$d_\pi = (1/\pi!)\textstyle\sum_{\sigma\vdash k}q_\sigma\chi^{-1}(K_\sigma)$$

where $\chi^{-1}(K_\sigma)=(1/|\sigma|)Q_1(\eta_1)^{\sigma_1}\cdots Q_k(\eta_1,\dots,\eta_k)^{\sigma_k}$ by Proposition 4.3. This proves the theorem.

For example, in the case when $k=3$ let us calculate the Doubilet functions $\omega_\pi=F(d_\pi)$:

$$\omega_{\{1^3\}} = a_1^3-2a_1a_2+a_3.$$

$$\omega_{\{3\}} = a_1^3-3a_1a_2+3a_3, \text{ and } \omega_{\{1,2\}} = 5a_1a_2-2a_1^3-3a_3.$$

Hence the projection of $\omega_{\{1,2\}}\in H_{,3}$ into $H_{3,3}$ is the symmetric function

$$-\{2(x_1^3+x_2^3+x_3^3)+x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_3+x_3^2x_1+x_3^2x_2\}.$$

If we denote by $M^{(2,1)}$ the Specht irreducible representation of S_3 (for definition, see §8), then $d_{\{1^3\}} = \eta_1^3-2\eta_1\eta_2+\eta_3 = [1_{S^3}]$, $d_{\{3\}} = \eta_1^3-3\eta_1\eta_2+3\eta_3 = [1_{S^3}]-$

$[M^{(2,1)}]+[\mathrm{Alt}\, S_3]$, and $d_{\{1,2\}}=5\eta_1\eta_2-2\eta_1^3-3\eta_3=[M]-2[1_{S^3}]$. It follows that $\langle d_\pi, \eta_{\pi'}\rangle=\delta_{\pi\pi'}$ as we should have.

§8. Inner plethysms

In this section, R denotes R_Z. Let M be a representation of S_n and let $\{e_1, e_2,\dots, e_\omega\}$ be a base for M. The k-th tensor product $M^{\otimes k}$ is considered as a representation of $S_n\times S_k$ when a linear operation is defined by

$$(\sigma, t)(e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_k}) = \sigma e_{i_{t(1)}}\otimes\sigma e_{i_{t(2)}}\otimes\cdots\otimes\sigma e_{i_{t(k)}},$$

for any $(\sigma, t)\in S_n\times S_k$ and for any basis element $e_{i_1}\otimes\cdots\otimes e_{i_k}$ with $1\leq i_1, i_2,\dots, i_k\leq\omega$. Since $R(S_n\times S_k)$ is isomorphic to $R(S_n)\otimes R(S_k)$, the map $\otimes k$: $R(S_n)\to R(S_n)\otimes R(S_k)$ is defined by

$$\otimes k([M]) = [M^{\otimes k}].$$

It is shown by Atiyah (Proposition 2.2 in [1]) that $\otimes k$ is well defined.

We notice that $\otimes k([M]-[N])$ for a general element $[M]-[N]\in R(S_n)$ is given by the following

PROPOSITION 8.1. $\otimes k([M]-[N])=\sum_{j=0}^{k}(-1)^j[\mathrm{Ind}_{S_{k-j}\times S_j}^{S_k} M^{\otimes(k-j)}\otimes N^{\otimes j}]$.

PROOF. It is sufficient to show that

$$(M, N)^k = (\textstyle\sum_{j=0,\, j:\text{even}}^{k} \mathrm{Ind}_{S_{k-j}\times S_j}^{S_k} M^{\otimes(k-j)}\otimes N^{\otimes j},$$
$$\textstyle\sum_{j=1,\, j:\text{odd}}^{k} \mathrm{Ind}_{S_{k-j}\times S_j}^{S_k} M^{\otimes(k-j)}\otimes N^{\otimes j}).$$

This can be proved by the induction on k.

DEFINITION 8.2. By an *inner plethysm* $T(\lambda)$ *associated with an element* $\lambda\in R_Z^*(S_k)$ we mean an operation

$$T(\lambda)\colon R(S_n) \longrightarrow R(S_n)\otimes \boldsymbol{Z} = R(S_n)$$

defined by $(1\otimes\lambda)(\otimes k)$.

