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Free boundary problems for some reaction-diffusion equations

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§ 1. Introduction

The present paper deals with a free boundary problem which models regional partition phenomena arising in population biology. Our problem is to look for a family of functions $\{u(x, t), s(t)\}$ $((x, t) \in [0, 1] \times [0, \infty))$ which satisfy

$$u_t = d_1 u_{xx} + uf(u) \quad \text{in} \quad S^-, \tag{1.1}$$

$$u_t = d_2 u_{xx} + ug(u) \quad \text{in} \quad S^+, \tag{1.2}$$

$$u(0, t) = 0 \quad \text{for} \quad t \in (0, \infty), \tag{1.3}$$

$$u(1, t) = 0 \quad \text{for} \quad t \in (0, \infty), \tag{1.4}$$

$$u(s(t), t) = 0 \quad \text{for} \quad t \in (0, \infty), \tag{1.5}$$

$$\begin{aligned} \dot{s}(t) = &- \mu_1 u_x(s(t)-0, t) + \mu_2 u_x(s(t)+0, t) \\ &\text{for} \quad t \in (0, \infty) \quad \text{where} \quad 0 < s(t) < 1, \end{aligned} \tag{1.6}$$

$$u(x, 0) = \varphi(x) \quad \text{for} \quad x \in I \equiv (0, 1), \tag{1.7}$$

$$s(0) = l, \tag{1.8}$$

where $x = s(t)$ corresponds to a free boundary, S^- (resp. S^+) is an open subset of $I \times (0, \infty)$ in which $x < s(t)$ (resp. $x > s(t)$), d_i and $\mu_i (i = 1, 2)$ are positive constants, $\dot{s}(t)$ denotes $(d/dt)s(t)$ and $u_x(s(t)-0, t)$ (resp. $u_x(s(t)+0, t)$) means the limit of $u(x, t)$ at $x = s(t)$ from the left (resp. right). For the derivation of the free boundary problem (1.1)–(1.8), we refer the reader to [8].

In (1.1) and (1.2), f and g are assumed to possess the following properties:

(A.1) f is locally Lipschitz continuous on $[0, \infty)$ and satisfies $f(1) = 0$ and $f(u) \leqq 0$ on $[1, \infty)$.

(A.2) g is locally Lipschitz continuous on $(-\infty, 0]$ and satisfies $g(1) = 0$ and $g(u) \leqq 0$ on $(-\infty, -1]$.

On the initial data $\{\varphi, l\}$ we put the following conditions:

(A.3) $0 \leqq l \leqq 1$.

(A.4) $\varphi \in H_0^1(I)$ satisfies $\varphi(l) = 0$ and $(l - x)\varphi(x) \geqq 0$ for $x \in \bar{I} = [0, 1]$.

Some related results for the problem (1.1)–(1.8), which is denoted by (P), can be found in our previous papers [8, 9] where we have studied the global existence, uniqueness, regularity and asymptotic behavior of solutions $\{u, s\}$ with non-homogeneous Dirichlet boundary conditions

$$u(0, t) = m_1 > 0 \quad \text{and} \quad u(1, t) = -m_2 < 0 \qquad \text{for} \quad t \in (0, \infty).$$

Owing to the non-homogeneity, it is proved there that the free boundary $x = s(t)$ never touches the fixed boundaries $x = 0, 1$.

However, when homogeneous Dirichlet boundary conditions are imposed, there is also the possibility that the free boundary hits one of the fixed ends $x = 0, 1$ in a finite time; that is, one phase disappears in a finite time. See Fig. 1, where some numerical experiments exhibit the disapperance of one phase in appropriate conditions. This is a very interesting phenomenon to discuss, though the analysis will be complicate.

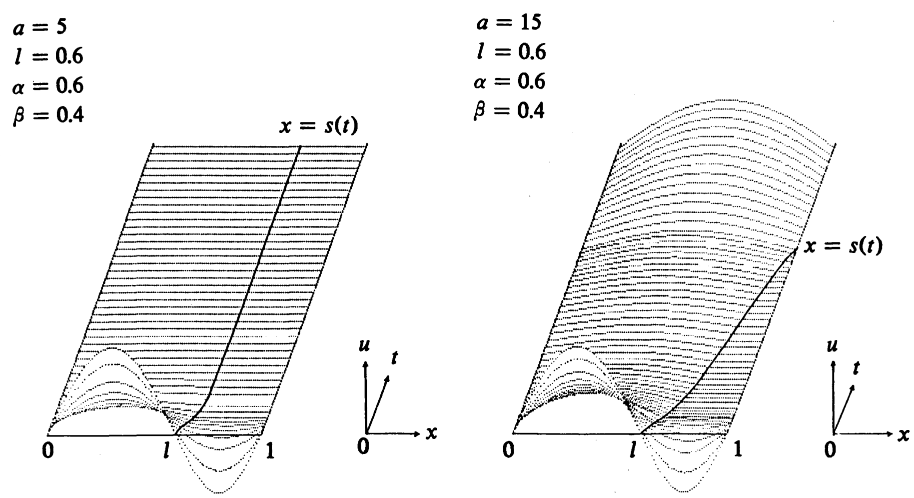

Fig. 1 (1A and 1B).

$d_1 = d_2 = \mu_1 = \mu_2 = 1, \quad f(u) = g(-u) = a(1-u),$

$$\phi(x) = \begin{cases} \alpha \sin \pi x/l & \text{for} \quad 0 \leqq x \leqq l, \\ -\beta \sin \pi(x-l)/(1-l) & \text{for} \quad l \leqq x \leqq 1. \end{cases}$$

Now the purpose of this paper is to study the global existence, uniqueness, regularity and asymptotic properties of solutions for (P) in the homogeneous case. The essential points of our analysis are almost the same as those developed by the authors in [8, 9]. Moreover, we intend to derive some conditions which guarantee the disappearance of one phase.

In §2, we construct a solution $\{u, s\}$ and derive its regularity properties over a time interval where the free boundary is distant from the fixed boundary. It will be shown in §3 that, if the free boundary hits the fixed boundary at a finite time $t=T^*$, the free boundary stays there after T^*. So we can continue to solve u as a solution of the usual initial boundary value problem for $t \geqq T^*$ (cf. Fig. 1B). The global existence result is stated in this section (Theorem 3.1). In §4, we give two comparison theorems for (P), which will help us to investigate the asymptotic properties of solutions for (P). In §5, we show some results about the dependence of $\{u, s\}$ on the initial data $\{\varphi, l\}$. These results will be used for studying the ω-limit set corresponding to the solution orbit for (P). Complete information on the ω-limit set is stated in §6 (Theorem 6.2). Especially, it is shown that any element of the ω-limit set satisfies the stationary problem associated with (P). So the analysis of the stationary problem becomes very important. It is carried out in §7 by putting some restrictions on the forms of f and g. In §8, stability or instability of each stationary solution is investigated with use of the comparison technique. Moreover, we give some sufficient conditions for the disappearance of one phase in a finite time. Finally, §9 is devoted to the study of bifurcation phenomena appearing for (P).

Notation

We summarize some notation used throughout this paper. We set

$$I = (0, 1) \quad \text{and} \quad Q = I \times (0, \infty).$$

For any set A in I or Q, we denote its closure by $\bar{A}$. Let s be a continuous function on $[0, \infty)$ with values in $\bar{I}$. For $0 \leqq \delta < T$, define

$$S^-_{\delta,T} = \{(x, t) \in Q;\ 0<x<s(t) \text{ and } \delta<t<T\},$$
$$S^+_{\delta,T} = \{(x, t) \in Q;\ s(t)<x<1 \text{ and } \delta<t<T\}.$$

If $T=\infty$, then $S^-_{\delta,T}$ and $S^+_{\delta,T}$ are simply denoted by S^-_δ and S^+_δ. Moreover, we write S^- and S^+ in place of S^-_0 and S^+_0, respectively. When $s(t) \equiv 0$ (resp. 1) for $\delta<t<T$, we understand that $S^-_{\delta,T}=\phi$ (resp. $S^+_{\delta,T}=\phi$) and $S^+_{\delta,T}=I \times (\delta, T)$ (resp. $S^-_{\delta,T}=I \times (\delta, T)$).

Let $u_i (i=1, 2)$ be continuous functions on $\bar{I}$ and let $s_i (i=1, 2)$ be numbers in $\bar{I}$. We write $\{u_1, s_1\} \geqq \{u_2, s_2\}$ if $u_1(x) \geqq u_2(x)$ for $x \in \bar{I}$ and $s_1 \geqq s_2$.

§ 2. Existence of solutions I

In this section we investigate existence and regularity properties of solutions for (P) by assuming

(A.3)′ $0<l<1$

in place of (A.3). Our first existence result is stated as follows.

THEOREM 2.1. *Under the assumptions* (A.1), (A.2), (A.3)$'$, *and* (A.4), *there exists a unique family* $\{T^*, u, s\} \in (0, \infty] \times C(\bar{I} \times [0, T^*)) \times C([0, T^*))$ *with the following properties.*

(i) $u(\cdot, 0) = \varphi$ *and* $s(0) = l$.

(ii) $\dot{s} \in L^3(0, T^*)$, $0 < s(t) < 1$ *for* $0 \leqq t < T^*$. *If* $T^* < \infty$, *then* $\lim_{t \to T^*} s(t)$ *exists and equals* 0 *or* 1.

(iii) $\{u, s\}$ *satisfies* (1.3), (1.4) *and* (1.5) *for* $0 \leqq t < T^*$ *and*

$$0 \leqq u \leqq \overline{M} \equiv \max\{1, \sup_{0 \leqq x \leqq l} \varphi(x)\} \quad in \quad S^-_{0,T^*}$$

$$0 \geqq u \geqq -\underline{M} \equiv \min\{-1, \inf_{l \leqq x \leqq 1} \varphi(x)\} \quad in \quad S^+_{0,T^*}.$$

(iv) $u^{\pm} \in C([0, T^*); H^1_0(I)) \cap L^\infty([0, T^*); H^1_0(I))$, *where* $u^+ = \max\{u, 0\}$ *and* $u^- = -\min\{u, 0\}$.

(v) $u_t \in L^2(S^-_{0,T^*}) \cap L^2(S^+_{0,T^*})$.

(vi) u_t, $u_{xx} \in C(S^-_{0,T^*}) \cap C(S^+_{0,T^*})$ *and* $\{u, s\}$ *satisfies* (1.1) *and* (1.2) *for* $0 < t < T^*$.

(vii) *For any* $\delta > 0$ *and* $\delta' > 0$, u_x *is Hölder continuous in* $(x, t) \in S^{\pm}_{\delta, T^* - \delta'}$ *and* $\dot{s}$ *is Hölder continuous in* $t \in [\delta, T^* - \delta']$.

(viii) $\{u, s\}$ *satisfies* (1.6) *for* $0 < t < T^*$.

PROOF. Since $0 < l < 1$, the arguments developed in [8, §§3–6] are valid to show that a pair of functions $\{u, s\}$ with the required regularity properties exists on some interval. Therefore, it exists on a maximal interval $[0, T^*)$. If $T^* = \infty$, there is nothing to prove more.
If $T^* < \infty$, we will prove the that $\lim_{t \to T^*} s(t)$ exists. For this purpose, we define

$$\begin{aligned} E(u, s) = {} & \frac{\mu_1^2}{2} \int_0^s u_x(x)^2 dx + \frac{\mu_2^2}{2} \int_s^1 u_x(x)^2 dx \\ & - \frac{\mu_1^2}{d_1} \int_0^s \tilde{F}(u(x))\, dx - \frac{\mu_2^2}{d_2} \int_s^1 \tilde{G}(u(x))\, dx, \end{aligned} \tag{2.1}$$

where $\tilde{F}(u) = \int_0^u v f(v) dv$ and $\tilde{G}(u) = \int_0^u v g(v) dv$. Then Lemma 6.1 in [8] yields

$$\begin{aligned} E(u(t), s(t)) & + \frac{\mu_1^2}{d_1} \int_0^t \int_0^{s(\tau)} u_t^2\, dx d\tau + \frac{\mu_2^2}{d_2} \int_0^t \int_{s(\tau)}^1 u_t^2\, dx d\tau \\ & + \frac{1}{2} \int_0^t |\dot{s}(\tau)|^3 d\tau \leqq E(\varphi, l) \end{aligned} \tag{2.2}$$

for $0 \leqq t < T^*$. Since $u(x, t)$ is bounded for all $(x, t) \in \bar{I} \times [0, T^*)$, (2.2) implies $\dot{s} \in L^3(0, T^*)$, from which it follows that $\lim_{t \to T^*} s(t)$ exists.

Now suppose that $0 < \lim_{t \to T^*} s(t) \equiv s(T^*) < 1$. Then one can show that

$$\lim_{t\to T^*} u(\cdot, t) \equiv u(\cdot, T^*) \text{ exists in } H_0^1(I)\text{-norm}$$

by reducing the boundary value problem (1.1), (1.3) and (1.5) in S_{0,T^*}^- (resp. (1.2), (1.4) and (1.5) in S_{0,T^*}^+) to the usual boundary value problem in an appropriate cylindrical domain (see, e.g., [8, §4]). Owning to the preceding local existence result, $\{u, s\}$ can be extended over $[0, T^{**}]$ with some $T^{**} > T^*$. This assertion contradicts the maximality of T^*; so that $s(T^*)$ must be one or zero.

Finally we will show the uniqueness of $\{T^*, u, s\}$. Let $\{T_1^*, u_1, s_1\}$ be another family satisfying (i)–(viii) in this theorem. Set $T^* = \min\{T^*, T_2^*\}$. By virtue of the comparison principle for (P) (see [8, Theorem 6.3]),

$$u(\cdot, t) = u_1(\cdot, t) \quad \text{and} \quad s(t) = s_1(t) \qquad \text{for} \quad 0 \leqq t < T_1^*.$$

Moreover, it is easy to see that $T^* = T_2^* = T_1^*$. q. e. d.

Further regularity properties of the solution $\{u, s\}$ can be obtained by using Moser's technique as in the paper of Evans [2].

