HIROSHIMA MATH. J.
26 (1996), 165–179

Global real analytic length parameters for Teichmüller spaces

Yoshihide OKUMURA

(Received November 10, 1994)

ABSTRACT. It is well-known that Teichmüller space is a global real analytic manifold. Using the geometry of Möbius transformations and the one-half power of a hyperbolic transformation, we consider the minimal number of global real analytic length parameters for Teichmüller space and such length parameter space.

1. Introduction

A Riemann surface S of genus g with m holes is called of *type* $(g, 0, m)$. If $2g + m \geq 3$, then S is conformally equivalent to the quotient space $\boldsymbol{D}/G$, where $\boldsymbol{D}$ is the unit disk in the complex plane and G is a Fuchsian group acting on $\boldsymbol{D}$. This G is also called of *type* $(g, 0, m)$. Teichmüller space $T(g, 0, m)$, $2g + m \geq 3$ is the set of equivalence classes of marked Fuchsian groups of type $(g, 0, m)$ and a global real analytic manifold of dimension $6g + 3m - 6$. It is well known that $T(g, 0, m)$ is parametrized global real analytically by some lengths of closed geodesics on a Riemann surface represented by a marked Fuchsian group (see for example, [1], [4], [6], [7], [8], [13] and [16]). Such lengths are called *length parameters*. In this paper, we consider the following problem.

PROBLEM. (i) What is the minimal number of global real analytic length parameters for $T(g, 0, 0)$, $g \geq 2$?
(ii) How is the parameter space described by such length parameters?

About the first problem, Wolpert [20] and [21] announced that the minimal number of these parameters is greater than $\dim(T(g, 0, 0)) = 6g - 6$. Next, Seppälä and Sorvali [17] and Okumura [8] showed that this minimal number is less than or equal to $6g - 4$. Finally, we concluded that *the minimal number of global real analytic length parameters for* $T(g, 0, 0)$, $g \geq 2$ *is* $6g - 5$ *and, further, that we can take these lengths from simple closed geodesics on a Riemann surface.* This was first proved by Schmutz [14]. In the

1991 *Mathematics Subject Classification. Primary* 32G15, 30F60; *Secondary* 30F10, 30F35.
Key words and phrases. Teichmüller space, moduli, Fuchsian group, Riemann surface.

same time, the author also obtained this result independently and concretely from different methods. Moreover, the author obtained a result about the second problem. Our results are derived from the consideration of the geometry of Möbius transformations and the one-half power of a hyperbolic transformation. A proof in the case of $g \geq 3$ is similar to that in the case of $g = 2$ and a parameter space of $T(g, 0, 0)$, $g \geq 3$ is defined by induction. Thus in this paper, we only consider the case of $g = 2$.

THEOREM 1.1. *$T(2, 0, 0)$ is parametrized global real analytically by seven length parameters which correspond to the absolute values of traces of the following hyperbolic elements of a marked Fuchsian group:*

$$A_1 , \quad B_1 , \quad B_1 A_1 ,$$
$$A_2 , \quad B_2 , \quad B_2 A_2 A_1 , \quad B_2 A_2 B_1^{-1} ,$$

where (A_1, B_1, A_2, B_2) is a canonical system of generators of this group (see Section 2 for the definition). Thus these length parameters are lengths of simple closed geodesics on a Riemann surface represented by a marked Fuchsian group. This parameter space is described as follows:

$$x_j > 2, \ y_j > 2 \ (j = 1, 2) , \qquad z_1 > 2, \ u > 2, \ v > 2 ,$$
$$x_1^2 + y_1^2 + z_1^2 - x_1 y_1 z_1 = x_2^2 + y_2^2 + |\mathrm{tr}\,(B_2 A_2)|^2 - x_2 y_2 |\mathrm{tr}\,(B_2 A_2)| < 0 ,$$
$$|\mathrm{tr}\,(B_2 A_2)| = \frac{1}{z_1^2 - 4}\{z_1 \sqrt{x_1 y_1 z_1 - (x_1^2 + y_1^2 + z_1^2) + 4}\sqrt{uvz_1 - (u^2 + v^2 + z_1^2) + 4}$$
$$+ 2(x_1 u + y_1 v) - z_1(y_1 u + x_1 v)\} > 2 ,$$

where $x_j := |\mathrm{tr}\,(A_j)|$, $y_j := |\mathrm{tr}\,(B_j)|$ $(j = 1, 2)$, $z_1 := |\mathrm{tr}\,(B_1 A_1)|$, $u := |\mathrm{tr}\,(B_2 A_2 A_1)|$ and $v := |\mathrm{tr}\,(B_2 A_2 B_1^{-1})|$.

The simple closed geodesics used in the above theorem are characterized as Theorems 3.7 and 4.9.

REMARK 1.2 ([11]). In the case of $T(g, n, m)$, $m \neq 0$, the minimal number of global real analytic length parameters is $\dim (T(g, n, m))$, where n means the number of branch points and punctures on a Riemann surface represented by a marked Fuchsian group.

The author would like to thank Professors F. Maeda, Y. Miki, K. Niino, H. Sato and M. Taniguchi who encourage him to publish this paper.

2. Preliminaries

Let X be the unit disk D or the upper half plane H in the complex plane. The group $M(X)$ of Möbius transformations preserving X is the group of isometries of X with respect to the Poincaré metric d. For distinct two points p_1 and p_2 in the closure $\bar{X}$ of X, let $L(p_1, p_2)$ be the full geodesic through p_1 and p_2 with the direction from p_1 to p_2, where this direction is sometimes ignored. Note that $L(p_1, p_2)$ divides $\bar{X}$ into two parts; the right-hand part and the left-hand part are denoted by $r - L(p_1, p_2)$ and $l - L(p_1, p_2)$, respectively.

