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On a function space related to the Hardy-Littlewood inequality for Riemannian symmetric spaces

Dedicated to Professor Kiyosato Okamoto on his 60th birthday

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ABSTRACT. On Riemannian symmetric spaces G/K we define an L^q type Schwartz space $\mathscr{J}^q(G)$ which corresponds to the Schwartz space with weight $|x|^{n(q-2)}$ on $\boldsymbol{R}^n$. We study some properties of $\mathscr{J}^q(G)$ and we prove if $2 \le q < 4$ and p and q are conjugate, then $J^q(G)$ equals to the L^p-type Schwartz space $\mathscr{S}^p(G)$ defined by Harish-Chandra.

1. Introduction

For a real number q $(2 \le q < \infty)$ and a Borel function f on $\boldsymbol{R}^n$ we put

$$\|f\|_{(q)} = \left(\int_{\boldsymbol{R}^n} |f(x)|^q |x|^{n(q-2)} dx\right)^{1/q}$$

and denote by $J^q(\boldsymbol{R}^n)$ the Banach space of all Borel functions f on $\boldsymbol{R}^n$ satisfying $\|f\|_{(q)} < \infty$. The Hardy-Littlewood theorem ([3]) says that if $f \in J^q(\boldsymbol{R}^n)$, then the Fourier transform $\tilde{f}$ of f is well-defined and there exists a constant $C_q > 0$ such that

$$\|\tilde{f}\|_q \le C_q \|f\|_{(q)} .$$

On the other hand, if $1 \le p \le 2$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$, then the Fourier transform $\tilde{f}$ of $f \in L^p(\boldsymbol{R}^n)$ is well-defined and there exists a constant $B_p > 0$ such that

$$\|\tilde{f}\|_q \le B_p \|f\|_p .$$

This is the Hausdorff-Young theorem. These two theorems suggest the resemblance between $L^p(\boldsymbol{R}^n)$ and $J^q(\boldsymbol{R}^n)$. In fact, if we put $f_\alpha(x) = (1 + |x|^2)^\alpha$ and $g_\beta(x) = |x|^\beta(|x| \le 1),\ = 0(|x| > 1)$, then

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$$f_\alpha \in L^p(\boldsymbol{R}^n) \Leftrightarrow \alpha < -\frac{n}{2p} \Leftrightarrow f_\alpha \in J^q(\boldsymbol{R}^n)$$

and

$$g_\beta \in L^p(\boldsymbol{R}^n) \Leftrightarrow \beta > -\frac{n}{p} \Leftrightarrow g_\beta \in J^q(\boldsymbol{R}^n) .$$

We have proved a Hardy-Littlewood theorem and a Hausdorff-Young theorem (Eguchi–Kumahara [1], [2]) for the spherical Fourier transform on Riemannian symmetric spaces G/K of noncompact type. The Euclidean space $\boldsymbol{R}^n$ is the symmetric space of the Euclidean motion group by the rotation group and is of rank one. The factor $|x|^n$ is the product of (distance from the origin)$^{\text{rank}}$ and the Jacobian with respect to the polar decomposition. For a noncompact type symmetric space $X = G/K$ we denote by $\sigma(x)$ the distance from the origin to x, by l the rank of X and by $\Omega(x)$ the Jacobian with respect to the polar decomposition. Then there exists a constant $C_q > 0$ such that

$$\|\tilde{f}\|_q \leq C_q\left(\int_X |f(x)|^q \sigma(x)^{l(q-2)} \Omega(x)^{q-2} d\mu(x)\right)^{1/q},$$

for any K-biinvariant measurable function f on G whose value of the integration on the right hand side is finite (Hardy-Littlewood theorem). There exists a constant $B_p > 0$ such that

$$\|\tilde{f}\|_q \leq B_p \|f\|_p ,$$

for any K-biinvariant L^p function f on G (Hausdorff-Young theorem). We define $J^q(G)$ as the Banach space of all K-biinvariant measurable functions f on G satisfying $\|f\|_{(q)} < \infty$, where $\|f\|_{(q)}$ is defined by the right hand side of the Hardy-Littlewood inequality (see § 3). Let $I^p(G) = L^p(K \backslash G/K)$ be the Banach space of K-biinvariant L^p-functions on G. If $\frac{1}{p} + \frac{1}{q} = 1$, then it can be proved that the spherical Fourier transforms of functions in $I^p(G)$ and $J^q(G)$ can be extended holomorphically to a certain tube domain ([1, Theorem 2], [2, Theorem 2]).