THEOREM 2.2. *Let $\{u, s\}$ be the solution of* (P) *satisfying conditions* (i)–(viii) *of Theorem* 2.1. *Then,*

$$u_x \in L^\infty(S_{\delta,T^*}^-) \cap L^\infty(S_{\delta,T^*}^+) \quad \textit{and} \quad \dot{s} \in L^\infty(\delta, T^*), \tag{2.3}$$

for any $\delta \in (0, T^*)$. Moreover, if $T^* < \infty$, then

$$u^\pm \in C([0, T^*]; H_0^1(I)) \quad \textit{and} \quad s \in C([0, T^*]). \tag{2.4}$$

PROOF. First we assume

$$\varphi_x \in L^\infty(I) \tag{2.5}$$

in addition to (A.4). Let p be any positive integer. Then

$$\begin{aligned} &\frac{d}{dt}\int_0^{s(t)} u_x(x, t)^{2p} dx \\ &= u_x(s(t)-0, t)^{2p}\dot{s}(t) + 2p\int_0^{s(t)} u_x^{2p-1}u_{xt}dx \\ &= u_x(s(t)-0, t)^{2p}\dot{s}(t) + 2pu_x(s(t)-0, t)^{2p-1}u_t(s(t)-0, t) \\ &\quad - 2p(2p-1)\int_0^{s(t)} u_x^{2p-2}u_{xx}u_t dx. \end{aligned} \tag{2.6}$$

Differentiation of (1.5) with respect to t gives

$$u_x(s(t)-0, t)\dot{s}(t) + u_t(s(t)-0, t) = 0.$$

Making use of (1.1) and integrating by parts we have

$$\int_0^{s(t)} u_x^{2p-2}u_{xx}u_t dx$$
$$= d_1 \int_0^{s(t)} u_x^{2p-2}u_{xx}^2 dx + \frac{1}{2p-1}\int_0^{s(t)} (u_x^{2p-1})_x u f(u) dx$$
$$= \frac{d_1}{p^2}\int_0^{s(t)} |(u_x^p)_x|^2 dx - \frac{1}{2p-1}\int_0^{s(t)} u_x^{2p}\tilde{f}(u)dx,$$

where $\tilde{f}(u)=(d/du)\,(uf(u))$, which is bounded almost everywhere on $[0, M]$ for $M>0$ (see (A.1)). Therefore, rearranging (2.6) one gets

$$\begin{aligned} &\frac{d}{dt}\left\{\int_0^{s(t)} u_x^{2p}dx\right\} + (2p-1)u_x(s(t)-0,\, t)^{2p}\dot{s}(t) \\ &\quad + \frac{2(2p-1)d_1}{p}\int_0^{s(t)} |(u_x^p)_x|^2 dx \\ &= 2p\int_0^{s(t)} u_x^{2p}\tilde{f}(u)dx. \end{aligned} \tag{2.7}$$

Similarly,

$$\begin{aligned} &\frac{d}{dt}\left\{\int_{s(t)}^1 u_x^{2p}dx\right\} - (2p-1)u_x(s(t)+0,\, t)^{2p}\dot{s}(t) \\ &\quad + \frac{2(2p-1)d_2}{p}\int_{s(t)}^1 |(u_x^p)_x|^2 dx \\ &= 2p\int_{s(t)}^1 u_x^{2p}\tilde{g}(u)dx, \end{aligned} \tag{2.8}$$

where $\tilde{g}(u)=(d/du)(ug(u))$. Here we observe that the following inequalities hold:

$$\begin{aligned} &\{(\mu_1 u_x(s(t)-0,\, t))^{2p} - (\mu_2 u_x(s(t)+0,\, t))^{2p}\}\dot{s}(t) \\ &\quad = \dot{s}(t)(-\mu_1 u_x(s(t)-0,\, t) + \mu_2(s(t)+0,\, t)) \\ &\qquad \times \{(-\mu_1 u_x(s(t)-0,\, t))^{2p-1} + \cdots + (-\mu_2 u_x(s(t)+0,\, t))^{2p-1}\} \\ &\quad \geqq \dot{s}(t)^2\{(-\mu_1 u_x(s(t)-0,\, t))^{2p-1} + (-\mu_2 u_x(s(t)+0,\, t))^{2p-1}\} \\ &\quad \geqq 2^{1-2p}\dot{s}(t)^2(-\mu_1 u_x(s(t)-0,\, t) - \mu_2 u_x(s(t)+0,\, t))^{2p-1} \\ &\quad \geqq 2^{1-2p}|\dot{s}(t)|^{2p+1}. \end{aligned}$$

Therefore, it follows from (2.7) and (2.8) that

$$\begin{aligned} &\frac{d}{dt}\left\{\int_0^{s(t)} \mu_1^{2p}u_x^{2p}dx + \int_{s(t)}^1 \mu_2^{2p}u_x^{2p}dx\right\} \\ &+ 2^{1-2p}(2p-1)\,|\dot{s}(t)|^{2p+1} \\ &+ \frac{2(2p-1)}{p}\left\{d_1\int_0^{s(t)} |(\mu_1^p u_x^p)_x|^2 dx + d_2\int_{s(t)}^1 |(\mu_2^p u_x^p)_x|^2 dx\right\} \end{aligned} \tag{2.9}$$

$$\leqq 2pM_0\left\{\int_0^{s(t)}\mu_1^{2p}u_x^{2p}dx+\int_{s(t)}^1\mu_2^{2p}u_x^{2p}dx\right\},$$

where $M_0=\max\{\text{ess. sup}_{0\leqq u\leqq M}|\tilde{f}(u)|, \text{ess. sup}_{-M\leqq u\leqq 0}|\tilde{g}(u)|\}$.

Here we prepare the following lemma whose proof is given at the end of this section.

LEMMA 2.3. *For $-\infty<\alpha<\beta<\infty$, let v be any $H^1(\alpha,\beta)$-function satisfying $v(\gamma)=0$ with some $\gamma\in[\alpha,\beta]$. Then there exists a positive constant C independent of v, α and β such that*

$$\|v\|_{L^2(\alpha,\beta)}\leqq C\|v_x\|_{L^2(\alpha,\beta)}^{1/3}\|v\|_{L^1(\alpha,\beta)}^{2/3}.$$

We continue the proof of Theorem 2.2. Since $u_x(0,t)\geqq 0$, $u_x(s(t)\pm 0,t)\leqq 0$ and $u_x(1,t)\geqq 0$, there exist $s_1(t)\in[0,s(t)]$ and $s_2(t)\in[s(t),1]$ such that $u_x(s_i(t),t)=0$, $i=1,2$. Hence Lemma 2.3 is applicable by taking $v=\mu_1^p u_x^p$ with $(\alpha,\beta)=(0,s(t))$ or $v=\mu_2^p u_x^p$ with $(\alpha,\beta)=(s(t),1)$. Consequently, Lemma 2.3 together with Young's inequality gives

$$\|\mu_1^p u_x^p\|_{L^2(0,s(t))}^2\leqq\varepsilon\|(\mu_1^p u_x^p)_x\|_{L^2(0,s(t))}^2+C_1\varepsilon^{-1/2}\|\mu_1^p u_x^p\|_{L^1(0,s(t))}^2$$

for any $\varepsilon>0$ with some C_1 independent of p and $s(t)$. This inequality is rewritten as

$$\int_0^{s(t)}|(\mu_1^p u_x^p)_x|^2dx\geqq\varepsilon^{-1}\int_0^{s(t)}\mu_1^{2p}u_x^{2p}\,dx-C_1\varepsilon^{-3/2}\left(\int_0^{s(t)}|\mu_1^p u_x^p|\,dx\right)^2. \tag{2.10}$$

Similarly,

$$\int_{s(t)}^1|(\mu_2^p u_x^p)_x|^2dx\geqq\varepsilon^{-1}\int_{s(t)}^1\mu_2^{2p}u_x^{2p}dx-C_1\varepsilon^{-3/2}\left(\int_{s(t)}^1|\mu_2^p u_x^p|\,dx\right)^2. \tag{2.11}$$

Now we take $p=2^k$ $(k=1,2,3,\ldots)$ and set

$$X_k(t)=\int_0^{s(t)}(\mu_1 u_x)^{2^k}dx+\int_{s(t)}^1(\mu_2 u_x)^{2^k}dx.$$

Then it follows from (2.9), (2.10) and (2.11) that, for any $\varepsilon>0$,

$$\frac{d}{dt}X_{k+1}(t)+(\varepsilon^{-1}d_0-2^{k+1}M_0)X_{k+1}(t)\leqq\varepsilon^{-3/2}d_0C_1X_k(t)^2, \tag{2.12}$$

where $d_0=2\min\{d_1,d_2\}$. Since $\varepsilon>0$ is arbitrary, we take $\varepsilon>0$ sufficiently small so that $\varepsilon^{-1}d_0>2^{k+1}M_0$. If we take $\varepsilon=d_0/(2^k 3M_0)$, (2.12) becomes

$$\frac{d}{dt}X_{k+1}(t)+2^kM_0X_{k+1}(t)\leqq 2^{3k/2}C_2X_k(t)^2, \tag{2.13}$$

with $C_2=C_1(3M_0)^{3/2}d_0^{-1/2}$. Solving differential inequality (2.13) one can find

$$X_{k+1}(t) \leqq \max\{X_{k+1}(0),\ 2^{k/2}C_3 \sup_{0\leqq t<T^*} X_k(t)^2\} \qquad \text{for} \quad 0\leqq t<T^*, \tag{2.14}$$

where $C_3=C_2/M_0$. We recall (2.2) to get

$$X_1(t) = \int_0^{s(t)} (\mu_1 u_x)^2 dx + \int_{s(t)}^1 (\mu_2 u_x)^2 dx \leqq K_0, \tag{2.15}$$

where K_0 is a positive constant depending on μ_1, μ_2, d_1, d_2, f, g, $\|\varphi\|_{H_0^1(I)}$ and l. Moreover, (2.5) implies

$$X_{k+1}(0) = \int_0^l (\mu_1\varphi_x)^{2^{k+1}} dx + \int_l^1 (\mu_2\varphi_x)^{2^{k+1}} dx \leqq K_1^{2^{k+1}}, \tag{2.16}$$

with some $K_1>0$. Here we may assume $2C_3\geqq 1$ and $K_1^4\leqq\sqrt{2}\,C_3K_0^2$ without loss of generality. In view of (2.15) and (2.16), it follows inductively from (2.14) that

$$X_{k+1}(t) \leqq 2^{a_k}C_3^{b_k}K_0^{c_k},$$

where

$$a_k = \frac{1}{2}\sum_{i=0}^{k-1} 2^i(k-i) = 2^k - \frac{k}{2} - 1,$$

$$b_k = \sum_{i=0}^{k-1} 2^i = 2^k - 1 \quad \text{and} \quad C_k = 2^k.$$

Therefore

$$\limsup_{k\to\infty} \{X_{k+1}(t)\}^{2^{-(k+1)}} \leqq \sqrt{2C_3K_0} \qquad \text{for} \quad 0\leqq t<T^*,$$

which implies

$$\|u_x(\cdot,t)\|_{L^\infty(0,s(t))} + \|u_x(\cdot,t)\|_{L^\infty(s(t),1)} \leqq C_4 \tag{2.17}$$

for all $0\leqq t<T^*$ with some $C_4>0$ independent of $s(t)$ and T^*. Moreover, (1.6) and (2.17) give

$$|\dot{s}(t)| \leqq C_4(\mu_1+\mu_2). \tag{2.18}$$

for all $0<t<T^*$.

We have put restriction (2.5) on φ to derive (2.17) and (2.18). For general φ satisfying (A.4), we observe that, for any $0<\delta<T^*$, $x=s(\delta)$ is distant from the fixed boundary and $u_x(\cdot,\delta)$ is Hölder continuous in $x\in[0, s(\delta)]$ (resp. $x\in[s(\delta), 1]$) by Theorem 2.1. Therefore, by taking $\{u(\cdot,\delta), s(\delta)\}$ in place of $\{\varphi, l\}$, it is sufficient to repeat the above procedure. Thus we see that (2.17) and (2.18) are valid for every $\delta\leqq t<T^*$ and, therefore, (2.3) follows.

It remains to show (2.4). Since Theorem 2.1 (ii) assures $s \in C([0, T^*])$, we have only to prove $u^{\pm} \in C([0, T^*]; H_0^1(I))$. We may take $s(T^*)=1$. We reduce initial boundary value problem (1.1), (1.3), (1.5) and (1.7) in $\{(x, t); 0<x<s(t), 0<t \leqq T^*\}$ to the corresponding problem in a cylindrical domain (see [8, §4]). Then the standard parabolic regularity result yields $u_+ \in C([0, T^*]; H_0^1(I))$. As to u_-, we use (2.3) to see

$$\int_{s(t)}^{1} u_x^2(x, t)dx \leqq \|u_x(\cdot, t)\|^2_{L^\infty(s(t),1)}(1-s(t)) \longrightarrow 0$$

as $t \to T^*$. Consequently, $u^- \in C([0, T^*]; H_0^1(I))$. q. e. d.

Proof of Lemma 2.3. We take $\alpha=0$ without loss of generality to prove this lemma.

Let w be any $H^1(I)$-function satisfying $w(\gamma')=0$ with some $\gamma' \in I$. Since $\|w\|_{H^1(I)} \leqq C_1\|w_x\|_{L^2(I)}$ with some $C_1>0$ independent of w, Gagliardo-Nirenberg's inequality gives

$$\|w\|_{L^2(I)} \leqq C_2\|w_x\|^{1/3}_{L^2(I)}\|w\|^{2/3}_{L^1(I)}, \tag{2.19}$$

where $C_2>0$ is a positive constant independent of w (see Nirenberg [11]).

Now let v be any function with the property stated in this lemma. If we set $w(x)=v(\beta x)$, $0 \leqq x \leqq 1$, a simple computation shows

$$\|w\|_{L^2(I)} = \beta^{-1/2}\|v\|_{L^2(0,\beta)}, \quad \|w\|_{L^1(I)} = \beta^{-1}\|v\|_{L^1(0,\beta)}$$
$$\text{and} \quad \|w_x\|_{L^2(I)} = \beta^{1/2}\|v_x\|_{L^2(0,\beta)}.$$

Substitution of these relations into (2.19) gives the assertion. q. e. d.

§ 3. Existence of solutions II

By Theorems 2.1 and 2.2, there exists a unique number T^* (which may be $+\infty$) such that (P) has the unique solution $\{u, s\}$ on $[0, T^*]$ whenever the initial dat $\{\varphi, l\}$ fulfills (A.3)$'$ and (A.4).

In the case $T^*<\infty$, the free boundary $x=s(t)$ hits one of the fixed ends at $t=T^*$, say, $s(T^*)=1$. Suppose that the solution $\{u, s\}$ can be extended beyond $t=T^*$ with the property $0<s(t)<1$ for $t \in (T^*, \tilde{T})$ with some $\tilde{T}>T^*$. Since $u(1, t)=u(s(t), t)=0$ for $t \in [T^*, \tilde{T})$, one can show $u(x, t) \equiv 0$ for $(x, t) \in S^+_{T^*,\tilde{T}}$ by applying the maximum principle to (1.2) (see, e.g., Nirenberg [10]). Moreover, since $u(0, t)=u(s(t), t)=0$ for $t \in [T^*, \tilde{T})$ and $u(x, T^*) \geqq 0$ for $x \in [0, 1]$, another application of the maximum principle to (1.1) yields $u(x, t) \geqq 0$ for $(x, t) \in [0, s(t)] \times [T^*, \tilde{T})$. Then it follows from (1.6) that $\dot{s}(t) \geqq 0$ for $t \in [T^*, \tilde{T})$, which contradicts the assumption. Thus we have shown that, if the free boundary $x=s(t)$ hits one of the fixed ends at $t=T^*$, it never leaves there for $t\geqq T^*$.

In order to study conditions under which T^* becomes finite, it is better to continue to solve (P) beyond $t=T^*<\infty$. In the case $T^*<\infty$, it seems natural from the preceding arguments to take $s(t)\equiv 0$ or 1 for $t\geqq T^*$ and neglect the free boundary condition (1.6) for $t>T^*$. For example, if $s(T^*)=1$ with $T^*<\infty$, we set $s(t)\equiv 1$ for $t\geqq T^*$ and construct $u(\cdot, t)$ $(t>T^*)$ as the solution of the usual initial boundary value problem (1.1), (1.3), (1.5) for $t>T^*$ with initial data $u(\cdot, T^*)$ at $t=T^*$.

The above procedure applies to the case $l=0$ or 1 in an obvious manner.

Then we can show the following theorem which generalizes Theorems 2.1 and 2.2.

THEOREM 3.1. *Under the assumptions* (A.1), (A.2), (A.3) *and* (A.4) *there exists a unique pair of functions* $\{u, s\}\in C(\bar{Q})\times C[0, \infty)$ *with the following properties*:

(i) $u(\cdot, 0)=\varphi$ *and* $s(0)=l$.

(ii) $\dot{s}\in L^3(0, \infty)\cap L^\infty(\delta, \infty)$ *for any* $\delta>0$.

(iii) $\{u, s\}$ *satisfies* (1.3), (1.4) *and* (1.5) *for* $t\in[0, \infty)$ *and*

$$0\leqq u\leqq \overline{M}\equiv \max\{1, \sup_{0\leqq x\leqq l} \varphi(x)\} \quad in \quad S^-,$$

$$0\geqq u\geqq -\underline{M}=\min\{-1, \inf_{l\leqq x\leqq 1} \varphi(x)\} \quad in \quad S^+.$$

(iv) $u^\pm\in C([0, \infty); H_0^1(I))\cap L^\infty(0, \infty; H_0^1(I))$, *and* $u_x\in L^\infty(S_\delta^-)\cap L^\infty(S_\delta^+)$ *for any* $\delta>0$.

(v) $u_t\in L^2(S^-)\cap L^2(S^+)$.

(vi) u_t, $u_{xx}\in C(S^-)\cap C(S^+)$ *and* $\{u, s\}$ *satisfies* (1.1) *and* (1.2) *everywhere.*

(vii) *For any* $\delta>0$, u_x *is Hölder continuous with respect to* (x, t) *in* $\{(y, \tau)\in \overline{S_\delta^-};\ s(\tau)\geqq\delta\}$ *and* $\{(y, \tau)\in \overline{S_\delta^+};\ s(\tau)\leqq 1-\delta\}$ *and* $\dot{s}$ *is Hölder continuous for* $t\in \{\tau\geqq\delta;\ \delta\leqq s(\tau)\leqq 1-\delta\}$.

(viii) $\{u, s\}$ *satisfies* (1.6) *for* $t\in\{\tau;\ 0<s(\tau)<1\}$.