An elliptic element $A \in M(X)$ has one fixed point in X. We denote it by $\mathrm{fp}(A)$. A hyperbolic element $A \in M(X)$ has the *attracting fixed point* $q(A)$, and the *repelling fixed point* $p(A)$, which are characterized by $q(A) = \lim_{n\to\infty} A^n(z)$ and $p(A) = \lim_{n\to\infty} A^{-n}(z)$ for any $z \in X$, respectively. For a hyperbolic element A, the *axis* of A $\mathrm{ax}(A) = L(p(A), q(A))$, and the *translation length* of A $\mathrm{tl}(A) = \inf\{d(z, A(z)) | z \in X\}$ are characterized by

$$\mathrm{ax}(A) = \{z \in X | d(z, A(z)) = \mathrm{tl}(A)\},$$

$$\cosh \frac{\mathrm{tl}(A)}{2} = \frac{|\mathrm{tr}(A)|}{2}.$$

We remark that $q(A) = p(A^{-1})$ and $l - \mathrm{ax}(A) = r - \mathrm{ax}(A^{-1})$.

Let A be a hyperbolic element of a Fuchsian group G acting on X. Then $\mathrm{ax}(A)$ projects on a closed geodesic on X/G whose length is $\mathrm{tl}(A)$ and corresponds to $|\mathrm{tr}(A)|$ real analytically.

To define a marked Fuchsian group, we give the following proposition.

PROPOSITION 2.1 (Keen [3]). *Let G be a Fuchsian group of type $(g, 0, m)$. Then G has a system of generators*

$$\Sigma = (A_1, B_1, \dots, A_g, B_g, E_1, \dots, E_m),$$

$$E_m E_{m-1} \dots E_1 C_g C_{g-1} \dots C_1 = \mathit{identity},$$

where A_j, B_j, $C_j = [B_j, A_j] = B_j^{-1} A_j^{-1} B_j A_j$ $(j = 1, \dots, g)$ and E_k $(k = 1, \dots, m)$ are hyperbolic elements with the axes illustrated as in Figure 2.1, and if $g = 0$ (resp. $m = 0$), then A_j, B_j and C_j (resp. E_k) are omitted.

A system Σ mentioned in Proposition 2.1 is called a *canonical system of generators of type* $(g, 0, m)$. A pair (G, Σ) is called a *marked Fuchsian group of type* $(g, 0, m)$. Two marked Fuchsian groups (G_1, Σ_1) and (G_2, Σ_2) are *equivalent*, if $G_2 = hG_1h^{-1}$ and $\Sigma_2 = h\Sigma_1h^{-1}$ for some $h \in M(X)$. *Teichmüller space* $T(g, 0, m)$ is the set of equivalence classes of (G, Σ) of type $(g, 0, m)$.

Let $A(z) = \dfrac{az + b}{cz + d}$, $ad - bc = 1$. Then $\tilde{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called the *matrix representation* of A. Since the matrix representations of A are $\pm\tilde{A}$, $\tilde{A}$ is determined up to the sign. If $\mathrm{tr}(\tilde{A})$ is positive (resp. negative), then $\tilde{A}$ is called the *positive* (*resp. negative*) *matrix representation.*

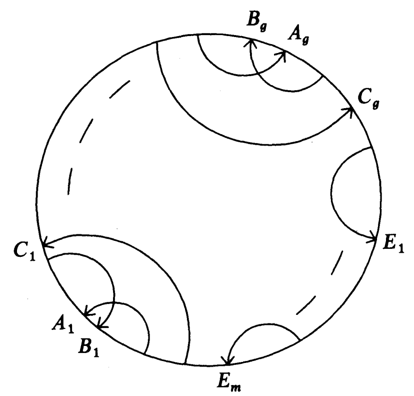

Figure 2.1

Remark 2.2. Similarly, we define the positive and negative matrix representations in the case of $\operatorname{tr}(\tilde{A}) = 0$, namely, A is elliptic of order 2 (see Lemma 4.2 and [12]).

Remark 2.3. Two Möbius transformations A and B uniquely determine the matrix $[\tilde{B}, \tilde{A}]$, which is independent of the choice of $\tilde{A}$ and $\tilde{B}$.

The following relations of commutator traces among $\tilde{X}$, $\tilde{Y}$, $\tilde{Z} = (\tilde{Y}\tilde{X})^{-1} \in SL(2, C)$ are useful: for ε, $\eta \in \{\pm 1\}$,

$$\begin{aligned} \operatorname{tr}([\tilde{X}, \tilde{Y}]) &= \operatorname{tr}([\tilde{X}^{\varepsilon}, \tilde{Y}^{\eta}]) = \operatorname{tr}([\tilde{Y}^{\varepsilon}, \tilde{X}^{\eta}]) \\ &= \operatorname{tr}([\tilde{Y}^{\varepsilon}, \tilde{Z}^{\eta}]) = \operatorname{tr}([\tilde{Z}^{\varepsilon}, \tilde{Y}^{\eta}]) \\ &= \operatorname{tr}([\tilde{Z}^{\varepsilon}, \tilde{X}^{\eta}]) = \operatorname{tr}([\tilde{X}^{\varepsilon}, \tilde{Z}^{\eta}]) . \end{aligned}$$

Finally, we define the one-half power of $A \in M(X)$.

Definition. Let $A \in M(X)$ be hyperbolic or parabolic. Then $X \in M(X)$ satisfying $X^2 = A$ is called the *one-half power* of A and denoted by $A^{1/2}$.

$A^{1/2}$ is determined by A as follows:

Proposition 2.4. *Let $A \in M(X)$ be hyperbolic or parabolic. If $\tilde{A}$ is the negative (resp. positive) matrix representation of A, then the matrix representations of $A^{1/2}$ are*

$$\frac{\pm 1}{\sqrt{|\operatorname{tr}(A)| + 2}}(\tilde{A} - I) \qquad \left(resp. \ \frac{\pm 1}{\sqrt{|\operatorname{tr}(A)| + 2}}(\tilde{A} + I) \right) .$$

Thus

$$|\operatorname{tr}(A^{1/2})| = \sqrt{|\operatorname{tr}(A)| + 2} .$$

3. Basic tools

In this section, we state some properties of hyperbolic transformations and a parametrization of $T(1, 0, 1)$.

First we consider the relationships between the positions of the axes of hyperbolic transformations and the traces of their matrix representations.