The purpose of the present paper is to point out more similarities between $I^p(G)$ and $J^q(G)$. There is a dense subset of $I^p(G)$ which plays an important role in harmonic analysis. That is the Schwartz space $\mathscr{I}^p(G)$ of L^p type (Trombi–Varadarajan [7]). We define the Schwartz space $\mathscr{J}^q(G)$ of J^q type and investigate some properties of $\mathscr{J}^q(G)$. This is a Fréchet space and dense in $J^q(G)$. Furthermore, $\mathscr{I}^q(G)$ is contained in $\mathscr{J}^q(G)$. If $2 \leq q < 4$, then we can prove that $\mathscr{J}^q(G) = \mathscr{I}^p(G)$. Moreover, we prove that $\mathscr{J}^q(G) = \mathscr{I}^p(G)$ for all $q \geq 2$ if the rank of G/K is one.

2. Notation and preliminaries

Let G be a connected semisimple Lie group with finite center and K a maximal compact subgroup of G. We denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$ with Cartan involution θ, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$, and Σ the corresponding set of restricted roots. Let M' and M be the normalizer and the centralizer of $\mathfrak{a}$ in K, respectively, and denote by $W = M'/M$, which is called the Weyl group of G/K, and let $|W|$ be its order. Fix a Weyl chamber $\mathfrak{a}^+$ and put $A^+ = \exp \mathfrak{a}^+$. Let Σ^+ be the corresponding set of positive restricted roots and $|\Sigma^+|$ be its order. For $\alpha \in \Sigma^+$, $\mathfrak{g}_\alpha$ denotes the root subspace and $m_\alpha = \dim \mathfrak{g}_\alpha$ the multiplicity of α. Let $\mathfrak{n} = \sum_{\Sigma^+} \mathfrak{g}_\alpha$ and $\rho = \frac{1}{2}\sum_{\Sigma^+} m_\alpha \alpha$. Then $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$. We denote by $G = KAN$ the corresponding decomposition of G. For $x \in G$, $H(x) \in \mathfrak{a}$ denotes the element uniquely determined by $x \in K \exp(H(x))N$. For $a \in A$, we write $\log a$ for $H(a)$.

Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$ and $\mathfrak{a}^*_{\mathbf{C}}$ its complexification. We denote by $\langle\ ,\ \rangle$ the Killing form of $\mathfrak{g}$. For $\lambda \in \mathfrak{a}^*$, let $H_\lambda \in \mathfrak{a}$ be the unique element determined by $\lambda(H) = \langle H_\lambda, H\rangle$ for all $H \in \mathfrak{a}$. For $\lambda, \mu \in \mathfrak{a}^*$, we put $\langle \lambda, \mu\rangle = \langle H_\lambda, H_\mu\rangle$ and $|\lambda| = \langle \lambda, \lambda\rangle^{1/2}$. Let $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$ and $\bar{N}$ denote the corresponding analytic subgroup of G. For $\varepsilon > 0$ we put $C_{\varepsilon\rho} = [w(\varepsilon\rho); w \in W]$, the convex hull of the set $\{w(\varepsilon\rho); w \in W\}$. For $0 < p < 2$ we define the tube domain T_p by $T_p = \mathfrak{a}^* + \sqrt{-1}C_{(2/p-1)\rho}$.

We denote by $C_c^\infty(G)$ the space of all compactly supported C^∞-functions on G and by $C_c^\infty(G/K)$ and $C_c^\infty(K\backslash G/K)$ the subspaces of $C_c^\infty(G)$ of right K-invariant and K-biinvariant functions, respectively. The Killing form induces euclidean measures on A and $\mathfrak{a}^*$. We normalize them by multiplying with the factor $(2\pi)^{-l/2}$ and denote them by da and $d\lambda$, respectively, where $l = \dim \mathfrak{a}$, the rank of G/K. Let dk be the normalized Haar measure on K so that the total measure is one. The Haar measures on N and $\bar{N}$ are normalized so that

$$\theta(dn) = d\bar{n}\,, \qquad \int_{\bar{N}} e^{-2\rho(H(\bar{n}))}d\bar{n} = 1\,.$$

Moreover, we normalize the Haar measure dx on G so that

$$\int_G f(x)dx = \int_{KAN} f(kan)e^{2\rho(\log a)}dkdadn\,, \qquad f \in C_c^\infty(G)\,.$$

We denote by vol (K/M) the volume of K/M with respect to the K-invariant measure $d\mu(b)$ induced from the restriction of $-\langle\ ,\ \rangle$ to $\mathfrak{k}$. Let dk_M be the K invariant measure on K/M defined by $dk_M = \mathrm{vol}\,(K/M)^{-1} d\mu(k_M)$.