PROOF OF THEOREM 3.1. In ivew of Theorems 2.1 and 2.2, there is nothing to prove if $T^*=\infty$. We consider the case $T^*<\infty$ and take $s(T^*)=1$. Since $u(\cdot, T^*)\in H_0^1(I)$ by Theorem 2.2, it is standard to solve the initial boundary value problem (1.1), (1.3), (1.5) for $t>T^*$ with the initial data $u(\cdot, T^*)$ at $t=T^*$ (see, Ladyženskaja et al [6] or Henry [3]). Observe that estimates (2.2) and (2.17) remain true for all $t\geqq T^*$. Therefore, recalling the results of Theorems 2.1 and 2.2, we have only to see the Hölder continuity of u_x with resect to $(x, t)\in \overline{S_\delta^-}$ for any $\delta>0$ to complete the proof. To do so, we reduce the initial boundary value problem (1.1), (1.3), (1.5), (1.7) in S_- to that in a cylindrical domain. Then the parabolic regularity result yields the Hölder continuity of u_x (see [8, §4]). q.e.d.

In what follows, we say that $\{u, s\}$ is a *smooth solution* of (P) if it has the properties stated in Theorem 3.1.

§4. Comparison principle

In this section we will give some comparison results which will be useful in the study of asymptotic behavior of smooth solutions for (P).

We prepare some terminology. Let $\mathscr{R}$ denote the set of all functions $\{u, s\} \in C(\bar{Q}) \times C([0, \infty))$ satisfying (i) u_x is continuous for (x, t) in $\overline{S^-} \cap \{(y, \tau); \tau>0, s(\tau)>0\}$ and $\overline{S^+} \cap \{(y, \tau); \tau>0, s(\tau)<1\}$, (ii) u_t, $u_{xx} \in C(S^-) \cap C(S^+)$, and (iii) $s(t)$ is continuously differentiable for $t \in \{\tau>0; 0<s(\tau)<1\}$.

According to our previous papers [8, 9], we will define super- and subsolutions for (P).

DEFINITION 4.1. A pair of functions $\{u, s\} \in \mathscr{R}$ is called a *supersolution* of (P) for the initial data $\{\varphi, l\}$ if it satisfies

(i)	$u_t \geqq d_1 u_{xx} + uf(u)$	in S^-,
(ii)	$u_t \geqq d_2 u_{xx} + ug(u)$	in S^+,
(iii)	$u(0, t) \geqq 0$	in $(0, \infty)$,
(iv)	$u(1, t) \geqq 0$	in $(0, \infty)$,
(v)	$u(s(t), t)=0$	for $t \in \{\tau>0, 0<s(\tau)<1\}$,
(vi)	$\dot{s}(t) \geqq -\mu_1 u_x(s(t)-0, t)+\mu_2 u_x(s(t)+0, t)$	for $t \in \{\tau>0; 0<s(\tau)<1\}$,
(vii)	$u(x, 0)=\varphi(x)$	in I,
(viii)	$s(0)=l$.	

A *subsolntion* of (P) for the initial data $\{\varphi, l\}$ is defined by reversing the inequality signs in (i), (ii), (iii), (iv) and (vi). If $\{u, s\}$ is a super- and subsolution of (P), it is called a *classical solution* of (P).

REMARK 4.1. Let $\{u, s\}$ be a supersolution of (P) for the initial data $\{\varphi, l\}$ satisfying (A.3) and (A.4). Then the maximum principle for parabolic equations assures $u \geqq 0$ in S^-. Now we observe that, if $s(T^*)=1$ for some $T^* \geqq 0$, then $s(t) \equiv 1$ for all $t \geqq T^*$. In fact, suppose that $s(t)<1$ for $t \in (T^*, \tilde{T})$. Then, by the maximum principle, $u \geqq 0$ in $S^+_{T^*, \tilde{T}}$; so that $\dot{s}(t) \geqq 0$ for $t \in (T^*, \tilde{T})$ (see (vi) in Definition 4.1). This result contradicts the assumption $s(t)<1$ for $t \in (T^*, \tilde{T})$.

Analogous results hold for a subsolution of (P); $u \leqq 0$ in S^+ and $s(t) \equiv 0$ for $t \geqq T^*$ when $s(T^*)=0$.

Our first comparison theorem corresponds to Theorem 5.1 in [8].

THEOREM 4.1. *Assume that* $\{\varphi^i, l^i\}$ $(i=1, 2)$ *satisfy* (A.3) *and* (A.4). *Let* $\{u^1, s^1\}$ *(resp.* $\{u^2, s^2\}$*) be a supersolution (resp. subsolution) of* (P) *for the initial data* $\{\varphi^1, l^1\}$ *(resp.* $\{\varphi^2, l^2\}$*). If* $\{\varphi^1, l^1\} \geqq \{\varphi^2, l^2\}$ *with* $l^1 \neq l^2$, *then*

$$\{u^1(\cdot, t), s^1(t)\} \geqq \{u^2(\cdot, t), s^2(t)\} \qquad \textit{for all} \quad t \geqq 0.$$

PROOF. We first consider the case when one of φ^i $(i=1, 2)$ is non-vanishing and $1>l^1>l^2>0$. Set $T^*=\inf\{t; s^i(t)=0$ or 1 for some $i\}$. Repeating the arguments in [8, Theorem 5.1] one can easily show that $s^1(t)>s^2(t)$ for $t\in[0, T^*)$ and $u^1(x, t)\geqq u^2(x, t)$ for $(x, t)\in \bar{I}\times[0, T^*]$. Therefore, $s^1(T^*)=1$ or $s^2(T^*)=0$. For example, let $s^1(T^*)=1$. Then it follows from Remark 4.1 that $s^1(t)\equiv 1$ for $t\geqq T^*$ and, therefore, $s^2(t)\leqq s^1(t)$ for $t\geqq T^*$. Moreover, since $u^1(0, t)\geqq 0\geqq u^2(0, t)$ and $u^1(s^1(t), t)\geqq 0\geqq u^2(s^2(t), t)$ for $t\geqq T^*$, the comparison principle for parabolic equations gives $u^1\geqq u^2$ in $(S^2)^-_{T^*}$; so that $u^1\geqq u^2$ in $\bar{Q}\times[T^*, \infty)$.

We next consider the case when $\varphi^1=\varphi^2\equiv 0$ and $1>l^1>l^2>0$. Making use of the comparison principle again we can see $u^1\geqq 0\geqq u^2$ in $\bar{Q}$. Hence $\dot{s}^1(t)\geqq 0\geqq \dot{s}^2(t)$ for $a.e. t\in(0, \infty)$; so that $s^1(t)>s^2(t)$ for all $t\in[0, \infty)$.

Finally it remains to consider the case when $l^1=1$ or $l^2=0$. We will treat the case $l^1=1$. By Remark 4.1, $s^1(t)\equiv 1$ for $t\geqq 0$ and $u^1\geqq 0$ in $\bar{Q}$. Since $s^2(t)\leqq 1$ and $u^2(s^2(t), t)\leqq 0$ for $t\geqq 0$, the comparison principle enables us to conclude $u^1\geqq u^2$ in $\bar{Q}$. Thus the proof is complete. q.e.d.

We will state another comparison theorem which gives a slightly preciser result than Theorem 4.1.

THEOREM 4.2. *In addition to the assumptions of Theorem* 4.1, *assume that one of* $\{u^i, s^i\}$ $(i=1, 2)$ *is a classical solution of* (P). *If* $\{\varphi^1, l^1\}\geqq\{\varphi^2, l^2\}$, *then*

$$\{u^1(\cdot, t), s^1(t)\} \geqq \{u^2(\cdot, t), s^2(t)\} \qquad \textit{for all} \quad t \geqq 0.$$

Moreover, if $\varphi^1\not\equiv\varphi^2$, *then*

$$u^1(x, t) > u^2(x, t) \qquad \textit{for} \quad (x, t)\in Q$$

and $s^1(t)>s^2(t)$ *for* $t\in\{\tau\in(0, \infty); 0<s^1(\tau)<1$ *or* $0<s^2(\tau)<1\}$.

PROOF. In order to prove the former half of this theorem, it suffices to combine the arguments used in [8, Theorem 6.3] and those in the proof of Theorem 4.1. The latter half can be derived by using the strong maximum principle for parabolic equations (see [10, Theorem 2]). q.e.d.

§ 5. Dependence on initial data

In this section we will study the dependence on initial data for smooth solutions of (P). Our result reads as follows.

THEOREM 5.1. *Suppose that* $\{\varphi^n, l^n\}$ $(n=1, 2,\dots)$ *and* $\{\varphi, l\}$ *satisfy* (A.3), (A.4),

$$\lim_{n\to\infty} l^n = l \quad \textit{and} \quad \lim_{n\to\infty} (\varphi^n)^{\pm} = \varphi^{\pm} \quad \textit{in} \quad H_0^1(I). \tag{5.1}$$

Let $\{u^n, s^n\}$ $(n=1, 2, \ldots)$ *and* $\{u, s\}$ *be the smooth solutions of* (P) *for the initial data* $\{\varphi^n, l^n\}$ *and* $\{\varphi, l\}$, *respectively. Then*

$$\lim_{n\to\infty} s^n(t) = s(t) \quad \textit{and} \tag{5.2}$$

$$\lim_{n\to\infty} (u^n)^{\pm}(\cdot, t) = u^{\pm}(\cdot, t) \quad \textit{in} \quad H_0^1(I) \tag{5.3}$$

for every $t \geqq 0$.

PROOF. First of all, we will give various uniform estimates for $\{u^n, s^n\}$. Observe that (2.2) remains valid for every $t\geqq 0$ with $\{u, s\}$ replaced by $\{u^n, s^n\}$; that is,

$$\begin{aligned} &E(u^n(t), s^n(t)) + \frac{\mu_1^2}{d_1}\int_0^t\int_0^{s^n(\tau)} (u_t^n)^2 dx d\tau \\ &\quad + \frac{\mu_2^2}{d_2}\int_0^t\int_{s^n(\tau)}^1 (u_t^n)^2 dx d\tau + \frac{1}{2}\int_0^t |\dot{s}^n(\tau)|^3 d\tau \\ &\leqq E(\varphi^n, l^n), \qquad \text{for} \quad t \geqq 0, \end{aligned} \tag{5.4}$$

where the right-hand side of (5.4) is bounded by a constant independent of n on account of (5.1). Since Theorem 3.1 (iii) together with (5.1) yields the uniform boundedness of $\{u^n\}$, it follows from (5.4) that

$$\{\dot{s}^n\}_{n=1}^{\infty} \text{ is bounded in } L^3(0, \infty), \tag{5.5}$$

$$\{u^n\}_{n=1}^{\infty} \text{ is bounded in } L^{\infty}(0, \infty; H_0^1(I)), \tag{5.6}$$

$$\{u_t^n\}_{n=1}^{\infty} \text{ is bounded in } L^2(Q). \tag{5.7}$$

In view of (5.7) one can also find that

$$\int_0^{\delta}\int_0^{s^n(\tau)} |u_{xx}^n|^2 dx d\tau + \int_0^{\delta}\int_{s^n(\tau)}^1 |u_{xx}^n|^2 dx d\tau \leqq C \tag{5.8}$$

for every $\delta > 0$ with some positive constant C independent of n. Then (5.8) implies that there exists some $t_n \in [0, \delta]$ satisfying

$$\int_0^{s^n(t_n)} |u_{xx}^n(x, t_n)|^2 dx + \int_{s^n(t_n)}^1 |u_{xx}^n(x, t_n)|^2 dx \leqq C/\delta. \tag{5.9}$$

Here we note that the following inequalities hold by Hölder's inequality:

$$\|u_x^n(t_n)\|_{L^{\infty}(0, s^n(t_n))} \leqq \|u_{xx}^n(t_n)\|_{L^2(0, s^n(t_n))} \cdot s^n(t_n)^{1/2},$$

$$\|u_x^n(t_n)\|_{L^\infty(s^n(t_n),1)} \leqq \|u_{xx}^n(t_n)\|_{L^2(s^n(t_n),1)} \cdot (1-s^n(t_n))^{1/2}.$$

Hence, it follows from (5.9) that

$$\|u_x^n(t_n)\|_{L^\infty(I)} \leqq C_1(\delta), \tag{5.10}$$

where $C_1(\delta)$ is independent of n. Repeating the arguments in the proof of Theorem 2.2 one can derive from (5.10)

$$\|u_x^n(t)\|_{L^\infty(0,s^n(t))} + \|u_x^n(t)\|_{L^\infty(s^n(t),1)} \leqq C_2(\delta) \tag{5.11}$$

for every $t \geqq \delta$ with some $C_2(\delta)>0$ independent of n (cf. (2.17)).

Let $T>0$ be any fixed positive number. Ascoli-Arzelà's theorem together with (5.4) implies that $\{s^n\}_{n=1}^\infty$ is relatively compact in $C([0, T])$. Using Ishii's result [4, Lemma 3.1], we see from (5.6) and (5.7) that $\{u^n\}_{n=1}^\infty$ is bounded in $C^{1/2,1/4}(\bar{I}\times[0, T])$; so that $\{u^n\}_{n=1}^\infty$ is relatively compact in $C(\bar{I}\times[0, T])$. These compactness results imply that there exists a subsequence $\{u^{n'}, s^{n'}\}$ of $\{u^n, s^n\}$ such that

$$\begin{aligned} &s^{n'} \longrightarrow \tilde{s} \quad \text{in} \quad C([0, T]) \\ &u^{n'} \longrightarrow \tilde{u} \quad \text{in} \quad C(\bar{I}\times[0, T]) \quad \text{as} \quad n' \longrightarrow \infty. \end{aligned} \tag{5.12}$$

We will accomplish the proof by dividing it into several cases. First we consider the case when $0<l<1$ and $0<\tilde{s}(t)<1$ for $t\in[0, T]$. Since $0<s^{n'}(t)<1$ for sufficiently large n', it can be shown that $\{(u^n)^\pm\}_{n=1}^\infty$ is relatively compact in $C([\varepsilon, T]; H_0^1(I))$ for any $\varepsilon>0$ by the method used in [8, Theorem 6.5]. Therefore,

$$(u^{n'})^\pm \longrightarrow (\tilde{u})^\pm \quad \text{as} \quad n' \longrightarrow \infty \quad \text{in} \quad C([\varepsilon, T]; H_0^1(I)) \tag{5.13}$$

for any $\varepsilon>0$. Since we have already obtained uniform estimates (5.6), (5.7) and (5.8), it is easily seen that the limiting function $\{\tilde{u}, \tilde{s}\}$ of $\{u^{n'}, s^{n'}\}$ satisfies (1.1), (1.2) and initial boundary conditions. Moreover, following the arguments by Yotsutani [14, Lemma 10.2] we find that $\{\tilde{u}, \tilde{s}\}$ satisfies the free boundary equation (1.6). The regularity properties of $\{\tilde{u}, \tilde{s}\}$ are derived as in [8]; so it becomes a smooth solution of (P). The uniqueness of smooth solutions for (P) (Theorem 4.2) yields $\{\tilde{u}, \tilde{s}\}=\{u, s\}$. This fact implies that (5.12) and (5.13) hold true with $\{u^{n'}, s^{n'}\}$ and $\{\tilde{u}, \tilde{s}\}$ replaced by $\{u^n, s^n\}$ and $\{u, s\}$. Thus (5.2) and (5.3) follow in the first case.