THEOREM 3.1 ([12]). *Let X, Y and Z be hyperbolic elements of $M(X)$ satisfying $ZYX =$ identity. Then about the positions of the axes of X, Y and Z, one of the following cases occurs:*

(a) *three axes are disjoint,*

(b) *three axes are parallel (namely, they have one common endpoint on the circle at infinity) or coincident,*

(c) *three axes do not intersect at one point and any two axes intersect each other.*

Thus, if some two axes are disjoint, parallel, coincident or intersecting, then three axes are also in the same situation. Furthermore, the orientations of the axes are determined as in Figure 3.1, where the pair (U, V, W) is any permutation of X, Y and Z. These cases are characterized by $\operatorname{tr}(\tilde{X})$, $\operatorname{tr}(\tilde{Y})$ *and* $\operatorname{tr}(\tilde{Y}\tilde{X})$ *as follows:*

$(\mathrm{a}_1) \Leftrightarrow \operatorname{tr}(\tilde{X})\operatorname{tr}(\tilde{Y})\operatorname{tr}(\tilde{Y}\tilde{X}) < 0,$

$(\mathrm{a}_2) \Leftrightarrow \operatorname{tr}(\tilde{X})\operatorname{tr}(\tilde{Y})\operatorname{tr}(\tilde{Y}\tilde{X}) > 0,\ \operatorname{tr}([\tilde{Y}, \tilde{X}]) > 2,$

$(\mathrm{b}) \Leftrightarrow \operatorname{tr}(\tilde{X})\operatorname{tr}(\tilde{Y})\operatorname{tr}(\tilde{Y}\tilde{X}) > 0,\ \operatorname{tr}([\tilde{Y}, \tilde{X}]) = 2,$

$(\mathrm{c}) \Leftrightarrow \operatorname{tr}(\tilde{X})\operatorname{tr}(\tilde{Y})\operatorname{tr}(\tilde{Y}\tilde{X}) > 0,\ \operatorname{tr}([\tilde{Y}, \tilde{X}]) < 2.$

REMARK 3.2.

$$\operatorname{tr}([\tilde{Y}, \tilde{X}]) = \operatorname{tr}^2(\tilde{X}) + \operatorname{tr}^2(\tilde{Y}) + \operatorname{tr}^2(\tilde{Y}\tilde{X}) - \operatorname{tr}(\tilde{X})\operatorname{tr}(\tilde{Y})\operatorname{tr}(\tilde{Y}\tilde{X}) - 2$$

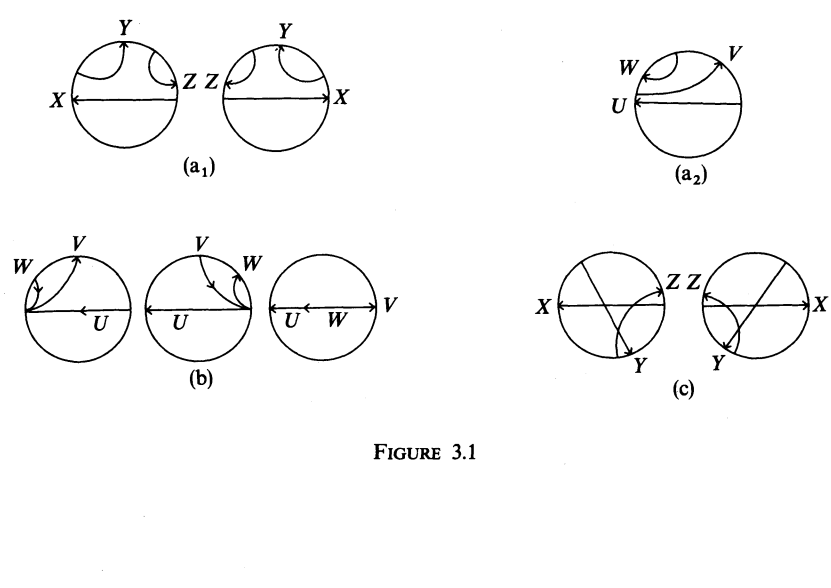

FIGURE 3.1

and $\operatorname{tr}(\tilde{X})\operatorname{tr}(\tilde{Y})\operatorname{tr}(\tilde{Y}\tilde{X})$ are invariant under the choice of matrix representations. In the case of (a_1), we have $\operatorname{tr}([\tilde{Y}, \tilde{X}]) > 18$.

REMARK 3.3. In the case of (c), the axes of X, Y and Z determine a triangle. This triangle does not shrink into one point, even if X and Y are deformated in such a way that $\operatorname{tr}([\tilde{Y}, \tilde{X}]) \leq k$ for some constant $k < 2$, or in particular, $[Y, X]$ is hyperbolic (namely, $\operatorname{tr}([\tilde{Y}, \tilde{X}]) < -2$).

REMARK 3.4. Similarly, for any non-trivial elements X, Y, $Z = (YX)^{-1} \in M(X)$, the positions of their fixed points and the directions of their actions are characterized by $\operatorname{tr}(\tilde{X})$, $\operatorname{tr}(\tilde{Y})$ and $\operatorname{tr}(\tilde{Y}\tilde{X})$ (see [12]).

THEOREM 3.5 ([12]). *Let A, $B \in M(X)$ be hyperbolic elements with the intersecting axes and p the intersection point of these axes. Let $R \in M(X)$ be the elliptic element of order 2 with the fixed point p.*

(i) *The axes of $A^{\varepsilon}B^{\eta}$, $B^{\varepsilon}A^{\eta}$, ε, $\eta \in \{\pm 1\}$ determine the quadrilateral with four sides $\operatorname{tl}(BA)/2$, $\operatorname{tl}(B^{-1}A)/2$, $\operatorname{tl}(BA)/2$ and $\operatorname{tl}(B^{-1}A)/2$ and four vertices $A^{-1/2}(p)$, $B^{1/2}(p)$, $A^{1/2}(p)$ and $B^{-1/2}(p)$.*

Further, suppose that $C = [B, A]$ is hyperbolic and $p(A)$, $q(B)$, $q(A)$ and $p(B)$ are arranged clockwise in order on the circle at infinity. Then the following claims hold:

(ii) *$(A, B^{-1}A^{-1}B, C^{-1})$, $(BA, B^{-1}A^{-1}, C^{-1})$ and $(A^{-1}BA, B^{-1}, C^{-1})$ are canonical systems of generators of type $(0, 0, 3)$.*

(iii) *A, B, C and R satisfy the following relations:*

$$A = RA^{-1}R = [R, A^{1/2}], \qquad B = RB^{-1}R = [R, B^{1/2}],$$
$$C^{1/2} = RBA,$$
$$\tilde{R} = \frac{\pm 1}{\det(\tilde{B}\tilde{A} - \tilde{A}\tilde{B})^{1/2}}(\tilde{B}\tilde{A} - \tilde{A}\tilde{B}).$$

(iv) *$C^{-1/2}A$, $C^{-1/2}B^{-1}$ and $C^{-1/2}BA$ are elliptic elements of order 2 and satisfy*

$$\operatorname{fp}(C^{-1/2}A) = (ABA)^{-1/2}(p) = (BA)^{-1/2}A^{-1/2}(p) = (BA)^{-1}B^{1/2}(p),$$
$$\operatorname{fp}(C^{-1/2}B^{-1}) = A^{-1/2}(p),$$
$$\operatorname{ax}(ABA) = L(\operatorname{fp}(C^{-1/2}A), p).$$

(v) *Let $A_{1/2}$ (resp. $A_{-1/2}$) be the elliptic element of order 2 with the fixed point $A^{1/2}(p)$ (resp. $A^{-1/2}(p)$), namely, $A_{1/2} = A^{1/2}RA^{-1/2}$ (resp. $A_{-1/2} = A^{-1/2}RA^{1/2}$). Similarly, let $B_{1/2}$ and $B_{-1/2}$ be defined. Then*

$$A = RA_{-1/2} = A_{1/2}R, \qquad B = RB_{-1/2} = B_{1/2}R,$$
$$BA = B_{1/2}A_{-1/2}, \qquad AB = A_{1/2}B_{-1/2},$$
$$C = B_{-1/2}A_{1/2}B_{1/2}A_{-1/2}.$$

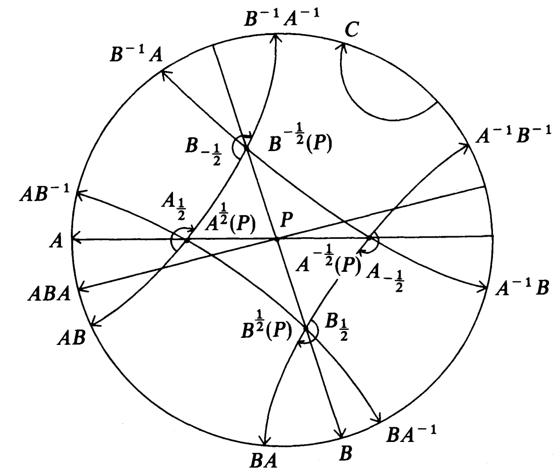

FIGURE 3.2

In particular, C is determined by four elliptic transformations of order 2 whose fixed points are four vertices of this quadrilateral (see Figure 3.2).

Now we state a parametrization of $T(1, 0, 1)$.

THEOREM 3.6 (Keen [4]). *Let $\Sigma_{(1,0,1)} = (A_1, B_1, C_1^{-1})$, $C_1 = [B_1, A_1]$ be a canonical system of generators of type $(1, 0, 1)$. Then $\Sigma_{(1,0,1)}$ is determined global real analytically by $x_1 := |\mathrm{tr}\,(A_1)|$, $y_1 := |\mathrm{tr}\,(B_1)|$ and $z_1 := |\mathrm{tr}\,(B_1 A_1)|$, up to conjugation by a Möbius transformation. Thus $T(1, 0, 1)$ is parametrized global real analytically by these three length parameters. This parameter space is described as follows:*

$$x_1 > 2\,, \qquad y_1 > 2\,, \qquad z_1 > 2\,,$$
$$x_1^2 + y_1^2 + z_1^2 - x_1 y_1 z_1 < 0\,.$$

This parameter space is determined by the following fact: A_1, B_1 and $B_1 A_1$ are hyperbolic and $\mathrm{tr}\,([\tilde{B}_1, \tilde{A}_1]) < -2$. We notice that

$$\mathrm{tr}\,(\tilde{A}_1)\,\mathrm{tr}\,(\tilde{B}_1)\,\mathrm{tr}\,(\tilde{B}_1 \tilde{A}_1) = x_1 y_1 z_1 > 0\,,$$

(see Theorem 3.1(c)).

Let $(a_1, b_1, (a_1 b_1 a_1^{-1} b_1^{-1})^{-1})$ be a canonical homotopy basis of the fundamental group of S corresponding to $\Sigma_{(1,0,1)}$. A closed curve on S and the closed geodesic freely homotopic to it are labeled the same symbol (see Figure 3.3). Then a_1, b_1 and $a_1 b_1$ are the projections of $\mathrm{ax}\,(A_1)$, $\mathrm{ax}\,(B_1)$ and $\mathrm{ax}\,(B_1 A_1)$, respectively. Thus three length parameters correspond to lengths

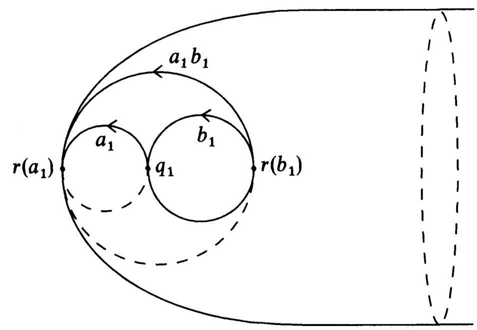

FIGURE 3.3

of a_1, b_1 and $a_1 b_1$. Let q_1 be the intersection point of a_1 and b_1. Let $r(a_1)$ be the unique point on a_1 satisfying $d(q_1, r(a_1)) = \mathrm{tl}(A_1)/2$, that is, two segments obtained from $a_1 - \{q_1, r(a_1)\}$ have the same length $\mathrm{tl}(A_1)/2$. Similarly, let $r(b_1)$ be defined. From Theorem 3.5(i), the positions of a_1, b_1 and $a_1 b_1$ are obtained as follows:

THEOREM 3.7. *$a_1 b_1$ intersects a_1 and b_1 at $r(a_1)$ and $r(b_1)$, respectively. Thus the geodesic through $r(a_1)$ and $r(b_1)$ is the closed geodesic $a_1 b_1$. Two segments obtained from $a_1 b_1 - \{r(a_1), r(b_1)\}$ have the same length* $\mathrm{tl}(B_1 A_1)/2$.