In the sequel, we denote $T(\lambda)([M])$ simply by $\lambda([M])$ for any S_n-representation M, if no confusion arises.

PROPOSITION 8.3. *For any $\lambda_\tau\in R^*(S_k)$ with $\tau\vdash k$ and for any S_n-representation M, we have*

$$\lambda_\tau([M]) = [\hom_{S_k}(\mathrm{Ind}_{S_\tau}^{S_k}\mathrm{Alt}\, S_\tau, M^{\otimes k})].$$

PROOF. It is well known that if $\{V_\sigma|\sigma\vdash k\}$ is a complete set of irreducible S_k-representations, then there exists a $(S_n\times S_k)$-representation decomposition

$$M^{\otimes k} \simeq \textstyle\sum_{\sigma\vdash k}\hom_{S_k}(V_\sigma, M^{\otimes k})\otimes V_\sigma,$$

where we consider $\hom_{S_k}(V_\sigma, M^{\otimes k})$ as an S_n-module with S_n-operations defined by $\sigma f = \sigma^{\otimes k} \circ f$ for $f \in \hom_{S_k}(V_\sigma, M^{\otimes k})$ and $\sigma \in S_n$. Then by definition

$$\begin{aligned} T(\lambda_\tau)([M]) &= \textstyle\sum_{\sigma \vdash k} \lambda_\tau([V_\sigma])[\hom_{S_k}(V_\sigma, M^{\otimes k})] \\ &= [\hom_{S_k}(\textstyle\sum_{\sigma \vdash k} \lambda_\tau([V_\sigma])V_\sigma, M^{\otimes k})]. \end{aligned}$$

However,

$$\begin{aligned} \textstyle\sum_{\sigma \vdash k} \lambda_\tau([V_\sigma])V_\sigma &= \textstyle\sum_{\sigma \vdash k} \mu(\eta_\tau)([V_\sigma])V_\sigma = \textstyle\sum_{\sigma \vdash k} \langle \mathrm{Ind}_{S_\tau}^{S_k} \mathrm{Alt}\, S_\tau, V_\sigma \rangle V_\sigma \\ &= \mathrm{Ind}_{S_\tau}^{S_k} \mathrm{Alt}\, S_\tau. \end{aligned}$$

Hence we obtain the proposition.

PROPOSITION 8.4. *For any partition $\tau \vdash k$ and for any S_n-representation M we have*

$$\lambda_\tau([M]) = \lambda_1([M])^{\tau_1} \lambda_2([M])^{\tau_2} \cdots \lambda_k([M])^{\tau_k}.$$

PROOF. By the Frobenius reciprocity law we have

$$\hom_{S_k}(\mathrm{Ind}_{S_\tau}^{S_k} \mathrm{Alt}\, S_\tau, M^{\otimes k}) \simeq \hom_{S_\tau}(\mathrm{Alt}\, S_\tau, \mathrm{Res}_{S_\tau}^{S_k} M^{\otimes k}).$$

Since $\mathrm{Alt}\, S_\tau \simeq (\mathrm{Alt}\, S_1)^{\otimes \tau_1} \otimes \cdots \otimes (\mathrm{Alt}\, S_k)^{\otimes \tau_k}$ and $\mathrm{Res}_{S_\tau}^{S_k} M^{\otimes k} \simeq M^{\otimes \tau_1} \otimes \cdots \otimes (M^{\otimes k})^{\otimes \tau_k}$, we obtain

$$\hom_{S_\tau}(\mathrm{Alt}\, S_\tau, \mathrm{Res}_{S_\tau}^{S_k} M^{\otimes k}) \simeq \otimes_{i=1}^{k} (\hom_{S_i}(\mathrm{Alt}\, S_i, M^{\otimes i}))^{\otimes \tau_i}.$$

Therefore we have the proposition by using Proposition 8.3.