We next consider the second case when $0<l<1$ and $x=\tilde{s}(t)$ hits a fixed end at some time in $(0, T)$. Let $\tilde{T}^*>0$ be the first time when $x=\tilde{s}(t)$ hits a fixed end, say, $\tilde{s}(\tilde{T}^*)=1$. As in the first case, we can prove that $\{\tilde{u}, \tilde{s}\}$ is a smooth solution of (P) on $[0, T^*]$. Therefore, the uniqueness result gives $s(t)=\tilde{s}(t)$ for $t\in[0, \tilde{T}^*]$ and $u(x, t)=\tilde{u}(x, t)$ for $(x, t)\in[0, 1]\times[0, \tilde{T}^*]$. Moreover, Theorem 2.1 assures $T^*=\tilde{T}^*$, where T^* is the first time when the free boundary $x=s(t)$ arrives at one of the fixed ends. We note that $s(t)\equiv 1$ and $u^-(\cdot, t)\equiv 0$ for $t\geqq T^*$. We will show that $\tilde{s}(t)\equiv 1$ for $t\geqq T^*$. Suppose that $0<\tilde{s}(t)<1$ for $t\in(T^*, \tilde{T})$ with some $\tilde{T}>T^*$. Since $\{\tilde{u}, \tilde{s}\}$ can be proved to satisfy (1.1)–(1.6) for $t\in(T^*, \tilde{T})$, the reasoning developed in §3 leads us to the contradiction. Therefore, $\tilde{s}(t)\equiv 1$ and $(\tilde{u})^-(\cdot, t)\equiv 0$ for $t\geqq T^*$. Since it is easy to see that $\tilde{u}$ satisfies (1.1), (1.3) and (1.5), one can conclude that $u(\cdot, t)\equiv\tilde{u}(\cdot, t)$ and $s(t)\equiv\tilde{s}(t)$ for $t\in[T^*, T]$. Hence, by virtue of (5.12) we get (5.2) and

$$\lim_{n\to\infty} u^n(\cdot, t) = u(\cdot, t) \quad \text{in} \quad C(\bar{I}\times[0, T]).$$

By mapping $(S^n_{0,T})^-$ or $(S^n_{0,T})^+$ to a cylindrical domain by a suitable change of variables, the regularity results for parabolic equations enable us to show that $\{u_n^+\}_{n=1}^{\infty}$ is relatively compact in $C([\varepsilon, T]; H_0^1(I))$ for any $\varepsilon>0$ and that $\{u_n^-\}_{n=1}^{\infty}$ is relatively compact in $C([\varepsilon, T^*-\varepsilon]; H_0^1(I))$ for any $\varepsilon>0$. Hence

$$\begin{aligned}
&\lim_{n\to\infty} u_n^+ = u^+ \quad \text{in} \quad C([\varepsilon, T]; H_0^1(I)) \\
&\lim_{n\to\infty} u_n^- = u^- \quad \text{in} \quad C([\varepsilon, T^*-\varepsilon]; H_0^1(I))
\end{aligned} \tag{5.14}$$

for any $\varepsilon>0$. Moreover, invoking (5.11) and $\lim_{n\to\infty} s^n(t)=1$ for $t\in[T^*, T]$ we find that

$$\int_{s^n(t)}^1 |(u_x^n)^-(x, t)|^2 dx \leqq C_2(T^*)(1-s^n(t)) \longrightarrow 0 \quad \text{as} \quad n\longrightarrow\infty \tag{5.15}$$

for every $t\in[T^*, T]$. Thus (5.14) and (5.15) yield (5.3).

Finally it remains to consider the case when $l=0$ or 1. For this case it is sufficient to repeat the procedure developed in the second case for $t\in[T^*, T]$.

q. e. d.

§6. Structure of ω-limit set

For every $\{\varphi, l\}$ satisfying (A.3) and (A.4), Theorems 3.1 and 4.2 give a unique smooth solution of (P), which is denoted by $\{u(x, t; \varphi, l), s(t; \varphi, l)\}$.

It is convenient to introduce the notion of ω-limit set associated with the solution orbit $\{\{u(\cdot, t; \varphi, l), s(t; \varphi, l); t\geqq 0\}$:

Definition 6.1. For the solution orbit $\{\{u(\cdot, t; \varphi, l), s(t; \varphi, l)\}; t\geqq 0\}$, the *$\omega$-limit set* $\omega(\varphi, l)$ is defined by

$$\begin{aligned}
\omega(\varphi, l) = \{&\{u^*, s^*\}\in H_0^1(I)\times\bar{I};\ \text{there exists a sequence } \{t_n\}\uparrow\infty \text{ such} \\
&\text{that } s(t_n; \varphi, l)\to s^* \text{ and } u^{\pm}(t_n; \varphi, l)\to(u^*)^{\pm} \text{ in } H_0^1(I) \text{ as } n\to\infty\}.
\end{aligned}$$

The product topology induced from $H_0^1(I)\times\bar{I}$ is called *Ω-topology*.

LEMMA 6.1.
(i) $\{s(t; \varphi, l); t \geqq 0\}$ *is relatively compact in* $\bar{I}$.
(ii) $\{u(\cdot, t; \varphi, l); t \geqq 0\}$ *is relatively compact in* $H_0^1(I)$.

PROOF. (i) Since $0 \leqq s(t; \varphi, l) \leqq 1$ for $t \geqq 0$, the assertion follows from Bolzano-Weierstrass's theorem.

(ii) We will complete the proof by dividing it into three cases.

(a) The case when the free boundary $x = s(t; \varphi, l)$ hits a fixed end in a finite time. For example, we take $s(T^*; \varphi, l) = 1$ and, therefore, $s(t; \varphi, l) \equiv 1$ for $t \geqq T^*$. Then $u(\cdot, t; \varphi, l)$ satisfies the usual boundary value problem (1.1), (1.3) and (1.5) for $t \geqq T^*$; so that the parabolic regularity results yield the compactness of $\{u(\cdot, t; \varphi, l); t \geqq T^*\}$ in $H_0^1(I)$ (see, e.g., Henry [3]).

(b) The case when $d \leqq s(t; \varphi, l) \leqq 1-d$ for all $t \geqq 0$ with some $d \in (0, 1)$. In this case the proof is the same as that of Lemma 7.1 (iii) in [8].

(c) The remaining case (that is, the free boundary neither hits the fixed boundary nor is distant from the fixed boundary by a positive constant). Let $\{u(\cdot, t_n; \varphi, l)\}_{n=1}^{\infty}$ be any sequence. We will prove that $\{u(t_n; \varphi, l)\}$ has a convergent subsequence (in $H_0^1(I)$) in the situation where $\{s(t_n; \varphi, l)\}$ satisfies

$$1 - \frac{1}{2n} \leqq s(t_n; \varphi, l) < 1.$$

(If $\{s(t_n; \varphi, l)\}$ is distant from the fixed boundary by a positive constant, the proof will become simpler). By Theorem 3.1,

$$\sup_{t \geqq \varepsilon} \|u_x(\cdot, t)\|_{L^\infty(s(t),1)} \leqq C(\varepsilon)$$

for each $\varepsilon > 0$ with some $C(\varepsilon) \geqq 0$. Therefore,

$$\int_{s(t_n; \phi, l)}^1 |u_x(x, t)|^2 dx \leqq C(\varepsilon)^2 (1 - s(t_n; \varphi, l)) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty;$$

that is, $\lim_{n \to \infty} u^-(\cdot, t_n) = 0$ in $H_0^1(I)$. We next show the compactness of $\{u^+(\cdot, t_n)\}$ in $H_0^1(I)$. By virtue of the uniform continuity of $t \to s(t; \varphi, l)$, there exists some $c > 0$ such that

$$1/4 \leqq s(t; \varphi, l) < 1 \qquad \text{for} \quad t \in [t_n - c, t_n + c].$$

Since $u^+ \in L^\infty(0, \infty; H_0^1(I))$ by Theorem 3.1, we can follow the arguments in [8, §4] to show the uniform Hölder continuity of

$$x \to u_x(x, t) \quad \text{in} \quad \cup_{n=1}^{\infty} \{(x, t); 0 \leqq x \leqq s(t), t_n - c \leqq t \leqq t_n + c\}.$$

Then it is easy to extract from $\{u^+(\cdot, t_n)\}$ a subsequence which converges in $H_0^1(I)$. q.e.d.

We are ready to give some information on the structure of $\omega(\varphi, l)$.

THEOREM 6.2. (i) *$\omega(\varphi, l)$ is non-empty, compact and connected in Ω-topology.*

(ii) *$\omega(\varphi, l)$ is positively invariant; if $\{u^*, s^*\} \in \omega(\varphi, l)$, then $\{u(\cdot, t; u^*, s^*), s(t; \varphi^*, s^*)\} \in \omega(\varphi, l)$ for every $t \geqq 0$.*

(iii) *If $\{u^*, s^*\} \in \omega(\varphi, l)$, then it satisfies*

$$
\text{(SP)} \quad \begin{cases} d_1 u^*_{xx} + u^* f(u^*) = 0, \quad u^* \geqq 0 \quad in \quad (0, s^*), \\ d_2 u^*_{xx} + u^* g(u^*) = 0, \quad u^* \leqq 0 \quad in \quad (s^*, 1), \\ u^*(0) = u^*(s^*) = u^*(1) = 0, \\ -\mu_1 u^*_x(s^*-0) + \mu_2 u^*_x(s^*+0) = 0 \quad if \quad 0 < s^* < 1. \end{cases}
$$

PROOF. (i) Lemma 6.1 assures that $\omega(\varphi, l)$ is non-empty. The compactness and connectedness of $\omega(\varphi, l)$ are derived from the definition of the ω-limit set (cf. [3, pp. 91–92]).

(ii) Let $\{u^*, s^*\} \in \omega(\varphi, l)$. Then there exists a sequence $\{t_n\} \uparrow \infty$ satisfying $\lim_{n\to\infty} \{u(\cdot, t_n, \varphi, l)\} = \{u^*, s^*\}$ in Ω-topology. Therefore, it follows from Theorem 5.1 that

$$
\lim_{n\to\infty} \{u(\cdot, t; u(t_n; \varphi, l), s(t_n; \varphi, l)), s(t; u(t_n; \varphi, l), s(t_n; \varphi, l))\} \tag{6.1}
$$
$$
= \{u(t; u^*, s^*), s(t; u^*, s^*)\} \text{ in } \Omega\text{-topology}
$$

for every $t \geqq 0$. Since the uniqueness of solutions for (P) implies $\{u(\cdot, t; u(\cdot, t_n; \varphi, l), s(t_n; \varphi, l)), s(t; u(\cdot, t_n; \varphi, l), s(t_n; \varphi, l))\} = \{u(t+t_n; \varphi, l), s(t+t_n; \varphi, l)\}$ for $t \geqq 0$, it follows from (6.1) that $\{u(t; u^*, s^*), s(t; u^*, s^*)\} \in \omega(\varphi, l)$.

(iii) We introduce the functional $E(u, s)$ defined by (2.1). By Lemma 6.1 in [8, I],

$$
\frac{d}{dt} E(u(t; \varphi, l), s(t; \varphi, l)) \tag{6.2}
$$
$$
\leqq -\frac{\mu_1^2}{d_1} \int_0^{s(t)} u_t^2 dx - \frac{\mu_2^2}{d_2} \int_{s(t)}^1 u_t^2 dx - \frac{1}{2} |\dot{s}(t)|^3
$$

for $t \in \{\tau > 0; 0 < s(\tau; \varphi, l) < 1\}$. Moreover, it is easily seen that

$$
\frac{d}{dt} E(u(t; \varphi, l), s(t; \varphi, l))
$$
$$
= \begin{cases} -\dfrac{\mu_1^2}{d_1} \displaystyle\int_0^1 u_t^2 dx \quad \text{if} \quad t \in \operatorname{Int} \{\tau > 0; s(\tau; \varphi, l) \equiv 1\} \\ -\dfrac{\mu_2^2}{d_2} \displaystyle\int_0^1 u_t^2 dx \quad \text{if} \quad t \in \operatorname{Int} (\tau > 0; s(\tau; \varphi, l) \equiv 0\}, \end{cases} \tag{6.3}
$$

where Int C means the interior of a set C. Since (6.2) and (6.3) imply that $t\to E(u(t;\varphi,l), s(t;\varphi,l))$ is monotone non-increasing, $E(u(t;\varphi,l), s(t;\varphi,l))$ converges to a constant E_∞ as $t\to\infty$. Therefore, it follows from Definition 6.1 that $E(u^*, s^*)=E_\infty$ for any $\{u^*, s^*\}\in\omega(\varphi, l)$. This fact, together with the positive invariance of $\omega(\varphi, l)$, leads to

$$E(u(t;u^*,s^*), s(t;u^*,s^*)) = E_\infty \quad \text{for every} \quad t\geqq 0. \tag{6.4}$$

Differentiating (6.4) with repect to t and making use of (6.2) and (6.3) we have

$$u_t(t;u^*,s^*)=0 \quad \text{for} \quad t>0 \quad \text{and} \quad \dot{s}(t;u^*,s^*)=0 \quad \text{if} \quad s(t;\varphi,l)\in I,$$

from which it follows that

$$u(t;u^*,s^*)\equiv u^* \quad \text{and} \quad s(t;u^*,s^*)\equiv s^* \qquad \text{for} \quad t\geqq 0.$$

Hence the conclusion of (iii) is easily derived. q.e.d.

The problem (SP) given in Theorem 6.2 is called a *stationary problem* associated with (P). In subsequent sections, we will investigate various properties of solutions for (SP) (*stationary solutions*).

REMARK 6.1. Let $\{u^*, s^*\}$ be a solution of (SP). If u^* is not identically zero in $(0, s^*)$, then it must be positive in $(0, s^*)$. Similarly, u^* must be negative in $(s^*, 1)$ if it is not identically zero in $(s^*, 1)$.

§7. Stationary problem

In this section we will study (SP). First observe that for every $\xi\in\bar{I}$, $\{0, \xi\}$ satisfies (SP). So we say that $\{0, \xi\}$ is a *trivial solution*. The set $\{\{0,\xi\};\ \xi\in\bar{I}\}$ is called the *set of trivial solutions*.

Our main task is to look for non-trivial solutions of (SP). Clearly, any nontrivial solution $\{u^*, s^*\}$ of (SP) satisfies one of the following three problems;

(i) If $s^*=1$, then

$$\text{(SP–1)} \qquad \begin{cases} d_1 u^*_{xx} + u^* f(u^*) = 0, & u^*>0 \quad \text{in} \quad I, \\ u^*(0)=u^*(1)=0. \end{cases}$$

(ii) If $s^*=0$, then

$$\text{(SP–2)} \qquad \begin{cases} d_2 u^*_{xx} + u^* g(u^*) = 0, & u^*<0 \quad \text{in} \quad I, \\ u^*(0)=u^*(1)=0. \end{cases}$$

(iii) If $0<s^*<1$, then

$$\text{(SP–3)} \qquad \begin{cases} d_1 u^*_{xx} + u^* f(u^*) = 0, & u^* > 0 \quad \text{in} \quad (0, s^*), \\ d_2 u^*_{xx} + u^* g(u^*) = 0, & u^* < 0 \quad \text{in} \quad (s^*, 1), \\ u^*(0) = u^*(s^*) = u^*(1) = 0, & \\ -\mu_1 u^*_x(s^* - 0) + \mu_2 u^*_x(s^* + 0) = 0. & \end{cases}$$

We will partition the set of non-trivial solutions for (SP) into three classes, $\mathscr{P}$, $\mathscr{N}$ and $\mathscr{O}$. These are the solutions of (SP–1), (SP–2) and (SP–3), respectively.

As in the paper of Smoller and Wasserman [13] (or [9, §3]) we set

$$F(u) = \frac{2}{d_1} \int_0^u v f(v) dv \tag{7.1}$$

and define the 'time' mapping

$$T_1(p) = 2 \int_0^{\alpha(p)} \{F(\alpha(p)) - F(u)\}^{-1/2} du \tag{7.2}$$

for $p > 0$ such that there exists some $\tilde{\alpha}(p) \in (0, 1)$ satisfying $p^2 = F(\tilde{\alpha}(p))$. In (7.2), $\alpha(p) = \min\{\tilde{\alpha}(p) \in (0, 1);\ p^2 = F(\tilde{\alpha}(p))\}$. In other words, $T_1(p)$ is the minimum of $x > 0$ at which the solution $v(x; p)$ of

$$\begin{cases} d_1 v_{xx} + v f(v) = 0, & x > 0, \\ v(0) = 0, \quad v_x(0) = p & (>0), \end{cases} \tag{7.3}$$

vanishes. Therefore, any solution $\{u^*, 1\}$ in $\mathscr{P}$ is obtained by looking for p^* such that $T_1(p^*) = 1$ and u^* is given by $u^*(x) = v(x; p^*)$.

Similarly, we define

$$T_2(q) = 2 \int_0^{\beta(q)} \{G(\beta(q)) - G(u)\} du, \quad q > 0 \tag{7.4}$$

$$\text{with} \quad G(u) = \frac{2}{d_2} \int_0^u v g(-v) dv,$$

where $\beta(q) = \min\{\tilde{\beta}(q) \in (0, 1);\ q^2 = G(\tilde{\beta}(q))\}$. Then it is seen that $1 - T_2(q)$ is the maximum of $x < 1$ where the solution $w(x; q)$ of

$$\begin{cases} d_2 w_{xx} + w g(w) = 0, & x < 1, \\ w(1) = 0, \quad w_x(1) = q & (>0), \end{cases} \tag{7.5}$$

vanishes. Therefore, every solution in $\mathscr{N}$ is given by $\{w(x; q^*), 0\}$, where q^* satisfies $T_2(q^*) = 1$.