4. A parametrization of $T(2, 0, 0)$

In this section, we consider global real analytic length parameters for $T(2, 0, 0)$ and prove Theorem 1.1.

First we show the following lemma.

LEMMA 4.1. *A system of generators (A_1, B_1, A_2, B_2) is canonical and of type* $(2, 0, 0)$ *if and only if $\Sigma_j = (A_j, B_j, C_j^{-1})$, $C_j = [B_j, A_j]$ $(j = 1, 2)$ is a canonical system of generators of type* $(1, 0, 1)$ *and $C_1 = C_2^{-1}$.*

In fact, the "only if" part is clear. The "if" part is obtained from the combination theorem about the amalgamated product of two Fuchsian groups generated by Σ_1 and Σ_2 with the amalgamated subgroup generated by $C_1 = C_2^{-1}$.

Let $\Sigma_{(2,0,0)} = (A_1, B_1, A_2, B_2)$ be a canonical system of generators of type $(2, 0, 0)$ acting on $\boldsymbol{D}$ normalized in such a way that $q(A_1) = -1$, $p(A_1) = 1$ and the intersection point of $\mathrm{ax}(A_1)$ and $\mathrm{ax}(B_1)$, say p_1, is 0. Since $\Sigma_1 = (A_1, B_1, C_1^{-1})$ is a canonical system of generators of type $(1, 0, 1)$, Theorem 3.6 implies that $x_1 := |\mathrm{tr}(A_1)|$, $y_1 := |\mathrm{tr}(B_1)|$ and $z_1 := |\mathrm{tr}(B_1 A_1)|$ determine A_1 and B_1 global real analytically.

Next we construct B_2A_2.

LEMMA 4.2. *B_2A_2 is determined global real analytically by A_1, B_1 and the absolute values of traces $u := |\mathrm{tr}\,(B_2A_2A_1)|$ and $v := |\mathrm{tr}\,(B_2A_2B_1^{-1})|$.*

PROOF. Let p_2 be the intersection point of $\mathrm{ax}\,(A_2)$ and $\mathrm{ax}\,(B_2)$ and $R_2 \in M(\boldsymbol{D})$ the elliptic element of order 2 with the fixed point p_2. Then by Theorem 3.5(iii), we have $R_2B_2A_2 = C_2^{1/2} = C_1^{-1/2}$. Thus if R_2 is determined, then B_2A_2 is determined. In order to simplify the calculation, we normalize in such a way that $\Sigma_{(2,0,0)}$ acts on $\boldsymbol{H}$, $\mathrm{ax}\,(B_1A_1) = L(\infty, 0)$ and $\mathrm{fp}\,(C_1^{-1/2}B_1^{-1}) = i$ (see Figure 4.1).

Then we have the positive matrix representation

$$\widetilde{B_1A_1} := \begin{bmatrix} k & 0 \\ 0 & \dfrac{1}{k} \end{bmatrix},$$

where $k = \dfrac{z_1 - \sqrt{z_1^2 - 4}}{2} \in (0, 1)$. Since $i = \mathrm{fp}\,(C_1^{-1/2}B_1^{-1}) \in \mathrm{ax}\,(A_1)$, $\mathrm{ax}\,(A_1)$ intersects the imaginary axis at i. Then we have $q(A_1)p(A_1) = -1$. Thus we obtain the negative matrix representation

$$\tilde{A}_1 := \begin{bmatrix} a & c \\ c & d \end{bmatrix},$$

where $ad = 1 + c^2$, $a + d = -x_1$. Since $A_1(\infty) = a/c < 0$ and $A_1(0) = c/d < 0$, we obtain $a < 0$, $d < 0$ and $c > 0$. Thus the matrix representation

$$\tilde{B}_1 := \widetilde{B_1A_1}\tilde{A}_1^{-1} = \begin{bmatrix} kd & -kc \\ -\dfrac{c}{k} & \dfrac{a}{k} \end{bmatrix}$$

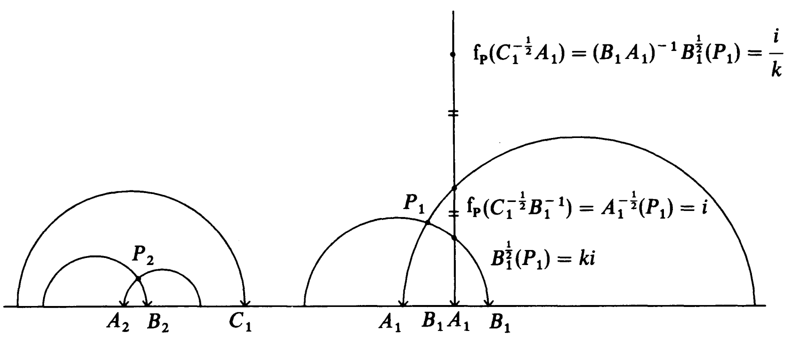

FIGURE 4.1

is negative. Since $a/k + kd = -y_1$, we have

$$a = \frac{kx_1 - y_1}{\dfrac{1}{k} - k}, \qquad d = \frac{-\dfrac{x_1}{k} + y_1}{\dfrac{1}{k} - k},$$

$$c = \frac{1}{\dfrac{1}{k} - k}\sqrt{x_1 y_1 z_1 - (x_1^2 + y_1^2 + z_1^2) + 4}.$$