Note that this proposition is stated by Atiyah as R^* is a subring of $Op(R)$. (See the first line on p. 178 in [1].)

Using the same methods as in the proofs of Propositions 8.3 and 8.4 we may prove the following

PROPOSITION 8.5. *For any $\sigma_\tau \in R^*(S_k)$ with $\tau \vdash k$ and for any S_n-representation M, we have*

$$\sigma_\tau([M]) = [\hom_{S_k}(\mathrm{Ind}_{S_\tau}^{S_k} 1_{S_\tau}, M^{\otimes k})] = \sigma_1([M])^{\tau_1} \sigma_2([M])^{\tau_2} \cdots \sigma_k([M])^{\tau_k}.$$

PROPOSITION 8.6. *For any S_n-representations M and N, we have*

$$\begin{aligned} \lambda_k([M]+[N]) &= \textstyle\sum_{i=0}^{k} \lambda_{k-i}([M])\lambda_i([N]), \\ \sigma_k([M]+[N]) &= \textstyle\sum_{i=0}^{k} \sigma_{k-i}([M])\sigma_i([N]), \\ \lambda_k([M]-[N]) &= \textstyle\sum_{i=0}^{k} (-1)^i \lambda_{k-i}([M])\sigma_i([N]), \\ \sigma_k([M]-[N]) &= \textstyle\sum_{i=0}^{k} (-1)^i \sigma_{k-i}([M])\lambda_i([N]). \end{aligned}$$

PROOF. These formulae can be proved by using Propositions 8.3–8.5 and 8.1, (cf. p. 178 in [1]).

Let H be a subgroup of a finite group G and let G/H be a G-set with the usual G action on the set of left cosets. Then it is easy to see that the permutation representation associated with the G-set G/H is isomorphic to a G-representation $\mathrm{Ind}_H^G 1_H$ of the trivial H-representation 1_H. Suppose that H contains no normal subgroup of G except $\{e\}$. Then the action of G on G/H is effective in the sense that if $g\bar{x}=\bar{x}$ for any $\bar{x}\in G/H$, then $g=e$. In this case G can be embedded in the permutation group $\mathrm{Aut}(G/H)$. Hence the G-set G/H is the G-restriction of the $\mathrm{Aut}(G/H)$-set G/H. It follows that the G-representation $\mathrm{Ind}_H^G 1_H$ is isomorphic to the G-restriction of an S_N-representation $\boldsymbol{F}^N$ with the natural S_N-action, where N is the index of H in G and $\boldsymbol{F}^N$ denotes the N dimensional complex vector space. Summarizing what we stated above, we obtain

PROPOSITION 8.7. *Let H be a subgroup of a finite group G with the property that H does not contain any normal subgroup of G except $\{e\}$. Then G can be embedded in the permutation group $\mathrm{Aut}\, G/H=S_N$, where N is the index of H in G. Considering G as a subgroup of S_N, the induced representation $\mathrm{Ind}_H^G 1_H$ of the trivial H-representation 1_H is isomorphic to the G-restriction of the S_N-permutation representation $\boldsymbol{F}^N$.*

LEMMA 8.8. *Let $\pi\vdash n$ and let $S_1^{\pi_1}\times\cdots\times S_n^{\pi_n}$ be a subgroup of S_n. If $\pi\neq\{n\}$, then S_π has no normal subgroup of S_n except the trivial group consisting of the identity.*

PROOF. Since $\pi\neq\{n\}$, there exists $k\,(1\leqq k<n)$ such that $S_\pi\subset S_{n-k}\times S_k$. If $n\geqq 5$, the only non-trivial normal subgroup of S_n is the alternating group A_n. Suppose that $S_\pi\supset A_n$. Then $(n-k)!k!\geq n!/2$, which is a contradiction. When $n=1$, 2, 3 and 4, it is easy to check the validity of the lemma. This completes the proof.