We can also look for solutions in $\mathscr{O}$ with the aid of time mappings T_1 and T_2. Consider the following auxiliary problem

$$\begin{cases} d_1 u_{xx} + uf(u) = 0, & u > 0 \quad \text{in} \quad (0, \xi), \\ d_2 u_{xx} + ug(u) = 0, & u < 0 \quad \text{in} \quad (\xi, 1), \\ u(0) = u(\xi) = u(1) = 0, \end{cases} \tag{7.6}$$

where $\xi \in I$ is any fixed number. Suppose that $p(\xi)$ and $q(\xi)$ satisfy

$$T_1(p(\xi)) = \xi \tag{7.7}$$

and

$$T_2(q(\xi)) = 1 - \xi, \tag{7.8}$$

respectively. Then the function

$$u(x; \xi) = \begin{cases} v(x; p(\xi)) & \text{for} \quad 0 \leqq x \leqq \xi, \\ w(x; q(\xi)) & \text{for} \quad \xi \leqq x \leqq 1, \end{cases} \tag{7.9}$$

satisfies (7.6). Since

$$-\mu_1 u_x(\xi - 0; \xi) = \mu_1 p(\xi) \quad \text{and} \quad \mu_2 u_x(\xi + 0; \xi) = -\mu_2 q(\xi), \tag{7.10}$$

it suffices to look for all $s^* \in I$ satisfying $\mu_1 p(s^*) = \mu_2 q(s^*)$ in order to get all solutions in $\mathcal{O}$.

From the preceding consideration it becomes important to study the qualitative nature of T_1 and T_2. In what follows, we will put some restriction on f and g to make our arguments clear and avoid technical complexity. The following two typical cases are considered here.

Case A. In addition to (A.1) (resp. (A.2)), $f(u)$ (resp. $g(u)$) is monotone non-increasing (resp. non-decreasing) and positive on $[0, 1)$ (resp. $(-1, 0]$.

Case B. f and g are quadratic polynomials of the form

$$f(u) = -\nu_1 (u - a)(u - 1) \quad \text{with} \quad \nu_1 > 0 \quad \text{and} \quad 0 \leqq a < 1/2, \tag{7.11}$$

$$g(u) = -\nu_2 (u + b)(u + 1) \quad \text{with} \quad \nu_2 > 0 \quad \text{and} \quad 0 \leqq b < 1/2. \tag{7.12}$$

7.1. Analysis of (SP) in Case A

In view of (7.2), $T_1(p)$ is defined for $0 < p < p_0 \equiv \sqrt{F(1)}$ and expressed as

$$T_1(p) = \sqrt{2d_1} \int_0^1 \left\{ \int_v^1 wf(\alpha(p)w)dw \right\}^{-1/2} dv. \tag{7.13}$$

Since $p \to \alpha(p)$ is monotone increasing for $p \in (0, p_0)$, we see from (7.13) and the monotonicity of f that $T_1(p)$ is continuous and monotone increasing for $p \in (0, p_0)$. Moreover, simple calculations show

$$\lim_{p\to 0} T_1(p) = \pi\sqrt{d_1/f(0)} \quad \text{and} \quad \lim_{p\to p_0} T_1(p) = \infty. \tag{7.14}$$

The graph of T_1 is given in Fig. 2.

In the same manner, one can prove that $T_2(q)$ is continuous and monotone increasing for $q \in (0, q_0)$ with $q_0 = \sqrt{G(1)}$ and that it satisfies

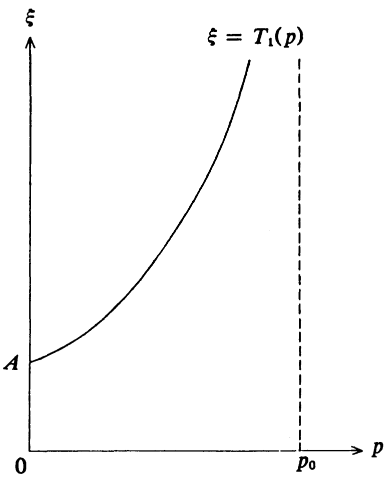

Fig. 2. Graph of T_1 in Case A.

$$\lim_{q\to 0} T_2(q) = \pi\sqrt{d_2/g(0)} \quad \text{and} \quad \lim_{q\to q_0} T_2(q) = \infty. \tag{7.15}$$

Now we are ready to state the existence result of non-trivial solutions for (SP).

THEOREM 7.1. *Set* $A = \pi\sqrt{d_1/f(0)}$ *and* $B = \pi\sqrt{d_2/g(0)}$.

(i) (SP) *has no solutions in* $\mathscr{P}$ *if* $A \geqq 1$ *and has a unique solution* $\{\bar{u}, 1\}$ *in* $\mathscr{P}$ *if* $A < 1$.

(ii) (SP) *has no solutions in* $\mathscr{N}$ *if* $B \geqq 1$ *and has a unique solution* $\{\underline{u}, 0\}$ *in* $\mathscr{N}$ *if* $B < 1$.

(iii) (SP) *has no solutions in* $\mathscr{O}$ *if* $A + B \geqq 1$ *and has a unique solution* $\{u_c, c\}$ in $\mathscr{O}$ *if* $A + B < 1$. *Moreover,*

$$\underline{u} < u_c < \bar{u} \quad in \quad I, \tag{7.16}$$

for the case $A + B < 1$ *where the existence of* $\underline{u}$ *and* $\bar{u}$ *is assured by* (i) *and* (ii).

PROOF. By virtue of the monotonicity of T_1 and (7.14), equation (7.7) determines uniquely a continuous and monotone increasing function $p(\xi)$ for $\xi \geqq A$ with $p(A) = 0$. Similary, since T_2 is also monotone increasing and (7.15) holds, a continuous and monotone decreasing function $q(\xi)$ is determined uniquely from (7.8) for $\xi \leqq 1-B$ with $q(1-B)=0$. So, (SP) has no solutions in $\mathscr{P}$ and $\mathscr{N}$ for $A \geqq 1$ and $B \geqq 1$, respectively. The unique solution $\{\bar{u}, 1\}$ in $\mathscr{P}$ for $A<1$ is expressed as

$$\bar{u}(x) = v(x;\, p(1)), \tag{7.17}$$

where $v(x;\, p)$ is the solution of (7.3). Similarly, the unique solution $\{\underline{u}, 0\}$ in $\mathscr{N}$ for $B<1$ is expressed as

$$\underline{u}(x) = w(x;\, q(0)), \tag{7.18}$$

where $w(x;\, q)$ is the solution of (7.5).

In order to look for solutions in $\mathscr{O}$, we draw two curves

$$C_1 = \{(\xi, \eta);\, \eta=\mu_1 p(\xi),\, \xi>A\}$$

and

$$C_2 = \{(\xi, \eta);\, \eta=\mu_2 q(\xi),\, \xi<1-B\}.$$

in (ξ, η)-plane. See Fig. 3. By the monotonicity of p and q, one can see that, if $A+B \geqq 1$, then C_1 and C_2 do not intersect; so there are no solutions in $\mathscr{O}$. If $A+B<1$, then C_1 and C_2 intersect at a unique point P_c, whose ξ-coordinate is denoted by c. So there is a unique solution in $\mathscr{O}$, which is expressed as $\{u(\cdot\,;\, c), c\}$, where $u(\cdot\,;\, \xi)$ is given by (7.9).

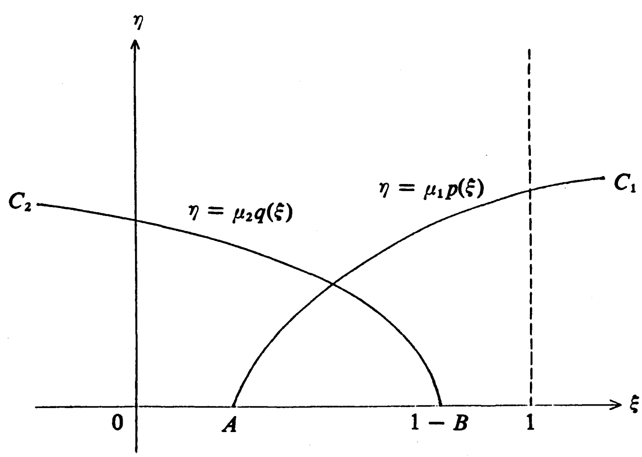

Fig. 3

We will show the order relation (7.16) in the case $A+B<1$. Since T_1 and T_2 are monotone increasing, we can employ the method in [9, Lemma 3.2] (which is based on the phase plane analysis) to show that

$$\bar{u}(x) = v(x;\, p(1)) > v(x;\, p(c)) > 0 \qquad \text{for} \quad 0 < x \leqq c$$

and

$$0 > w(x;\, q(c)) > w(x;\, q(0)) = \underline{u}(x) \qquad \text{for} \quad c \leqq x < 1.$$

(Recall (7.17) and (7.18)). Hence (7.16) easily follows in view of the expression of $u(x;\, c)$. q.e.d.

7.2. Analysis in Case B.

We will study the time mapping T_1 by substituting (7.11) into (7.13). It is easy to see that $T_1(p)$ is defined for $0<p<p_0\equiv=\{\nu_1(1-2a)/6d_1\}^{1/2}$. The result of Smoller and Wasserman ([13, Theorem 2.1]) tells us that $T_1(p)$ has exactly one critical point at $p=p^*\in(0,\, p_0)$ (see also [12]). Hence $T_1(p)$ is monotone decreasing on $(0,\, p^*)$ and monotone increasing on $(p^*,\, p_0)$. Moreover, a simple calculation yields

$$\lim_{p\to 0} T_1(p) = \lim_{p\to p_0} T_1(p) = \infty.$$

Setting $A^*=T_1(p^*)$, we define $p_1(\xi)\in[p^*,\, p_0)$ (resp. $p_2(\xi)\in(0,\, p^*]$) for $\xi\geqq A^*$ by (7.7). Then $p_1(\xi)$ is continuous and monotone increasing for $\xi\geqq A^*$ and $p_2(\xi)$ is continuous and monotone decreasing for $\xi\geqq A^*$. See Fig. 4.

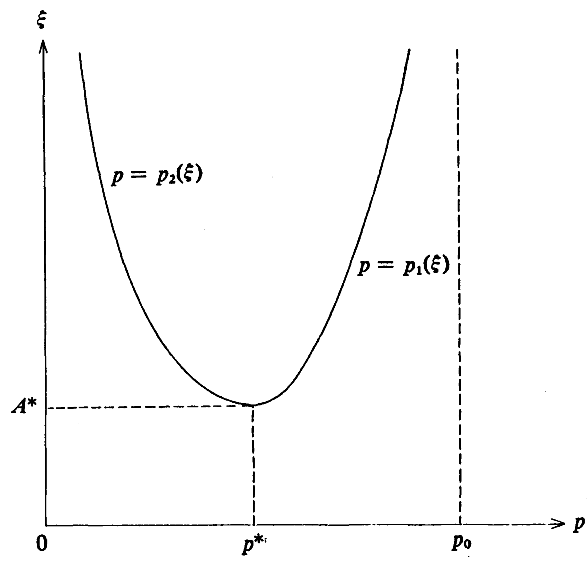

Fig. 4. Graph of T_1 in Case B.

Analogously, $T_2(q)$ has exactly one critical point at $q=q^*\in(0,\, q_0)$ with $q_0=\{\nu_2(1-2b)/6d_2\}^{1/2}$. Set $B^*=T_2(q^*)$. Then (7.8) determines a unique function

$q_1(\xi)$ (resp. $q_2(\xi)$) with values in $[q^*, q_0)$ (resp. $(0, q^*]$) for $\xi \leqq 1-B^*$. Then $q_1(\xi)$ (resp. $q_2(\xi)$) is monotone deceasing (resp. increasing) for $\xi \leqq 1-B^*$.

As in the proof of Theorem 7.1, we will draw the following curves in (ξ, η)-plane:

$$C_1 = C_{11} \cup C_{12} \quad \text{with} \quad C_{1i} = \{(\xi, \eta);\, \eta = \mu_1 p_i(\xi) \text{ for } \xi \geqq A^*\} \quad (i=1, 2),$$

and

$$C_2 = C_{21} \cup C_{22} \quad \text{with} \quad C_{2i} = \{(\xi, \eta);\, \eta = \mu_2 q_i(\xi) \text{ for } \xi \leqq 1-B^*\} \quad (i=1, 2).$$

See Fig. 5.

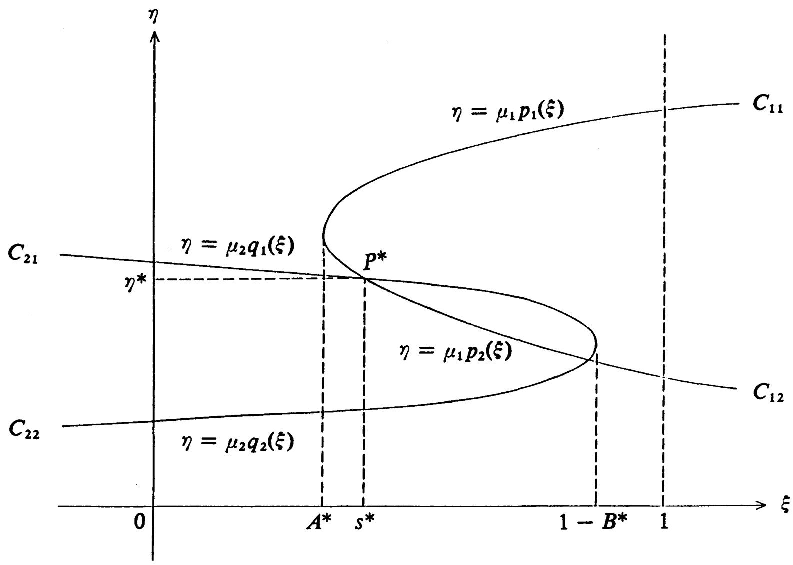

Fig. 5

The existence results for (SP) can be stated as follows.

THEOREM 7.2.

(i) (SP) *has no solutions in* $\mathscr{P}$ *if* $A^*>1$, *a unique solution* $\{\bar{u}_1, 1\}$ *in* $\mathscr{P}$ *if* $A^*=1$ *and two solutions* $\{\bar{u}_1, 1\}$, $\{\bar{u}_2, 1\}$ *in* $\mathscr{P}$ *with* $\bar{u}_1>\bar{u}_2$ *if* $A^*<1$.

(ii) (SP) *has no solutions in* $\mathscr{N}$ *if* $B^*>1$, *a unique solution* $\{\underline{u}_1, 0\}$ *in* $\mathscr{N}$ *if* $B^*=1$ *and two solution* $\{\underline{u}_1, 0\}$, $\{u_2, 0\}$ *in* $\mathscr{N}$ *with* $\underline{u}_1>\underline{u}_2$ *if* $B^*<1$.

(iii) *When* $A^*+B^*>1$, (SP) *has no solutions in* $\mathscr{O}$.

(iv) *When* $A^*+B^*\leqq 1$, (SP) *has the same number of solutions in* $\mathscr{O}$ *as that of intersecting points of* C_1 *and* C_2. *Moreover any solution* $\{u^*, s^*\}$ *in* $\mathscr{O}$ *satisfies*

$$\underline{u}_1(x) < u^*(x) < \bar{u}_1(x) \qquad \text{for} \quad x \in I, \tag{7.19}$$

where $\underline{u}_1$ *and* $\bar{u}_1$ *are the solutions in* (i) *and* (ii), *whose existence is assured for* $A^*+B^*<1$.

PROOF. The idea of the proof is almost the same as that of Theorem 7.1. We note that the number of solutions in $\mathscr{P}$ is identical with that of p's satisfying $T_1(p)=1$. So (SP) has no solutions in $\mathscr{P}$ for $A^*>1$. For $A^*\leqq 1$, there are solutions $\{\bar{u}_1, 1\}$, $\{\bar{u}_2, 1\}$ in $\mathscr{P}$, which are expressed as

$$\bar{u}_1(x) = v(x;\, p_1(1)) \quad \text{and} \quad \bar{u}_2(x) = v(x;\, p_2(1)),$$

where $v(x;\, p)$ is the solution of (7.3). These solutions are identical for $A^*=1$ because $p_1(A^*)=p_2(A^*)$. If $A^*<1$, then $p_1(1)>p_2(1)$; so that it is possible to show $\bar{u}_1>\bar{u}_2$ with the aid of phase plane analysis (see [9, Lemma 3.1]). Thus the proof of (i) is complete.