The matrix representation

$$\tilde{C}_1^{-1} := [\tilde{B}_1, \tilde{A}_1]^{-1} = \begin{bmatrix} ad - \dfrac{c^2}{k^2} & \dfrac{1}{k}\left(\dfrac{1}{k} - k\right)ac \\ -k\left(\dfrac{1}{k} - k\right)cd & ad - k^2c^2 \end{bmatrix}$$

is negative, since $\operatorname{tr}(\tilde{C}_1^{-1}) = 2 - (1/k - k)^2 c^2 < 2$ and C_1^{-1} is hyperbolic. Thus by Proposition 2.4, we have the negative matrix representation

$$\widetilde{C_1^{-1/2}} := \begin{bmatrix} -\dfrac{c}{k} & \dfrac{a}{k} \\ -kd & kc \end{bmatrix}.$$

Put $\tilde{R}_2 = \begin{bmatrix} e & f \\ g & -e \end{bmatrix}$, where $gf = -(e^2 + 1)$ and $g > 0$ (this is called the negative matrix representation (see [12])). Since $\operatorname{Re}(\operatorname{fp}(R_2)) < 0$, we have $e < 0$. Then

$$\tilde{R}_2\widetilde{C_1^{-1/2}} = \begin{bmatrix} -\dfrac{ce}{k} - kdf & \dfrac{ae}{k} + kcf \\ -\dfrac{cg}{k} + kde & \dfrac{ag}{k} - kce \end{bmatrix}.$$

Hence we have

$$\operatorname{tr}(\tilde{R}_2\widetilde{C_1^{-1/2}}\tilde{A}_1) = g/k - kf > 0,$$

$$\operatorname{tr}(\tilde{R}_2\widetilde{C_1^{-1/2}}\tilde{B}_1^{-1}) = g - f > 0.$$

Since $u = |\operatorname{tr}(B_2A_2A_1)| = \operatorname{tr}(\tilde{R}_2\widetilde{C_1^{-1/2}}\tilde{A}_1)$ and $v = |\operatorname{tr}(B_2A_2B_1^{-1})| = \operatorname{tr}(\tilde{R}_2\widetilde{C_1^{-1/2}}\tilde{B}_1^{-1})$, we have

$$f = \frac{u - \frac{v}{k}}{\frac{1}{k} - k}, \qquad g = \frac{u - kv}{\frac{1}{k} - k},$$

$$e = \frac{-1}{\frac{1}{k} - k}\sqrt{uvz_1 - (u^2 + v^2 + z_1^2) + 4}.$$

Therefore R_2 is determined global real analytically.

Q.E.D.

REMARK 4.3. The intersection point p_2 is also determined global real analytically.

LEMMA 4.4.

$$|\mathrm{tr}\,(B_2 A_2)| = \mathrm{tr}\,(\tilde{R}_2 \widetilde{C_1^{-1/2}})$$

$$= \frac{1}{z_1^2 - 4}\{z_1\sqrt{x_1 y_1 z_1 - (x_1^2 + y_1^2 + z_1^2) + 4}\sqrt{uvz_1 - (u^2 + v^2 + z_1^2) + 4}$$

$$+ 2(x_1 u + y_1 v) - z_1(y_1 u + x_1 v)\}.$$

Since a and f are negative, we obtain the following proposition.

PROPOSITION 4.5.

$$k < \frac{|\mathrm{tr}\,(B_1)|}{|\mathrm{tr}\,(A_1)|} < \frac{1}{k} \qquad \textit{and} \qquad k < \frac{|\mathrm{tr}\,(B_2 A_2 B_1^{-1})|}{|\mathrm{tr}\,(B_2 A_2 A_1)|} < \frac{1}{k},$$

where $k = \dfrac{|\mathrm{tr}\,(B_1 A_1)| - \sqrt{|\mathrm{tr}\,(B_1 A_1)|^2 - 4}}{2}$.

REMARK 4.6. In order to calculate $|\mathrm{tr}\,(B_2 A_2)|$, we determined $C_1^{-1/2}$. We can prove Lemma 4.2 without determining it.

In fact, under the same normalization as in Lemma 4.2, we consider the matrix representations of $C_1^{-1/2}A_1$ and $C_1^{-1/2}B_1^{-1}$. Since $\mathrm{fp}\,(C_1^{-1/2}A_1) \in \mathrm{ax}\,(B_1 A_1)$, we put $\mathrm{fp}\,(C_1^{-1/2}A_1) = i/l$ for some $l \in (0, 1)$. Then we have the matrix representations

$$\widetilde{C_1^{-1/2}A_1} := \begin{bmatrix} 0 & \frac{1}{l} \\ -l & 0 \end{bmatrix} \qquad \text{and} \qquad \widetilde{C_1^{-1/2}B_1^{-1}} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Thus we have $l = k$, since $|\mathrm{tr}\,(B_1 A_1)| = |\mathrm{tr}\,(\widetilde{C_1^{-1/2}B_1^{-1}})^{-1}\widetilde{C_1^{-1/2}A_1})| = 1/l + l$. Hence we obtain $\tilde{R}_2$ as in the later half of the above proof.

LEMMA 4.7. *A_2 and B_2 are determined by A_1, B_1, B_2A_2 and the absolute values of traces $x_2 := |\mathrm{tr}(A_2)|$ and $y_2 := |\mathrm{tr}(B_2)|$ global real analytically.*

PROOF. Let $\hat{\Sigma}_2 = (\hat{A}_2, \hat{B}_2, \hat{C}_2^{-1})$ be a canonical system of generators of type (1, 0, 1) acting on $\boldsymbol{D}$ such that $q(\hat{A}_2) = -1$, $p(\hat{A}_2) = 1$ and the intersection point of $\mathrm{ax}(\hat{A}_2)$ and $\mathrm{ax}(\hat{B}_2)$ is 0. Further, let $\hat{\Sigma}_2$ satisfy

$$|\mathrm{tr}(\hat{A}_2)| = |\mathrm{tr}(A_2)|, \qquad |\mathrm{tr}(\hat{B}_2)| = |\mathrm{tr}(B_2)| \qquad \text{and} \qquad |\mathrm{tr}(\hat{B}_2\hat{A}_2)| = |\mathrm{tr}(B_2A_2)|.$$

Then Theorem 3.6 implies that $\hat{\Sigma}_2$ is determined by these three traces and conjugate to Σ_2. Thus we have

$$T\hat{\Sigma}_2T^{-1} = \Sigma_2 \qquad \text{for some} \quad T \in M(\boldsymbol{D}).$$

Since $q(C_2) = p(C_1)$ and $p(C_2) = q(C_1)$, T satisfies

$$T(q(\hat{C}_2)) = p(C_1), \qquad T(p(\hat{C}_2)) = q(C_1) \quad \text{and} \quad T(0) = p_2.$$

Thus T is determined global real analytically and so are A_2 and B_2.