Combining Proposition 8.7 and Lemma 8.8, we obtain

PROPOSITION 8.9. *Any basis element $\rho_\pi=[\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi}]$ in $R(S_n)$ is $[\mathrm{Res}_{S_n}^{S_N} \boldsymbol{F}^N]$, where N is the index of S_π in S_n.*

By the Specht irreducible representation $M^{(N-1,1)}$ we mean the subrepresentation of $\boldsymbol{F}^N$ consisting of $(z_1,\dots, z_N)$ with $z_1+\cdots+z_N=0$ in $\boldsymbol{F}^N$. Since we have the decomposition $\boldsymbol{F}^N=M^{(N-1,1)}\oplus 1_{S_N}$, we have $\mathrm{Ind}_{S_\pi}^{S_n} 1_{S_\pi}\simeq \mathrm{Res}_{S_n}^{S_N}\boldsymbol{F}^N=\mathrm{Res}_{S_n}^{S_N} M^{(N-1,1)}\oplus 1_{S_n}$.

THEOREM 8.10. *For any basis element $\rho_\pi\in R(S_n)\,(\pi\vdash n)$ and for any basis $\lambda_\tau\in R^*(S_k)\,(\tau\vdash k)$, $\lambda_\tau(\rho_\pi)$ can be computed effectively provided the character of* *i-th exterior powers of Specht irreducible representations $M^{(N-1,1)}$ for any i and N, can be computed.*

PROOF. From Propositions 8.6 and 8.9 we obtain

$$\begin{aligned}\lambda_i(\rho_\pi) &= \lambda_i([\mathrm{Res}^{S_N}_{S_n} M^{(N-1,1)}]+[1_{S_n}]) = \textstyle\sum_{j=0}^{i} \lambda_{i-j}(\mathrm{Res}^{S_N}_{S_n} M^{(N-1,1)}])\lambda_j([1_{S_n}])\\ &= \lambda_i([\mathrm{Res}^{S_N}_{S_n} M^{(N-1,1)}])+\lambda_{i-1}([\mathrm{Res}^{S_N}_{S_n} M^{(N-1,1)}])\\ &= \mathrm{Res}^{S_N}_{S_n} \lambda_i([M^{(N-1,1)}])+\mathrm{Res}^{S_N}_{S_n} \lambda_{i-1}([M^{(N-1,1)}]).\end{aligned}$$

Proposition 8.4 allows us to proceed $\lambda_\tau(\rho_\pi)=\lambda_1(\rho_\pi)^{\tau_1}\cdots\lambda_k(\rho_\pi)^{\tau_k}$. Hence the proof is complete.

Now we calculate the character of $\lambda_i([M^{(N-1,1)}])=[\hom_{S_i}(\mathrm{Alt}\, S_i, (M^{(N-1,1)})^{\otimes i})]$ for all N and i.

PROPOSITION 8.11. *Suppose that $\sigma\in S_N$ has the shape $\tau\vdash N$ with $\tau_l=0\,(l<k)$ and $\tau_k>0$. Then*

$$\begin{aligned}&\chi(\lambda_i[M^{(N-1,1)}])(\sigma)\\ &= \textstyle\sum_{\omega=0}^{k-1}\sum_{\pi\vdash i-\omega}(-1)^\omega \,\mathrm{Sgn}\,\pi \begin{pmatrix}\tau_1\\ \pi_1\end{pmatrix}\cdots\begin{pmatrix}\tau_{k-1}\\ \pi_{k-1}\end{pmatrix}\begin{pmatrix}\tau_k-1\\ \pi_k\end{pmatrix}\begin{pmatrix}\tau_{k+1}\\ \pi_{k+1}\end{pmatrix}\cdots\begin{pmatrix}\tau_{i-\omega}\\ \pi_{i-\omega}\end{pmatrix}.\end{aligned}$$

PROOF. $M^{(N-1,1)}$ is the S_N submodule of the permutation representation $\boldsymbol{F}^N$ spanned by $e_1=(1, 0, 0,\ldots, 0, -1)$, $e_2=(0, 1, 0,\ldots, 0, -1),\ldots,$ and $e_{N-1}=(0, 0,\ldots, 0, 1, -1)$. The action of S_N on $M^{(N-1,1)}$ is given by

$$\sigma e_j = e_{\sigma(j)} - e_{\sigma(N)} \qquad (1\le j\le N-1)$$

for any $\sigma\in S_N$, where e_N is considered as 0 whenever e_N occurs in the formula.