The following integral formula corresponds to the Cartan decomposition $G = KAK$ (Helgason [6]).

$$\int_G f(x)dx = \frac{(2\pi)^{l/2}\,\mathrm{vol}\,(K/M)}{|W|}\int_{\mathfrak{a}} \prod_{\alpha\in\Sigma^+} |\sinh \alpha(H)|^{m(\alpha)} dH$$

$$\times \iint_{K\times K} f(k_1 \exp(H) k_2) dk_1 dk_2\,, \qquad f \in C_c^\infty(G)\,.$$

We put

$$\Omega(\exp H) = \frac{(2\pi)^{l/2}\,\mathrm{vol}\,(K/M)}{|W|}\prod\nolimits_{\alpha\in\Sigma^+} |\sinh \alpha(H)|^{m(\alpha)}\,, \qquad H \in \mathfrak{a}\,.$$

By the W-invariance of $\Omega(a)(a \in A)$ we can extend it to G by $\Omega(x) = \Omega(a)$ for $x = k_1 a k_2$, k_1, $k_2 \in K$, $a \in A$.

Finally, we put $\sigma(x) = \sqrt{\langle X, X\rangle}$ for $x = k \exp X$, $k \in K$, $X \in \mathfrak{p}$.

3. Schwartz space of L^p type

Let $I^p(G)$ be the Banach space of all K-biinvariant measurable functions f on G such that

$$\|f\|_p = \left(\int_G |f(x)|^p dx\right)^{1/p} < \infty\,.$$

Of course, we identify two functions which differ only on a set of measure zero. Let

$$\varphi_\lambda(x) = \int_K e^{(\sqrt{-1}\lambda - \rho)(H(xk))} dk\,, \qquad x \in G\,,$$

be the elementary spherical function. Then φ_λ is bounded if and only if $\lambda \in T_1$. We put $\Xi = \varphi_0$. The Harish-Chandra c-function is defined by

$$c(\lambda) = \int_{\bar{N}} e^{(-\sqrt{-1}\lambda + \rho)(H(\bar{n}))} d\bar{n}\,.$$

We define the spherical Fourier transform $\tilde{f}$ of $f \in I^1(G)$ by

$$\tilde{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx\,, \qquad \lambda \in \mathfrak{a}^*\,.$$

Let $L^2\left(\mathfrak{a}^*, \dfrac{1}{|W||c(\lambda)|^2}d\lambda\right)^W$ be the Hilbert space of W-invariant square integrable functions on $\mathfrak{a}^*$ with respect to the measure $\dfrac{1}{|W||c(\lambda)|^2}d\lambda$. Then the Plancherel theorem can be stated as follows (see e.g. Warner [8], p. 338).

LEMMA 1. *For $f \in I^1(G) \cap I^2(G)$, we have*

$$\|f\|_2 = \left(\frac{1}{|W|}\int_{\mathfrak{a}^*} |\tilde{f}(\lambda)|^2 \frac{1}{|c(\lambda)|^2} d\lambda\right)^{1/2}.$$

Moreover, the map $f \mapsto \tilde{f}$ can be extended to an isometry of $I^2(G)$ onto $L^2\left(\mathfrak{a}^*, \dfrac{1}{|W||c(\lambda)|^2}d\lambda\right)^W$.

The following is the Hausdorff-Young theorem (cf. Eguchi-Kumahara [1]).

LEMMA 2. *Let $1 \le p < 2$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$. Then the spherical Fourier transform can be defined for functions in $I^p(G)$ and, for each $f \in I^p(G)$, the spherical Fourier transform $\tilde{f}$ can be extended to a holomorphic function on* Int T_p *and, for any $\eta \in$* Int $C_{(2/p-1)\rho}$, *there exists a constant $B_{p,\eta} > 0$ such that*

$$\left(\frac{1}{|W|}\int_{\mathfrak{a}^*} |\tilde{f}(\xi + \sqrt{-1}\eta)|^q |c(\xi)|^{-2} d\xi\right)^{1/q} \le B_{p,\eta}\|f\|_p, \qquad f \in I^p(G).$$