Q.E.D.

From Theorem 3.6 and Lemmas 4.1, 4.2, 4.4 and 4.7, we obtain the following lemma.

LEMMA 4.8. *A parameter space of $T(2, 0, 0)$ is described as follows:*

$$x_j > 2,\ y_j > 2\ (j = 1, 2), \qquad z_1 > 2,\ u > 2,\ v > 2,$$

$$x_1^2 + y_1^2 + z_1^2 - x_1y_1z_1 = x_2^2 + y_2^2 + |\mathrm{tr}(B_2A_2)|^2 - x_2y_2|\mathrm{tr}(B_2A_2)| < 0,$$

$$|\mathrm{tr}(B_2A_2)| = \frac{1}{z_1^2-4}\{z_1\sqrt{x_1y_1z_1-(x_1^2+y_1^2+z_1^2)+4}\sqrt{uvz_1-(u^2+v^2+z_1^2)+4}$$

$$+ 2(x_1u + y_1v) - z_1(y_1u + x_1v)\} > 2.$$

Finally, we consider the segments in $\boldsymbol{X}$ and closed geodesics on a Riemann surface corresponding to these seven length parameters. By Theorem 3.5(i), we have

$$\mathrm{ax}(A_j) = L(A_j^{-1/2}(p_j), p_j), \qquad \frac{\mathrm{tl}(A_j)}{2} = d(A_j^{-1/2}(p_j), p_j),$$

$$\mathrm{ax}(B_j) = L(p_j, B_j^{1/2}(p_j)), \qquad \frac{\mathrm{tl}(B_j)}{2} = d(p_j, B_j^{1/2}(p_j)),$$

$$\mathrm{ax}(B_jA_j) = L(A_j^{-1/2}(p_j), B_j^{1/2}(p_j)), \qquad \frac{\mathrm{tl}(B_jA_j)}{2} = d(A_j^{-1/2}(p_j), B_j^{1/2}p_j)).$$

Since R_2, $C_1^{-1/2}A_1$ and $C_1^{-1/2}B_1^{-1}$ are elliptic elements of order 2 and

$$B_2A_2A_1 = R_2C_1^{-1/2}A_1\ ,$$
$$B_2A_2B_1^{-1} = R_2C_1^{-1/2}B_1^{-1}\ ,$$

we have

$$\mathrm{ax}\,(B_2A_2A_1) = L(\mathrm{fp}\,(C_1^{-1/2}A_1), p_2)\,, \qquad \frac{\mathrm{tl}\,(B_2A_2A_1)}{2} = d(\mathrm{fp}\,(C_1^{-1/2}A_1), p_2)\,,$$

$$\mathrm{ax}\,(B_2A_2B_1^{-1}) = L(\mathrm{fp}\,(C_1^{-1/2}B_1^{-1}), p_2)\,, \qquad \frac{\mathrm{tl}\,(B_2A_2B_1^{-1})}{2} = d(\mathrm{fp}\,(C_1^{-1/2}B_1^{-1}), p_2)\,.$$

Thus these seven length parameters are obtained from the lengths of the thick segments drawn in Figure 4.2.

Let S be the Riemann surface represented by $\Sigma_{(2,0,0)}$. Let (a_1, b_1, a_2, b_2) be a canonical homotopy basis of the fundamental group of S corresponding to $\Sigma_{(2,0,0)}$. Let q_j, $r(a_j)$ and $r(b_j)$ $(j = 1, 2)$ be the points defined as in Section 3, which are the fixed points of the *hyperelliptic involution* of S, namely, the *Weierstrass points* of S. Then a_j, b_j and a_jb_j $(j = 1, 2)$ are the simple closed geodesics positioned as in Figure 4.3(i).

Since $\mathrm{fp}\,(C_1^{-1/2}A_1) = (B_1A_1)^{-1}B_1^{1/2}(p_1)$ and $\mathrm{fp}\,(C_1^{-1/2}B_1^{-1}) = A_1^{-1/2}(p_1)$ are projected onto $r(b_1)$ and $r(a_1)$ respectively, we obtain the following theorem.

THEOREM 4.9. *The geodesic through $r(b_1)$ (resp. $r(a_1)$) and q_2 is the simple closed geodesic $a_1a_2b_2$ (resp. $b_1^{-1}a_2b_2$) such that two segments obtained*