Since two elements in S_N are conjugate if and only if they have the same shape and since characters are constant on conjugacy classes, we may assume without loss of generality that the disjoint cycle decomposition of σ is arranged such that the cycles appear in descending order with respect to cycle lengths and the integers occur in ascending order. For example, if the shape of σ is $\{2^2, 3, 4\}$, then σ is assumed to be

$$(1, 2, 3, 4)(5, 6, 7)(8, 9)(10, 11).$$

Since $\lambda_i([M^{(N-1,1)}])$ is represented by the i-th exterior power $\Lambda^{(i)}(M^{(N-1,1)})$ of $M^{(N-1,1)}$ with a base $B=\{e_{\alpha_1}\wedge\cdots\wedge e_{\alpha_i}|1\le\alpha_1<\cdots<\alpha_i\le N-1\}$, the action of S_N is given by

$$\begin{aligned}\sigma(e_{\alpha_1}\wedge\cdots\wedge e_{\alpha_i}) &= \sigma e_{\alpha_1}\wedge\cdots\wedge\sigma e_{\alpha_i} = (e_{\sigma(\alpha_1)}-e_{\sigma(N)})\wedge\cdots\wedge(e_{\sigma(\alpha_i)}-e_{\sigma(N)})\\ &= e_{\sigma(\alpha_1)}\wedge\cdots\wedge e_{\sigma(\alpha_i)} - \textstyle\sum_{j=1}^{i} e_{\sigma(\alpha_1)}\wedge\cdots\wedge e_{\sigma(\alpha_{j-1})}\wedge e_{\sigma(N)}\wedge e_{\sigma(\alpha_{j+1})}\wedge\cdots\wedge e_{\sigma(\alpha_i)}.\end{aligned}$$

By our hypothesis on σ whose shape is $\tau \vdash N$ with $\tau_l = 0\,(l<k)$ and $\tau_k > 0$, we have $\sigma(N) = N-k+1$.

If $\{\alpha_1,\dots,\alpha_i\} = \{\sigma(\alpha_1),\dots,\sigma(\alpha_i)\}$, then $\{\alpha_1,\dots,\alpha_i\} \subset \{1, 2,\dots, N-k\}$ and σ restricted to $\{\alpha_1,\dots,\alpha_i\}$ gives rise to a "subpermutation" of σ. If the shape of the subpermutation is denoted by π, then $\pi \vdash i$ and $\sigma(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_i}) = (\mathrm{Sgn}\,\pi)e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_i} - \cdots$. If $\pi \vdash i$, then the total number of subpermutations of the shape π is

$$n(\pi) = \binom{\tau_1}{\pi_1}\cdots\binom{\tau_{k-1}}{\pi_{k-1}}\binom{\tau_k - 1}{\pi_k}\binom{\tau_{k+1}}{\pi_{k+1}}\cdots\binom{\tau_i}{\pi_i}.$$

If $\{\alpha_1,\dots,\alpha_i\} = \{\sigma(\alpha_1),\dots,\sigma(\alpha_{j-1}), N-k+1, \sigma(\alpha_{j+1}),\dots,\sigma(\alpha_i)\}$, then there exists an integer ω with $k>\omega>0$ such that $\{\alpha_1,\dots,\alpha_i\} = \{\alpha_1,\dots,\alpha_{i-\omega}, N-k+1, \dots, N-k+\omega\}$ and $\{\alpha_1,\dots,\alpha_{i-\omega}\} = \{\sigma(\alpha_1),\dots,\sigma(\alpha_{i-\omega})\} \subset \{1, 2,\dots, N-k\}$. Denoting by $\pi \vdash i-\omega$ the shape of the subpermutation of σ restricted to $\{\alpha_1,\dots,\alpha_{i-\omega}\}$, we obtain $\sigma(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_i}) = \sigma(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{i-\omega}} \wedge e_{N-k+1} \wedge \cdots \wedge e_{N-k+\omega}) = \cdots - e_{\sigma(\alpha_1)} \wedge \cdots \wedge e_{\sigma(\alpha_{i-\omega})} \wedge e_{N-k+2} \wedge \cdots \wedge e_{N-k+\omega} \wedge e_{N-k+1} = \cdots + (-1)^{\omega}(\mathrm{Sgn}\,\pi)e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{i-\omega}} \wedge e_{N-k+1} \wedge \cdots \wedge e_{N-k+\omega} = \cdots + (-1)^{\omega}(\mathrm{Sgn}\,\pi)\,(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_i})$. Again the total number of subpermutations of σ with the shape $\pi \vdash i-\omega$ is $n(\pi)$.