One can prove (ii) in a similar manner.

If we want to know the number of solutions in $\mathscr{O}$, we may count the number of intersecting points of C_1 and C_2. Therefore, it is easy to see (iii) and the first half of (iv).

It remains to prove that any solution $\{u^*, s^*\}\in\mathscr{O}$ satisfies (7.19) for $A^*+B^*\leqq 1$. In (7.19), $\bar{u}_1$ and $\underline{u}_1$ are given by

$$\bar{u}_1(x) = v(x;\, p_1(1)) \quad \text{and} \quad \underline{u}_1(x) = w(x;\, q_1(0)),$$

where $v(\cdot\,;\, p)$ and $w(\cdot\,;\, q)$ are the solutions of (7.3) and (7.5), respectively. For the sake of convenience, let $\{u^*, s^*\}$ correspond to an intersecting point $P^*=(s^*, \eta^*)$ of C_{12} and C_{21} as in Fig. 5; that is, $\eta^*=\mu_1 p_2(s^*)=\mu_2 q_1(s^*)$ and u^* is expressed as

$$u^*(x) = \begin{cases} v(x;\, p_2(s^*)) & \text{for} \quad 0\leqq x\leqq s^* \\ w(x;\, q_1(s^*)) & \text{for} \quad s^*\leqq x\leqq 1, \end{cases} \tag{7.20}$$

by (7.9). Since $p_1(s^*)\geqq p_2(s^*)$, the same reasoning as in the proof of (i) yields

$$v(x;\, p_2(s^*)) \leqq v(x;\, p_1(s^*)) \qquad \text{for} \quad 0\leqq x\leqq s^*. \tag{7.21}$$

We now make use of Lemma 3.2 in [9]. Since $p_1(\xi)$ is monotone increasing for $\xi>A^*$, we can show

$$v(x;\, p_1(s^*)) \leqq v(x;\, p_1(1)) = \bar{u}_1(x) \qquad \text{for} \quad 0\leqq x\leqq s^*. \tag{7.22}$$

Similarly,

$$\underline{u}_1(x) = w(x;\, q_1(0)) \leqq w(x;\, q_1(s^*)) \qquad \text{for} \quad s^*\leqq x\leqq 1, \tag{7.23}$$

because $q_1(\xi)$ is monotone decreasing for $\xi<1-B^*$. Therefore, the order relation (7.19) is derived from (7.20), (7.21), (7.22) and (7.23) q. e. d.

REMARK 7.1. Theorem 7.2 and its proof imply the maximality of $\{\bar{u}_1, 1\}\in\mathscr{P}$

with $\bar{u}_1 = v(\cdot\,;\,p_1(1))$ (whenever it exists) in the sense that $\{\bar{u}_1, 1\} \geqq \{u^*, s^*\}$ for every solution $\{u^*, s^*\}$ of (SP). Similarly, $\{\underline{u}_1, 0\} \in \mathscr{N}$ with $\underline{u}_1 = w(\cdot\,;\,q_1(0))$ is minimal whenever it exists.

REMARK 7.2. We have carried out the analysis of (SP) by putting the restrictions $0 \leqq a < 1/2$ and $0 \leqq b < 1/2$ in (7.11) and (7.12). For $a \leqq -1$, f satisfies the conditions in Case A; so the results in 7.1 are valid. For $a \geqq 1/2$, (SP) has no solutions in $\mathscr{P}$; so the analysis of (SP) becomes very simple. For $-1 < a < 0$, we can show that T_1 has only one critical point (see [13, Theorem 2.2]). Therefore, our method developed in 7.2 remains valid. Since the similar results hold for g, one can get complete information on the structure of $\mathscr{P}$, $\mathscr{N}$ and $\mathscr{O}$ for general f and g.

§ 8. Asymptotic behavior

In this section we will investigate asymptotic properties of smooth solutions of (P) in connection with stability or instability of stationary solutions.

We first prepare some terminology. For $\{u^i, s^i\} \in H_0^1(I) \times \bar{I}$ $(i=1, 2)$, we sometimes use the metric defined by

$$\rho(\{u^1, s^1\}, \{u^2, s^2\}) = \|u^1 - u^2\|_{H_0^1(I)} + |s^1 - s^2|.$$

DEFINITION 8.1. Let $\{u^*, s^*\}$ be any solution of (SP). Then it is said that $\{u^*, s^*\}$ is *asymptotically stable from above* (resp. *from below*) if there exists a positive number δ with the following property: whenever the initial data $\{\varphi, l\}$ fulfills

$$\rho(\{\varphi, l\}, \{u^*, s^*\}) < \delta \quad \text{and} \quad \{\varphi, l\} \geqq \{u^*, s^*\}$$
$$(\text{resp. } \{\varphi, l\} \leqq \{u^*, s^*\}),$$

the smooth solution $\{u(\cdot, t; \varphi, l), s(t; \varphi, l)\}$ of (P) satisfies

$$\lim_{t\to\infty} \rho(\{u(\cdot, t; \varphi, l), s(t; \varphi, l)\}, \{u^*, s^*\}) = 0.$$

In particular, it is said that $\{u^*, s^*\}$ is *asymptotically stable* if it is asymptotically stable from above and below.

DEFINITION 8.2. Let $\{u^*, s^*\}$ be any solution of (SP). It is said that $\{u^*, s^*\}$ is *asymptotically unstable* if for any $\varepsilon > 0$ there exists some $\{\varphi, l\}$ and $\delta > 0$ which satisfy $\rho(\{u^*, s^*\}, \{\varphi, l\}) < \varepsilon$ and $\limsup_{t\to\infty} \rho(\{u(\cdot, t; \varphi, l)\}, \{u^*, s^*\}) \geqq \delta$.

We also introduce the notion of asymptotic stability for a subset of $H_0^1(I) \times \bar{I}$ in place of a stationary solution.

DEFINITION 8.3. Let K be any set in $H_0(I) \times \bar{I}$. Then it is said that K is *asymptotically stable as a set* if there is a positive number δ such that, if

$$\operatorname{dist}(\{\varphi, l\}, K) \equiv \inf_{\{u,s\} \in K} \rho(\{\varphi, l\}, \{u, s\}) < \delta,$$

then

$$\lim_{t \to \infty} \operatorname{dist}(\{u(\cdot, t; \varphi, l), s(t; \varphi, l)\}, K) = 0.$$

8.1. Stability analysis in Case A

First we will study Case A for which Theorem 7.1 gives complete information on the set of stationary solutions.

THEOREM 8.1. *Let f and g satisfy the conditions in Case A. Set $A = \pi\sqrt{d_1/f(0)}$, $B = \pi\sqrt{d_2/g(0)}$.*

(i) *If $A \geqq 1$ and $B \geqq 1$, then the set of trivial solutions is asymptotically stable as a set in the sense that*

$$\lim_{t \to \infty} u(\cdot, t; \varphi, l) = 0 \quad \text{in} \quad H_0^1(I) \tag{8.1}$$

for all $\{\varphi, l\}$. Moreover, if $A > 1$ and $B > 1$, then

$$\lim_{t \to \infty} s(t; \varphi, l) = s^* \tag{8.2}$$

with some s^* depending on $\{\varphi, l\}$.

(ii) *If $A < 1$ and $B \geqq 1$, then $\{\bar{u}, 1\} \in \mathscr{P}$ is asymptotically stable and any trivial solution $\{0, \xi\}$ with $A < \xi \leqq 1$ is asymptotically unstable in the sense that*

$$\lim_{t \to \infty} \{u(\cdot, t; \varphi, l), s(t; \varphi, l)\} = \{\bar{u}, 1\} \quad \text{in} \quad \Omega\text{-topology} \tag{8.3}$$

for any $\{\varphi, l\}$ satisfying $\varphi \geqq 0$, $\varphi \not\equiv 0$ and $A < l \leqq 1$. Moreover, the set $\{\{0, \xi\};\ 0 \leqq \xi < A$ and $\pi^2(d_1\xi^{-2} - d_2(1-\xi)^{-2}) > f(0) - g(0)\}$ is asymptotically stable as a set.

(iii) *If $A \geqq 1$ and $B < 1$, then $\{\underline{u}, 0\} \in \mathscr{N}$ is asymptotically stable and any trivial solution $\{0, \xi\}$ with $0 \leqq \xi < 1 - B$ is asymptotically unstable in the sense that*

$$\lim_{t \to \infty} \{u(\cdot, t; \varphi, l), s(t; \varphi, l)\} = \{\underline{u}, 0\} \quad \text{in} \quad \Omega\text{-topology} \tag{8.4}$$

for any $\{\varphi, l\}$ satisfying $\varphi \leqq 0$, $\varphi \not\equiv 0$ and $0 \leqq l < 1 - B$. Moreover, the set $\{\{0, \xi\};\ 1 - B < \xi \leqq 1$ and $\pi^2(d_1\xi^{-2} - d_2(1-\xi)^{-2}) < f(0) - g(0)\}$ is asymptotically stable as a set.

(iv) *If $A < 1$, $B < 1$, and $A + B > 1$, then $\{\bar{u}, 1\} \in \mathscr{P}$ and $\{\underline{u}, 1\} \in \mathscr{N}$ are* *asymptotically stable and any trivial solution* $\{0, \xi\}$ *satisfying* $0 \leqq \xi < 1 - B$ *or* $A < \xi < 1$ *is asymptotically unstable in the sense of* (ii) *or* (iii).

(v) *If* $A + B < 1$, *then both* $\{\bar{u}, 1\} \in \mathscr{P}$ *and* $\{\underline{u}, 0\} \in \mathscr{N}$ *are asymptotically stable and any trivial solution is asymptotically unstable. Moreover,* $\{u_c, c\} \in \mathscr{O}$ *is asymptotically unstable in the sense that* (8.3) (*resp.* (8.4)) *holds true for any* $\{\varphi, l\}$ *satisfying* $\{\varphi, l\} \geqq \{u_c, c\}$ (*resp.* $\{\varphi, l\} \leqq \{u_c, c\}$) *with* $\{\varphi, l\} \not\equiv \{u_c, c\}$.

PROOF. (i) In this case (SP) has no non-trivial solutions by Theorem 7.1. Therefore, Theorem 6.2 implies that $\omega(\varphi, l)$ is contained in the set of trivial solutions; so that (8.1) easily follows. In order to show (8.2) in the case $A > 1$ and $B > 1$, we consider the following functions

$$\bar{U}(x, t) = a(t) \sin \pi x, \tag{8.5}$$

$$\underline{U}(x, t) = -\, b(t) \sin \pi x, \tag{8.6}$$

as comparison functions. We take

$$\begin{aligned} a(t) &= a(0) \exp\{(f(0) - \pi^2 d_1)t\}, \\ b(t) &= b(0) \exp\{(g(0) - \pi^2 d_2)t\}, \end{aligned} \tag{8.7}$$

where $a(0) > 0$ and $b(0) > 0$ are sufficiently large numbers such that

$$-\, b(0) \sin \pi x \leqq \varphi(x) \leqq a(0) \sin \pi x \qquad \text{for} \quad x \in \bar{I}, \tag{8.8}$$

then $\{\bar{U}, 1\}$ and $\{\underline{U}, 0\}$, respectively, become a supersolution and a subsoluton of (P). Hence, by virtue of (8.8), Theorem 4.2 gives

$$\{\underline{U}(\cdot, t), 0\} \leqq \{u(\cdot, t; \varphi, l), s(t; \varphi, l)\} \leqq \{\bar{U}(\cdot, t), 1\} \quad \text{for all} \quad t \geqq 0, \tag{8.9}$$

which, in particular, implies the exponential decay of $|u(\cdot, t; \varphi, l)|$ to zero as $t \to \infty$ (use $A > 1$, $B > 1$ and (8.7)). Now (8.2) is trivial in the case when $s(t; \varphi, l)$ arrives at a fixed end in a finite time T^* (and, therefore, stays there for $t \geqq T^*$). Hence we may assume $0 < s(t; \varphi, l) < 1$ for all $t \geqq 0$. Multiplying (1.1) by $\mu_1 x/d_1$ and integrating the resulting expression over S_t^- one gets

$$\begin{aligned} &\frac{\mu_1}{d_1} \int_0^{s(t)} x u(x, t)\, dx - \frac{\mu_1}{d_1} \int_0^l x \varphi(x) dx \\ &= \mu_1 \int_0^t s(\tau) u_x(s(\tau) - 0, \tau) d\tau + \frac{\mu_1}{d_1} \int_0^t d\tau \int_0^{s(\tau)} x u f(u) dx. \end{aligned} \tag{8.10}$$

Similarly, multiplying (1.2) by $\mu_2 x/d_2$ and integrating over S_t^+ we have

$$\frac{\mu_2}{d_2} \int_{s(t)}^1 x u(x, t) dx - \frac{\mu_2}{d_2} \int_l^1 x \varphi(x) dx$$

$$= \mu_2 \int_0^t u_x(1, \tau)d\tau - \mu_2 \int_0^t s(\tau)u_x(s(\tau)+0, \tau)d\tau \tag{8.11}$$

$$+ \frac{\mu_2}{d_2} \int_0^t d\tau \int_{s(\tau)}^1 xug(u)dx.$$

By virtue of (1.6), addition of (8.10) and (8.11) yields

$$\frac{1}{2} s(t)^2 = \left\{\frac{1}{2} l^2 + \frac{\mu_1}{d_1} \int_0^l x\varphi(x)dx + \frac{\mu_2}{d_2} \int_l^1 x\varphi(x)dx\right\}$$

$$+ \left\{- \frac{\mu_1}{d_1} \int_0^{s(t)} xu(x, t)dx - \frac{\mu_2}{d_2} \int_{s(t)}^1 xu(x, t)dx\right\} \tag{8.12}$$

$$+ \mu_2 \int_0^t u_x(1, \tau)d\tau + \int_0^t d\tau \left\{\frac{\mu_1}{d_1} \int_0^{s(\tau)} xuf(u)dx + \frac{\mu_2}{d_2} \int_{s(\tau)}^1 xug(u)dx\right\}.$$

for every $t \geqq 0$. It follows from (8.9) that the second term in the right-hand side of (8.12) approaches to zero as $t \to \infty$ and that the fourth term converges as $t \to \infty$. Moreover, (8.9) assures

$$0 \leqq u_x(1, \tau) \leqq \pi b(0) \exp\{(g(0) - \pi^2 d_2)t\} \qquad \text{for} \quad t \geqq 0;$$

so that the third term also converges as $t \to \infty$. Thus we find from (8.12) that (8.2) holds true with some s^*.