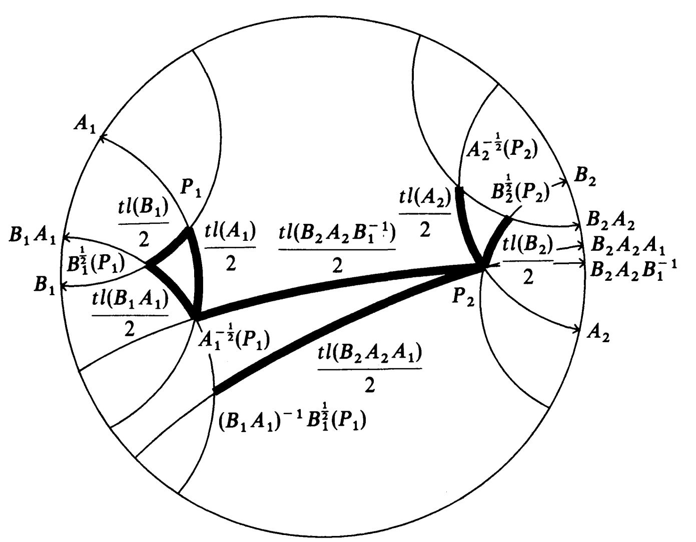

FIGURE 4.2

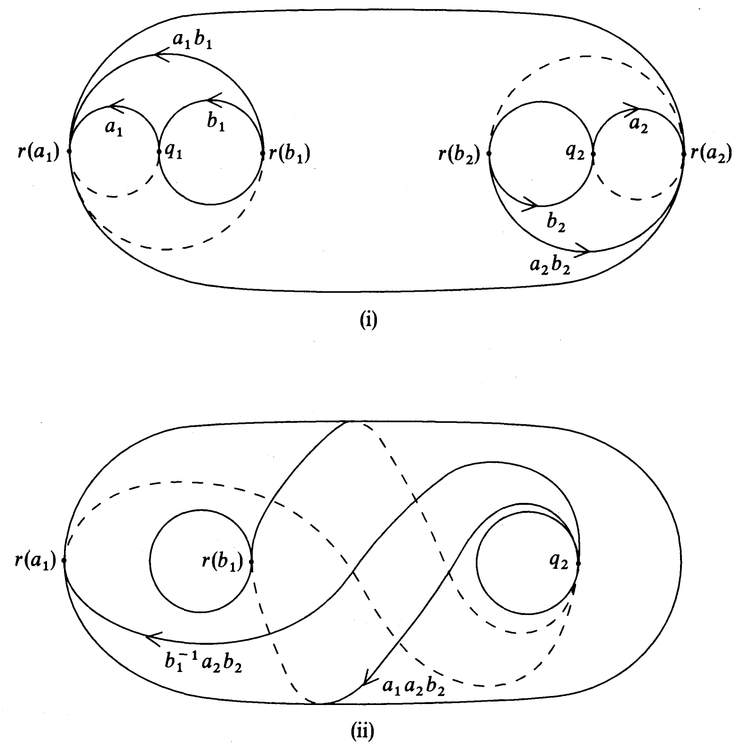

FIGURE 4.3

from $a_1a_2b_2 - \{r(b_1), q_2\}$ *(resp.* $b_1^{-1}a_2b_2 - \{r(a_1), q_2\}$*) have the same length* $\dfrac{\mathrm{tl}\,(B_2A_2A_1)}{2}$ $\left(\textit{resp.}\ \dfrac{\mathrm{tl}\,(B_2A_2B_1^{-1})}{2}\right)$ *(see Figure* 4.3(ii)).

Hence Lemma 4.8 and Theorem 4.9 complete the proof of Theorem 1.1.

References

[1] Y. Imayoshi and M. Taniguchi, An Introduction to Teichmüller Spaces, Springer-Verlag, Tokyo, Berlin, Heidelberg, New York, Paris, Hong Kong, Barcelona, 1992.

[2] L. Keen, Canonical polygons for finitely generated Fuchsian Groups, Acta Math., **115** (1966), 1–16.

[3] L. Keen, Intrinsic moduli on Riemann surfaces, Ann. of Math., **84** (1966), 404–420.

[4] L. Keen, On Fricke moduli, in Advances in the Theory of Riemann Surfaces, (L. V. Ahlfors ed. et al.), Ann. Math. Studies **66**, Princeton Univ. Press, Princeton, N. J., 1971, 205–224.

[5] L. Keen, A correction to "On Fricke moduli", Proc. Amer. Math. Soc., **40** (1973), 60–62.
[6] L. Keen, A rough fundamental domain for Teichmüller spaces, Bull. Amer. Math. Soc., **83** (1977), 1199–1226.
[7] T. Nakanishi and M. Näätänen, Length parameters for Teichmüller space of punctured surfaces, RIMS Kokyuroku **882**, Research Institute for Mathematical Science, Kyoto Univ., Kyoto, Japan, 1994, 57–69.
[8] Y. Okumura, On the global real analytic coordinates for Teichmüller spaces, J. Math. Soc. Japan, **42** (1990), 91–101.
[9] Y. Okumura, Global real analytic length parameters and angle parameters for Teichmüller spaces and the geometry of hyperbolic transformations, RIMS Kokyuroku **882**, Research Institute for Mathematical Science, Kyoto Univ., Kyoto, Japan, 1994, 77–88.
[10] Y. Okumura, Global real analytic angle parameters for Teichmüller spaces, to appear in J. Math. Soc. Japan, **49** (1997).
[11] Y. Okumura, On the global real analytic coordinates for Teichmüller spaces II, in preparation.
[12] Y. Okumura, Geometry of Möbius transformations acting on the hyperbolic plane and characterization of simple dividing loops on Riemann surfaces, in preparation.
[13] K. Saito, Algebraic representations of Teichmüller space, Kodai Math. J., **17** (1994), 609–626.
[14] P. Schmutz, Die Parametrisierung des Teichmüllerraumes durch geodätische Längenfunktionen, Comment. Math. Helv., **68** (1993), 278–288.
[15] M. Seppälä and T. Sorvali, On geometric parametrization of Teichmüller spaces, Ann. Acad. Sci. Fenn. Ser. A I Math., **10** (1985), 515–526.
[16] M. Seppälä and T. Sorvali, Parametrization of Möbius groups acting in a disk, Comment. Math. Helv., **61** (1986), 149–160.
[17] M. Seppälä and T. Sorvali, Parametrization of Teichmüller spaces by geodesic length functions, in Holomorphic Functions and Moduli II (D. Drasin ed. et al.), Mathematical Sciences Research Institute Publications **11**, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1988, 267–284.
[18] M. Seppälä and T. Sorvali, Geometry of Riemann Surfaces and Teichmüller Spaces, Mathematics Studies **169**, North-Holland, Amsterdam, London, New York, Tokyo, 1992.
[19] T. Sorvali, Parametrization of free Möbius groups, Ann. Acad. Sci. Fenn. Ser. A I Math., **579**, 1974, 1–12.
[20] S. A. Wolpert, On the symplectic geometry of deformations of a hyperbolic surface, Ann. of Math., **117** (1983), 207–234.
[21] S. A. Wolpert, Geodesic length functions and the Nielsen problem, J. Differential Geom., **25** (1987), 275–296.

Faculty of Technology
Kanazawa University
2-40-20, *Kodatsuno*
Kanazawa 920
Japan