By the above arguments, the diagonal entries of the matrix representation of σ with respect to $B = \{e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_i}\}$ contain $n(\pi)$ numbers of $(-1)^{\omega}\mathrm{Sgn}\,\pi$ for each $\pi \vdash i-\omega$ with $0 \le \omega < k$. This completes the proof.

For any integer N and any sequence $\mu = \{\mu_1,\dots,\mu_j\}$ of positive integers with $N \ge \mu_1 \ge \cdots \ge \mu_j$, we define a partition $\mu(N)$ as

$$\mu(N) = \{N-\mu_1, \mu_1-\mu_2,\dots, \mu_{j-1}-\mu_j, \mu_j\} \vdash N.$$

We now evaluate

$$\sigma_i([\boldsymbol{F}^N]) = [\hom_{S_i}(1_{S_i}, (\boldsymbol{F}^N)^{\otimes i})].$$

PROPOSITION 8.12. *$\sigma_i([\boldsymbol{F}^N]) = \sum_\mu [\mathrm{Ind}\, S_{S_{\mu(N)}}^{S_N} 1_{S_{\mu(N)}}]$, where the summation is taken over all sequences $\mu = \{\mu_1,\dots,\mu_j\}$ of positive integers with $N \ge \mu_1 \ge \cdots \ge \mu_j$ and $\mu_1 + \cdots + \mu_j = i$, and $S_{\mu(N)} = S_{N-\mu_1} \times S_{\mu_1-\mu_2} \times \cdots S_{\mu_{j-1}-\mu_j} \times S_{\mu_j}$.*

PROOF. Let $\{e_1,\dots,e_N\}$ be a base for $\boldsymbol{F}^N$. It is known that $\hom_{S_i}(1_{S_i}, (\boldsymbol{F}^N)^{\otimes i})$ is isomorphic to the i-th symmetric product of $\boldsymbol{F}^N$. A base for the i-th symmetric product of $\boldsymbol{F}^N$ consists of canonical elements $e_{\alpha_1}^{m_1} \otimes \cdots \otimes e_{\alpha_N}^{m_N}$ with $\{\alpha_1,\dots,\alpha_N\} = \{1,\dots,N\}$ and $0 \le m_1 \le \cdots \le m_N$ such that $m_1 + \cdots + m_N = i$ and if $m_a = m_b$ and $a<b$, then $\alpha_a < \alpha_b$. The action of S_N is given by $\sigma(e_{\alpha_1}^{m_1} \otimes \cdots \otimes e_{\alpha_N}^{m_N}) = e_{\sigma(\alpha_1)}^{m_1} \otimes \cdots \otimes e_{\sigma(\alpha_N)}^{m_N}$ which is considered as a canonical element by exchanging factors if necessary. Then two basis elements $e_{\alpha_1}^{m_1} \otimes \cdots \otimes e_{\alpha_N}^{m_N}$ and $e_{\beta_1}^{n_1} \otimes \cdots \otimes e_{\beta_N}^{n_N}$ are in the same orbit under the action of S_N if and only if $m_k = n_k$ for all k.