(ii) Let $\xi \in I$ be fixed. We make use of the following functions

$$U(x, t) = \begin{cases} a(t) \sin \pi x/\xi & \text{for } 0 \leqq x \leqq \xi \text{ with } a > 0, \\ -b(t) \sin \pi(1-x)/(1-\xi) & \text{for } \xi \leqq x \leqq 1 \text{ with } b > 0 \end{cases} \tag{8.13}$$

as comparison functions in place of (8.5) or (8.6). It is easy to see that $\{U, \xi\}$ is a supersolution of (P) if a and b satisfy

$$\begin{gathered} \dot{a} = a\left(f(0) - \frac{d_1\pi^2}{\xi^2}\right), \quad \dot{b} = b\left(g(-b) - \frac{d_2\pi^2}{(1-\xi)^2}\right) \\ \text{and} \quad \frac{a\mu_1}{\xi} \leqq \frac{b\mu_2}{1-\xi} \quad \text{with} \quad a \leqq 1 \quad \text{and} \quad b \leqq 1, \end{gathered} \tag{8.14}$$

while $\{U, \xi\}$ is a subsolution of (P) if

$$\begin{gathered} \dot{a} = a\left(f(a) - \frac{d_1\pi^2}{\xi^2}\right), \quad \dot{b} = b\left(g(0) - \frac{d_2\pi^2}{(1-\xi)^2}\right) \\ \text{and} \quad \frac{a\mu_1}{\xi} \geqq \frac{b\mu_2}{1-\xi} \quad \text{with} \quad a \leqq 1 \quad \text{and} \quad b \leqq 1. \end{gathered} \tag{8.15}$$

First we fix $\xi \in (A, 1)$ and construct subsolutions of (P) by using (8.15). It follows from the assumptions $A < 1$ and $B \geqq 1$ that

$$b(t) = b(0)\exp\left\{\left(g(0) - \frac{d_2\pi^2}{(1-\xi)^2}\right)t\right\} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty, \tag{8.16}$$

and

$$\lim_{t\to\infty} a(t) = u_\xi \tag{8.17}$$

for some $u_\xi \in \Phi_\xi \equiv \{u \in I; f(u) = d_1\pi^2/\xi^2\}$. Observe $a(t) \geqq \min\{a(0), \min\{v_\xi; v_\xi \in \Phi_\xi\}\}$. Therefore, in order to get a subsolution of (P), it suffices to choose $a(0) \leqq 1$ and $b(0) \leqq 1$ such that

$$\mu_1(1-\xi)\min\{a(0), \min\{v_\xi; v_\xi \in \Phi_\xi\}\} \geqq \mu_2\xi b(0)$$

(use (8.15)). Theorem 4.2 assures that, if $\{\varphi, l\} \geqq \{U(\cdot, 0), \xi\}$ then $\{u(\cdot, t; \varphi, l), s(t; \varphi, l)\} \geqq \{U(\cdot, t), \xi\}$ for every $t \geqq 0$. Therefore, in view of (8.16) and (8.17), one can assert that for any $\{u^*, s^*\} \in \omega(\phi, l)$

$$\{u^*, s^*\} \geqq \{U_\xi, \xi\} \tag{8.18}$$

where $U_\xi(x) = u_\xi \sin \pi x/\xi$ for $x \in [0, \xi]$ and $U_\xi(x) = 0$ for $x \in [\xi, 1]$. On the other hand, it follows from Theorems 6.2 and 7.1 that stationary solution satisfying (8.18) must be $\{\bar{u}, 1\}$; so that $\omega(\varphi, l) = \{\{\bar{u}, 1\}\}$. Therefore, we have shown the validity of (8.3) for any $\{\varphi, l\}$ such that $\{\varphi, l\} \geqq \{U(\cdot, 0), \xi\}$. This fact implies the asymptotic instability of $\{0, \xi\}$ with $A < \xi < 1$ as well as the asymptotic stability of $\{\bar{u}, 1\}$. (The assertion obtained here is stronger than that of (ii).)

We next fix $\xi \in (0, A)$ and construct supersolutions of the form (8.13). In view of (8.14),

$$a(t) = a(0)\exp\left\{\left(f(0) - \frac{d_1\pi^2}{\xi^2}\right)t\right\} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty$$

and

$$b(t) \leqq b(0)\exp\left\{\left(g(0) - \frac{d_2\pi^2}{(1-\xi)^2}\right)t\right\} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow \infty.$$

Moreover, one can see that for any $\varepsilon > 0$,

$$b(t) \geqq b(T_\varepsilon)\exp\left\{\left(g(0) - \frac{d_2\pi^2}{(1-\xi)^2} - \varepsilon\right)t\right) \quad \text{for} \quad t \geqq T_\varepsilon$$

with some $T_\delta > 0$. If ξ satisfies

$$f(0) - g(0) < \pi^2\left(\frac{d_1^2}{\xi^2} - \frac{d_2\pi^2}{(1-\xi)^2}\right), \tag{8.19}$$

it is possible to choose sufficiently small $a(0)$ so that $\{U(\cdot, t), \xi\}$ becomes a supersolution of (P). Therefore, Theorem 4.2 implies

$$\{u(\cdot, t; \varphi, l), s(t; \varphi, l)\} \leqq \{U(\cdot, t), \xi\}$$

for any $\{\varphi, l\} \leqq \{U(\cdot, 0), \xi\}$. The same resoning as in the case $A<\xi<1$ shows that $\omega(\varphi, l)$ is contained in a set of trivial solutions $\{0, \xi\}$ satisfying $\xi \in (0, A)$ and (8.19). This fact yields the asymptotic stability of this set.

The proofs of (iii) and (iv) are accomplished in the same way as that of (ii) with use of the comparison principle.

The proof of (v) is completed as follows. For $\xi \in (A, 1)$, take a subsolution $\{U(\cdot, t), \xi\}$ defined by (8.13) with $b(t) \equiv 0$. Repeating the preceding procedure leads us to conclude the asymptotic stability of $\{\bar{u}, 1\}$ and the asymptotic instability of $\{0, \xi\}$ with $A<\xi<1$. The asymptotic stability of $\{\underline{u}, 0\}$ and the asymptotic instability of $\{0, \xi\}$ with $0<\xi<1-B$ are derived in an analogous manner. It remains to show the asymptotic instability of $\{u_c, c\}$. In place of (8.13), we consider

$$U(x) = \begin{cases} v(x; p(\xi)) & \text{for} \quad 0 \leqq x \leqq \xi \\ w(x; q(\xi)) & \text{for} \quad \xi \leqq x \leqq 1, \end{cases}$$

where $v(x; p)$ (resp. $w(x; q)$) is the solution of (7.3) (resp. (7.5)) with $A<\xi<1-B$. Since $\{U, \xi\}$ is a supersolution (resp. subsolution) of (P) for any $\xi \in (A, c)$ (resp. $\xi \in (c, 1-B)$), the standard method based on the comparison principle yields the instability of $\{u_c, c\}$ in the sense of (v) (see [8, §9]). q.e.d.

We will study conditions under which the free boundary arrives at one of the fixed ends in a finite time.

THEOREM 8.2. *Let $\{u(\cdot, t), s(t)\}$ be a smooth solution of* (P). *If*

$$\lim_{t\to\infty} \{u(t), s(t)\} = \{\bar{u}, 1\} \ (\textit{resp.}\ \{\underline{u}, 0\}) \ \textit{in } \Omega\textit{-topology}, \tag{8.20}$$

then there exists a non-negative number $T_1<\infty$ (resp. $T_2<\infty$) such that

$$s(t) \equiv 1 \quad \textit{for} \quad t \geqq T_1 \quad (\textit{resp.}\ s(t) \equiv 0 \textit{ for } t \geqq T_2).$$

PROOF. Assume that $\lim_{t\to\infty} \{u(t), s(t)\} = \{\bar{u}, 1\}$ (in Ω-topology) and that $0<s(t)<1$ for all $t \geqq 0$. Then we can make use of the identity (8.12). Since $u \geqq 0$ in S^-, $u \leqq 0$ in S^+ and $u_x(1, t) \geqq 0$ for $t \geqq 0$, it follows from (8.12) that

$$\begin{aligned} \frac{1}{2} s(t)^2 \geqq & \left\{\frac{1}{2} l^2 + \frac{\mu_2}{d_2} \int_l^1 x\varphi(x)dx - \frac{\mu_1}{d_1} \int_0^{s(t)} xu(x, t)\, dx\right\} \\ & + \int_0^t d\tau \left\{\frac{\mu_1}{d_1} \int_0^{s(\tau)} xuf(u)dx + \frac{\mu_2}{d_2} \int_{s(\tau)}^1 xug(u)dx\right\}, \end{aligned} \tag{8.21}$$

where the first term in the right-hand side of (8.21) is bounded from below by some constant. Since $\{\bar{u}, 1\} \in \mathscr{P}$ satisfies (SP–1),

$$\frac{\mu_1}{d_1}\int_0^1 x\bar{u}f(\bar{u})dx = -\mu_1\int_0^1 x\bar{u}_{xx}dx = -\mu_1\bar{u}_x(1) \equiv c_1 < 0.$$

Therefore, by virtue of (8.21), there exists a positive number T_0 such that

$$\frac{\mu_1}{d_1}\int_0^{s(t)} xu(x, t)f(u(x, t))dx + \frac{\mu_2}{d_2}\int_{s(t)}^1 xu(x, t)g(u(x, t))dx$$
$$\geqq \frac{1}{2}c_1 \qquad \text{for all} \quad t \geqq T_0.$$

Therefore, letting $t\to\infty$ we see that the right-hand side of (8.21) tends to ∞. This result contradicts the boundedness of $s(t)$; so that the free boundary hits a fixed end in a finite time.

If we wish to get the assertion in the case where $\lim_{t\to\infty}\{u(t), s(t)\} = \{\underline{u}, 0\}$ in Ω-topology, we have only to use the following identity in place of (8.12):

$$\begin{aligned}
&\frac{1}{2}(1-s(t))^2 \\
&= \left\{\frac{1}{2}(1-l)^2 - \frac{\mu_1}{d_1}\int_0^l (1-x)\varphi(x)dx - \frac{\mu_2}{d_2}\int_l^1 (1-x)\varphi(x)dx\right\} \\
&+ \left\{\frac{\mu_1}{d_1}\int_0^{s(t)} (1-x)u(x, t)dx + \frac{\mu_2}{d_2}\int_{s(t)}^1 (1-x)u(x, t)dx\right\} \\
&+ \mu_1\int_0^t u_x(0, \tau)d\tau - \int_0^t d\tau\left\{\frac{\mu_1}{d_1}\int_0^{s(\tau)} (1-x)uf(u)dx\right. \\
&\left. + \frac{\mu_2}{d_2}\int_{s(t)}^1 (1-x)ug(u)dx\right\},
\end{aligned}$$

whose derivation is almost the same as that of (8.12). q.e.d.

8.2. Stability analysis in Case B

We next study stability or instability properties of stationary solutions given by Theorem 7.2 in Case B.

THEOREM 8.3. *Let f and g be of the forms* (7.11) *and* (7.12), *respectively.*

(i) *The set of trivial solutions is asymptotically stable as a set. Moreover if $A^* > 1$ (resp. $B^* > 1$), then*

$$\lim_{t\to\infty} u^+(\cdot, t; \varphi, l) = 0 \; (\textit{resp.} \lim_{t\to\infty} u^-(\cdot, t; \varphi, l) = 0) \quad \textit{in} \quad H_0^1(I) \tag{8.22}$$

for any $\{\varphi, l\}$. In particular, if $A^* > 1$ and $B^* > 1$, then

$$\lim_{t\to\infty} \{u(\cdot,\, t;\, \varphi,\, l),\, s(t;\, \varphi,\, l)\} = \{0,\, s^*\} \quad \textit{in} \quad \Omega\textit{-topology} \tag{8.23}$$

with some $s^* \in \bar{I}$ *depending on* $\{\varphi, l\}$.

(ii) *If* $A^* \leqq 1$ (*resp.* $B^* \leqq 1$), *then* $\{\bar{u}_1, 1\} \in \mathscr{P}$ (*resp.* $\{\underline{u}_1, 0\} \in \mathscr{N}$) *is asymptotically stable from aobve* (*resp. from below*) *in the sense that*

$$\lim_{t\to\infty} \{u(\cdot,\, t;\, \varphi,\, l),\, s(t;\, \varphi,\, l)\} = \{\bar{u}_1,\, 1\} \tag{8.24}$$

(*resp.* $\{\underline{u}_1, 0\}$) *in* Ω*-topology*)

for any $\{\varphi, 1\}$ *satisfying* $\varphi \geqq \bar{u}_1$ (*resp.* $\varphi \leqq \underline{u}_1$).

In paticular if $A^* < 1$ (*resp.* $B^* < 1$), *then* $\{\bar{u}_1, 1\} \in \mathscr{P}$ (*resp.* $\{\underline{u}_1, 0\} \in \mathscr{N}$) *is asymptotically stable and* $\{\bar{u}_2, 1\} \in \mathscr{P}$ (*resp.* $\{\underline{u}_2, 0\} \in \mathscr{N}$) *is asymptotically unstable in the following sense*: (8.24) *is valid for any* $\{\varphi, 1\}$ *satisfying* $\varphi \geqq \bar{u}_2$ *with* $\varphi \not\equiv \bar{u}_2$ (*resp.* $\varphi \leqq \underline{u}_2$ *with* $\varphi \not\equiv \underline{u}_2$), *while* (8.22) *is valid for any* $\{\varphi, l\}$ *satisfying* $\varphi \leqq \bar{u}_2$ *with* $\varphi \not\equiv \bar{u}_2$ (*resp.* $\varphi \geqq \underline{u}_2$ *with* $\varphi \not\equiv \underline{u}_2$).

(iii) *When* (8.24) *holds true, there exists a non-negative number* $T_1 < \infty$ (*resp.* $T_2 < \infty$) *such that*

$$s(t;\, \varphi,\, l) \equiv 1 \qquad \textit{for} \quad t \geqq T_1 \quad (\textit{resp.}\ s(t;\, \varphi,\, l) \equiv 0\ \textit{for}\ t \geqq T_2).$$

(iv) *Any solution* $\{u^*, s^*\}$ *in* $\mathscr{O}$ *is asymptotically unstable.*

Proof. First it is better to state some results on the asymptotic behavior of solutions for

$$\begin{cases} V_t = d_1 V_{xx} + Vf(V), & (x, t) \in Q, \\ V(0, t) = V(1, t) = 0, & t > 0, \end{cases} \tag{8.25}$$

with initial conditions $V(\cdot, 0) = V_0 \geqq 0$.

For $A^* > 1$, the stationary problem associated with (8.25) has no positive solutions; so that every solution of (8.25) decays to zero in $H_0^1(I)$ as $t \to \infty$.

For $A^* = 1$, the stationary problem has exactly two non-negative solutions $V = \bar{u}_1$ and $V = 0$. Note that $\bar{u}_1(x) = v(x;\, p_1(1)) = v(x;\, p_2(1))$, where $v(x;\, p)$ is the solution of (7.3). For $\xi > 1$, one can see

$$\bar{u}_1 > v(\cdot\,;\, p_2(\xi)) \quad \text{in} \quad (0, x_0) \quad \text{and} \quad \bar{u}_1 < v(\cdot\,;\, p_2(\xi)) \quad \text{in} \quad (x_0, 1)$$

with some $x_0 \in I$; so that $v(x;\, p_2(\xi))$ and its suitable translations become supersolutions for (8.25). Therefore, by the comparison principle any solution V of (8.25) satisfies

$$\lim_{t\to\infty} V(\cdot,\, t) = \bar{u}_1 \quad (\text{resp.}\ 0) \quad \text{in} \quad H_0^1(I) \tag{8.26}$$

for V_0 such that $V_0 \geqq \bar{u}_1$ (resp. $V_0 \leqq \bar{u}_1$) with $V_0 \not\equiv \bar{u}_1$ (See, e.g., Aronson et al [1] or Matano [7]).

For $A^* < 1$, (8.25) has exactly three non-negative stationary solutions $V = \bar{u}_1$, $V = \bar{u}_2$ and $V = 0$ with $0 < \bar{u}_2 = v(\cdot, p_2(1)) < \bar{u}_1 = v(\cdot, p_1(1))$ on I. We observe that $v(\cdot; p_2(\xi))$ and its suitable translations are supersolutions (resp. subsolutions) of (8.25) for $\xi > 1$ (resp. $A^* \leqq \xi < 1$). Therefore, the comparison argument allows us to see the validity of (8.26) for any V_0 satisfying $V_0 \geqq \bar{u}_2$ (resp. $V_0 \leqq \bar{u}_2$) with $V_0 \not\equiv \bar{u}_2$.

These results assure that $V = 0$ is asymptotically stable from above. Moreover, if a solution of (8.25) decays to zero as $t \to \infty$, then its decaying rate is of exponential type because $-d_1\pi^2 + f(0) < 0$ (see, e.g. Kielhöfer [5]).

Analogous results hold for the following initial boundary value problem

$$\begin{cases} W_t = d_2 W_{xx} + Wg(W), & (x, t) \in Q, \\ W(0, t) = W(1, t) = 0, & t \geqq 0, \\ W(x, 0) = W_0(x) \leqq 0, & x \in I. \end{cases} \tag{8.27}$$

Making use of these asymptotic properties for (8.25) and (8.27) one can easily derive the assertions of (i) and (ii). Especially, the convergence of $s(t; \varphi, l)$ in (8.23) is proved in the same manner as Theorem 8.1 (i) with use of the exponential decay of $|u(\cdot, t; \varphi, l)|$ as $t \to \infty$.

The proof of (iii) is the same as that of Theorem 8.2.

Finally, we will show (iv). For the sake of convenience, let $\{u^*, s^*\}$ correspond to an intersecting point of C_{12} and C_{21} and let u^* be expressed by (7.20). We define

$$U_{21}(x; \xi) = \begin{cases} v(x; p_2(\xi)) & \text{for} \quad 0 \leqq x \leqq \xi \\ w(x; q_1(\xi)) & \text{for} \quad \xi \leqq x \leqq 1. \end{cases}$$

Since both $p_2(\xi)$ and $q_1(\xi)$ are monotone decreasing, one can show that, if $\xi \in [A^*, s^*)$ then

$$U_{21}(\cdot\,; \xi) > u^* \quad \text{on} \quad (0, x_1) \quad \text{and} \quad U_{21}(\cdot\,; \xi) < u^* \quad \text{on} \quad (x_1, 1)$$

with some $x_1 \in (0, s^*)$ and, if $\xi \in (s^*, 1 - B^*]$

$$U_{21}(\cdot\,; \xi) < u^* \quad \text{on} \quad (0, x_2) \quad \text{and} \quad U_{21}(\cdot\,; \xi) > u^* \quad \text{on} \quad (x_2, 1)$$

with some $x_2 \in (0, s^*)$ (see [9]). Note that $\mu_1 p_2(\xi) - \mu_2 q_1(\xi)$ changes sign at $\xi = s^*$ when ξ moves in its neighborhood. Then $\{U_{21}(\cdot\,; \xi), \xi\}$ becomes a supersolution (resp. subsolution) of (P) if $\mu_1 p_2(\xi) < \mu_2 q_1(\xi)$ (resp. $\mu_1 p_2(\xi) > \mu_2 q_1 \cdot (\xi)$). Therefore, Theorem 4.2 enables us to drive the asymptotic instability of $\{u^*, s^*\}$. q.e.d.

RREMARK 8.1. Suppose that (8.22) holds for u^+ and u^-. Then $|u(\cdot, t; \varphi, l)|$

decays exponentially to zero (uniformly in I) as $t\to\infty$. Therefore, we can show the validity of (8.23) in this situation (see the proof of Theorem 8.1 (i)).

REMARK 8.2. The comparison principle enables us to get preciser information on the asymptotic stability or instability for stationary solutions if one can choose suitable comparison functions. For instance, take

$$U(x, t; \xi) = \begin{cases} V(x, t; \xi) & \text{if} \quad 0 \leqq x \leqq \xi, \\ W(x, t; \xi) & \text{if} \quad \xi \leqq x \leqq 1, \end{cases}$$

where $V(x, t; \xi)$ (resp. $W(x, t; \xi)$) is the solution of (8.25) (resp. (8.27)) with $0<x<1$ replaced by $0<x<\xi$ (resp. $\xi<x<1$). Suppose that $A^*<1$ and $A^*+B^*>1$. We fix $\xi\in(A^*, 1)$ and take V_0 such that $V_0\geqq v(\cdot\,; p_2(\xi))$ ($V_0\not\equiv v(\cdot\,; p_2(\xi))$) on $(0, \xi)$. It is possible to show $\lim_{t\to\infty} V(\cdot, t; \xi)=v(\cdot, t; p_1(\xi))$ uniformly on $(0, \xi)$ and $\lim_{t\to\infty} W(\cdot, t; \xi)=0$ uniformly on $(\xi, 1)$ for any W_0 (see the proof of Theorem 8.3). Hence $\{U(\cdot, t; \xi), \xi\}$ becomes a subsolution of (P) with an appropriate choice of W_0. Since no stationary solutions $\{u^*, s^*\}$ other than $\{\bar{u}_1, 1\}$ satisfy $\{u^*, s^*\}\geqq\{\lim_{t\to\infty} U(\cdot, t; \xi), \xi\}$, we can decide that

$$\lim_{t\to\infty} \{u(\cdot, t; \varphi, l), s(t; \varphi, l)\} = \{\bar{u}_1, 1\} \quad \text{in} \quad \Omega\text{-topology}$$

for any $\{\varphi, l\}$ satisfying $\{\varphi, l\}\geqq\{U(\cdot, 0; \xi), \xi\}$.

§ 9. Bifurcation results

In this section we will state some bifurcation results by taking $d_1=d_2=d$, $\mu_1=\mu_2=\mu$ and $f(u)=g(-u)$.

9.1. Case A

Set

$$f(u) = ah(u),$$

where h is a monotone decreasing and Lipschitz continuous function on $[0, 1]$ such that $h(0)=1$ and $h(1)=0$. We fix d_1, μ and regard a as a parameter. Theorem 7.1 gives us complete information about the structure of the set of solutions for (SP).

When a is smaller than $\pi^2 d$, (SP) has no nontrivial solutions; so that the set of trivial solutions is asymptotically stable (as a set) by Theorem 8.1 (i) (Fig. 6A). As a becomes larger than $\pi^2 d$, two nontrivial solutions $\{\bar{u}, 1\}\in\mathscr{P}$ and $\{-\underline{u}, 0\}\in\mathscr{N}$ bifurcate from the set of trivial solutions as in Fig. 6B. These bifurcating solutions are asymptotically stable and trivial solution $\{0, \xi\}$ satisfying $0\leqq\xi<1-\pi\sqrt{d/a}$ or $\pi\sqrt{d/a}<\xi\leqq 1$ becomes asymptotically unstable (see Theorem 8.1 (iv)). Further-

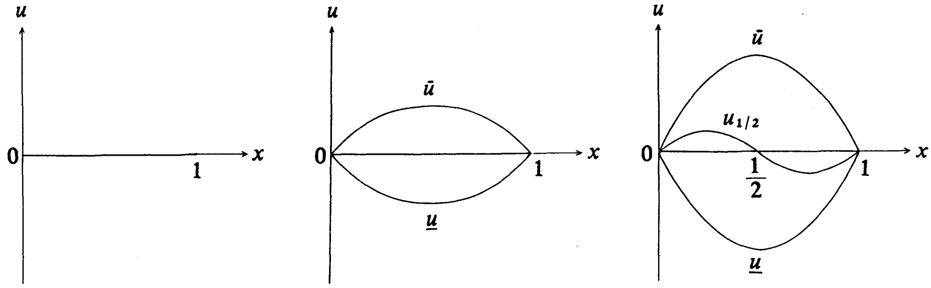

Fig. 6 A $a \leqq \pi^2 d$ Fig. 6 B $\pi^2 d < a \leqq 4\pi^2 d$ Fig. 6 C $4\pi^2 d < a$

Fig. 6 (6A, 6B and 6C). Solutions of (SP) in Case A.

more, as a becomes larger than $4\pi^2 d$, another nontrivial stationary solution $\{u_{1/2}, 1/2\}$ bifurcates from the set of trivial solutions (see Fig. 6C). By virtue of Theorem 8.1 (v), $\{u_{1/2}, 1/2\}$ together with all trivial solutions is asymptotically unstable. Fig. 7 shows some numerical experiments which exhibit the asymptotic behavior of the solutions $\{u, s\}$ of (P). Here we have carried out the numerical analysis by taking $d_1 = d_2 = \mu_1 = \mu_2 = 1$, $f(u) = g(-u) = a(1-u)$ and

$$\varphi(x) = \begin{cases} \alpha \sin \pi x / l & \text{for } 0 \leqq x \leqq l, \\ -\beta \sin \pi(1-x)/(1-l) & \text{for } l \leqq x \leqq 1. \end{cases}$$

9.2. Case B

We take

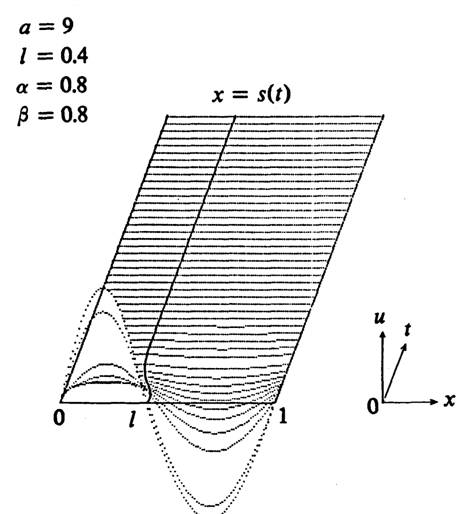

Fig. 7A. For $a = 9 < \pi^2$, every solution of (P) converges to one of trivial stationary solutions.

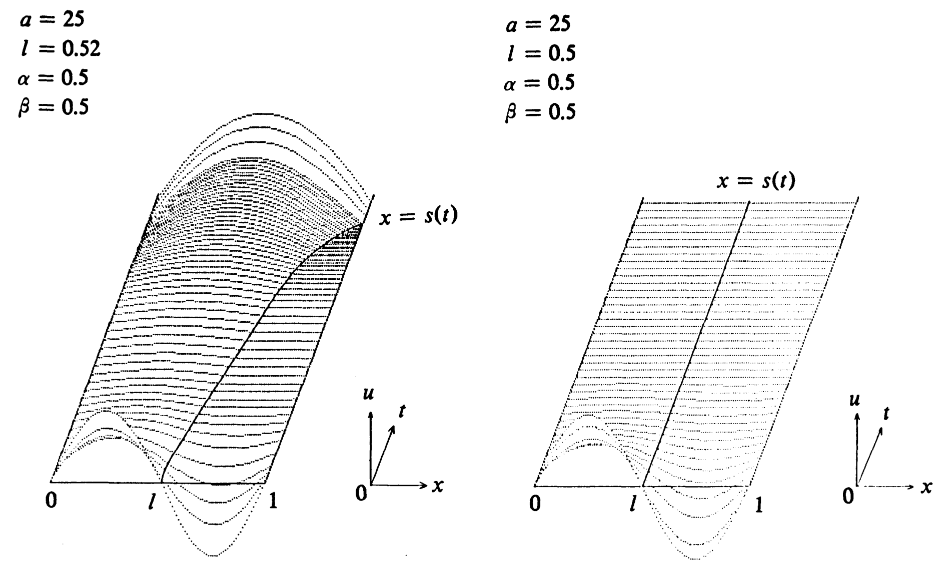

Fig. 7B. For $a=25\in(\pi^2, 4\pi^2)$, the maximal solution $\{\bar{u}, 1\}$ is asymptotically stable and the free boundary $x=s(t)$ hits the fixed boundary $x=1$ in a finite time if $\{u, s\}$ converges to $\{\bar{u}, 1\}$.

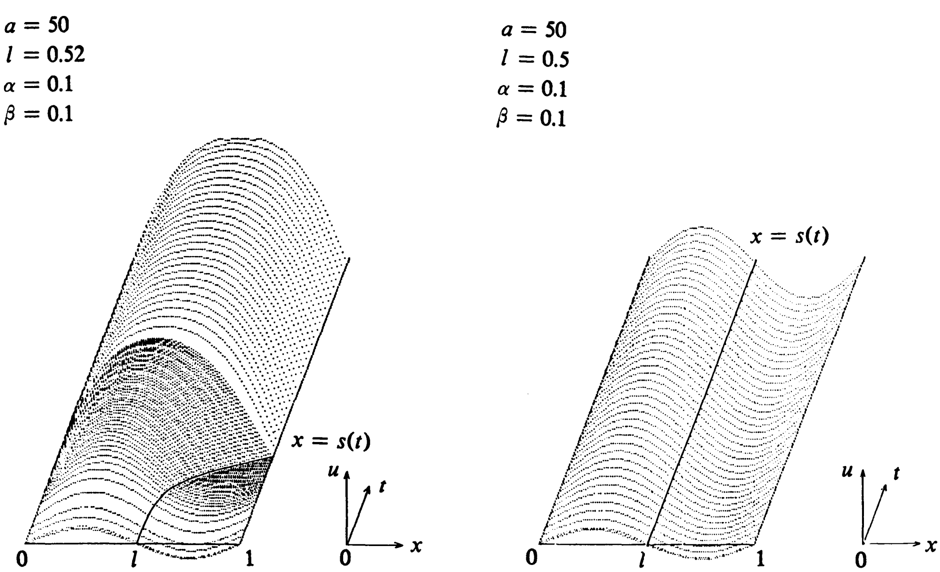

Fig. 7C. For $a=50>4\pi^2$, the maximal solution $\{\bar{u}, 1\}$ is asymptotically stable, while another non-trivial solution $\{u_{1/2}, 1/2\}$ is realized if the solution of (P) starts from a symmetric initial data.

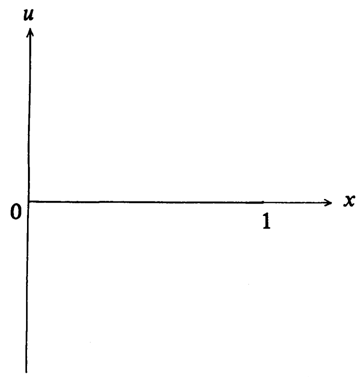

Fig. 8A $\nu<a^*d$

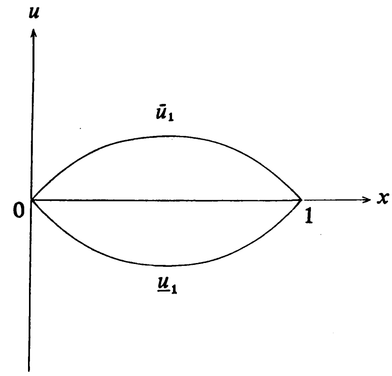

Fig. 8B $\nu=a^*d$

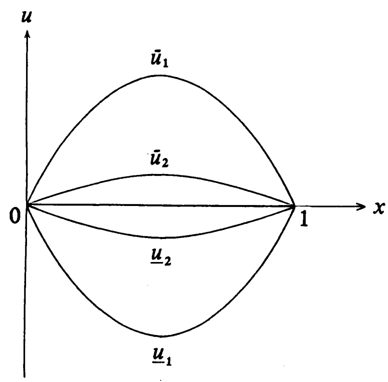

Fig. 8C $a^*d<\nu<4a^*d$

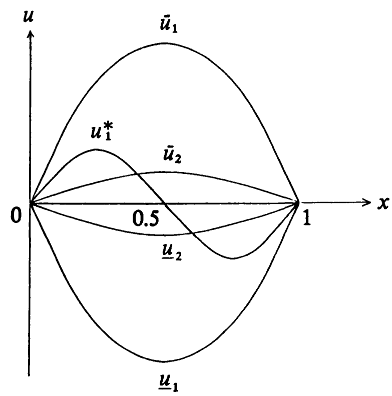

Fig. 8D $\nu=4a^*d$

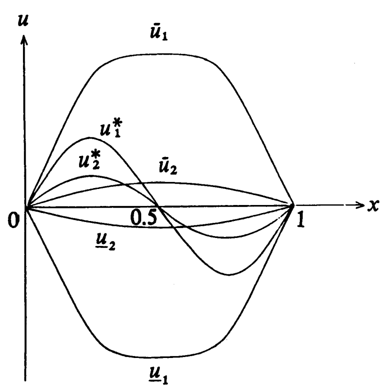

Fig. 8E $4a^*d<\nu$

Fig. 8 (8A, 8B, 8C, 8D and 8E). Solutions of (SP) in Case B.

$$f(u) = -\nu(u-a)(u-1) \quad \text{with} \quad 0 < a < 1/2.$$

Fixing d, μ_1 and a we regard ν as a parameter. By Theorem 7.2, the structure of $\mathscr{P}$, $\mathscr{N}$ and $\mathscr{O}$ changes depending on A^* $(=B^*)$, which is the minimum of $T_1(p)$ defined by (7.2). Note that A^* is written in the form

$$A^* = \sqrt{a^* d/\nu},$$

where a^* is a positive constant depending only on a.

By Theorem 8.3 (i), the set of trivial solutions is always asymptotically stable as a set. For, $\nu = a^* d$, a pair of nontrivial stationary solutions in $\mathscr{P}$ and $\mathscr{N}$ suddenly appear as in Fig. 8B (In [13], Smoller calls such a phenomenon a spontaneous bifurcation). Then, for $\nu > a^* d$, each non-trivial solution bifurcates into two non-trivial solutions in the same class, one of which (a maximal one $\{\bar{u}_1, 1\}$ or a minimal one $\{\underline{u}_1, 0\}$) is asymptotically stable and the other is asymptotically unstable (use Theorem 8.3 (ii)). See Fig. 8C. Moreover, for $\nu = 4ad^*$, a new non-trival solution in $\mathscr{O}$, which is asymptotically unstable, appears as in Fig. 8D and, for $\nu > 4ad^*$, it bifurcates into two non-trivial solutions in $\mathscr{O}$ which are asymptotically unstable by Theorem 8.3 (iv) (Fig. 8E).

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