Now, for a basis element $v = e_{\alpha_1}^{m_1} \otimes \cdots \otimes e_{\alpha_N}^{m_N}$, let μ_l be the number of k's with

$m_k \geq l$ $(l=0, 1, 2,\ldots)$. Then $\mu_0 = N$ and we obtain a sequence $\mu(=\mu(v))=\{\mu_1,\ldots, \mu_j\}$ with $N \geq \mu_1 \geq \cdots \geq \mu_j \geq 1$ and $\mu_1+\cdots+\mu_j = m_1+\cdots+m_N = i$. For $\sigma \in S_N$, $\sigma(v)=v$ if and only if $m_{\sigma(k)} = m_k$ for all k. Thus we see that the stabilizer of v is $S_{\mu(N)}$. It follows that the orbit of v under S_N is $\mathrm{Ind}_{S_{\mu(N)}}^{S_N} 1_{S_{\mu(N)}}$ and that two basis elements v and v' are in the same orbit if and only if $\mu(v)=\mu(v')$. This completes the proof.

Littlewood has done these calculations in Propositions 8.11 and 8.12. (See Theorems I and II in [6] and p. 139 in [5].)

PROPOSITION 8.13. *For any basis element $\rho_\pi \in R(S_n)$ with $\pi \vdash n$,*

$$\sigma_i(\rho_\pi) = \textstyle\sum_\mu \mathrm{Res}_{S_n}^{S_N} \rho_{\mu(N)} \qquad (N \textit{ is the index of } S_\pi \textit{ in } S_n)$$

where the summation is taken over all sequences $\mu=\{\mu_1,\ldots,\mu_j\}$ with $N \geq \mu_1 \geq \cdots \geq \mu_j > 0$ and $\mu_1+\cdots+\mu_j = i$, and $\mu(N)=\{N-\mu_1, \mu_1-\mu_2,\ldots, \mu_{j-1}-\mu_j, \mu_j\} \vdash N$.

PROOF. It is immediate from Propositions 8.9 and 8.12.

TEHOREM 8.14. *Any inner plethysm $T(\lambda)$: $R_{\boldsymbol{Z}} \to R_{\boldsymbol{Z}}$ can be evaluated by the procedures established in this section.*

PROOF. For any element $\xi \in R(S_N)$ and for any $\lambda \in R^*(S_k)$ with $\lambda = \sum_{\tau \vdash k} a_\tau \lambda_\tau$ $(a_\lambda \in \boldsymbol{Z})$, we have

$$\lambda(\xi) = \textstyle\sum_{\tau \vdash k} a_\tau \lambda_\tau(\xi) = \sum_{\tau \vdash k} a_\tau \lambda_1(\xi)^{\tau_1} \lambda_2(\xi)^{\tau_2} \cdots \lambda_k(\xi)^{\tau_k}$$

by Proposition 8.4. If $\xi = [M]-[N]$, then Proposition 8.6 shows that

$$\lambda_i(\xi) = \textstyle\sum_{j=0}^{i} (-1)^j \lambda_{i-j}([M]) \sigma_j([N]) .$$

Since the S_n-representations M and N are direct sums of basis elements of ρ_π's, $\lambda_{i-j}([M])$ and $\sigma_j([N])$ are calculated by Propositions 8.6, 8.11, 8.13, and Theorem 8.10. This completes the proof.

Finally, we would like to comment about the character of $\sigma_i(\rho_\pi)$. Since $\rho_{\mu(N)} = \rho_{N-\mu_1} \rho_{\mu_1-\mu_2} \cdots \rho_{\mu_{j-1}-\mu_j} \rho_{\mu_j}$,

$$\chi(\rho_{\mu(N)}) = \chi(\rho_{N-\mu_1}) \chi(\rho_{\mu_1-\mu_2}) \cdots \chi(\rho_{\mu_j})$$

can be effectively calculated by the facts that $\chi(\rho_i) = \sum_{\pi \vdash i} K_\pi$ and

$$K_\pi \cdot K_\sigma = ((\pi \vee \sigma)!/\pi!\sigma!) K_{\pi \vee \sigma} \quad \text{(Proposition 2.1)}.$$

This, in turn, enables us to evaluate the character of $\sigma_i(\rho_\pi)$ by Proposition 8.13.

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