Let $U(\mathfrak{g}_C)$ be the universal enveloping algebra of the complexification $\mathfrak{g}_C$ of $\mathfrak{g}$. Let $p > 0$. We denote by $\mathscr{I}^p(G)$ the space of all $f \in C^\infty(K\backslash G/K)$ such that for any $u \in U(\mathfrak{g}_C)$ and any integer $m \ge 0$,

$$\mu^p_{u,m}(f) = \sup_{x \in G} (1 + \sigma(x))^m |(uf)(x)| \Xi(x)^{-2/p} < \infty.$$

Then $\mathscr{I}^p(G)$ is a Fréchet space by the system of seminorms $\{\mu^p_{u,m}\}$ and is dense in $I^p(G)$ (see Trombi-Varadarajan [7]).

Let $S(\mathfrak{a}^*_C)$ be the symmetric algebra over $\mathfrak{a}^*_C$ and for $s \in S(\mathfrak{a}^*_C)$ denote by $\partial(s)$ the corresponding differential operator on $\mathfrak{a}^*_C$. Let $0 < p < 2$. We define the space $\overline{\mathscr{Z}}(T_p)$ to be the set of all W-invariant holomorphic functions F on Int T_p such that for any $s \in S(\mathfrak{a}^*_C)$ and any integer $m \ge 0$,

$$\zeta^p_{s,m}(F) = \sup_{\lambda \in \mathrm{Int}\, T_p} (1 + |\lambda|^2)^m |(\partial(s)F)(\lambda)| < \infty.$$

Then the following important theorem due to Trombi–Varadarajan holds true.

LEMMA 3 (TROMBI-VARADARAJAN [7]). *Let $0 < p < 2$. Then, for $f \in \mathscr{I}^p(G)$, the integral $\tilde{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx$ converges absolutely for all $\lambda \in T_p$.*

The function $\tilde{f}$ *lies in* $\overline{\mathscr{Z}}(T_p)$ *and the spherical Fourier transform* $f \mapsto \tilde{f}$ *is a linear topological isomorphism of* $\mathscr{I}^p(G)$ *onto* $\overline{\mathscr{Z}}(T_p)$.

4. Schwartz Space of J^q type

For $q \geq 2$ we define the Banach space $J^q(G)$ of all K-biinvariant measurable functions f on G such that

$$\|f\|_{(q)} = \left(\int_G |f(x)|^q \sigma(x)^{l(q-2)} \Omega^{q-2} dx\right)^{1/q} < \infty .$$

Then the following Hardy-Littlewood theorem holds (Eguchi–Kumahara [2]).

LEMMA 4. *Let* $2 \leq q < \infty$. *Then the spherical Fourier transform can be defined for* $f \in J^q(G)$ *and there exists a constant* $C_q > 0$, *independent of* f, *such that*

$$\left(\frac{1}{|W|}\int_{\mathfrak{a}^*} |\tilde{f}(\lambda)|^q |c(\lambda)|^{-2} d\lambda\right)^{1/q} \leq C_q \|f\|_{(q)} .$$

We denote by $\mathscr{J}^q(G)$ the set of all $f \in C^\infty(K\backslash G/K)$ such that for any $u \in U(\mathfrak{a}_{\mathbf{C}}^*)$ and any integer $m \geq 0$,

$$\nu_{u,m}^q(f) = \sup_{x \in G} (1 + \sigma(x))^m |(uf)(x)| \sigma(x)^{l(1-2/q)} \Omega(x)^{1-2/q}(x) \Xi(x)^{-2/q} < \infty .$$

Then $\mathscr{J}^q(G)$ is a Fréchet space by the system of the seminorms $\{\nu_{u,m}^q\}$.

5. Some inclusion properties

The following estimate in (1) is an immediate consequence of the definition of $\Omega(x)$. The statement (2) is due to Harish-Chandra (see [4] Theorem 3).

LEMMA 5. (1) *We put* $c_1 = 2^{-|\Sigma^+|}(2\pi)^{l/2} \operatorname{vol}(K/M)|W|^{-1}$. *Then*

$$\Omega(a) \leq c_1 e^{2\rho(\log a)} \qquad a \in A^+ ,$$

$$\Omega(a) \sim c_1 e^{2\rho(\log a)} \qquad a \in A^+ \quad \text{and} \quad a \to \infty .$$

(2) *There exist constants* $c_2 >$ *and* $d > 0$ *such that*

$$1 \leq e^{\rho(\log a)} \Xi(a) \leq c_2(1 + |\log a|)^d \qquad a \in A^+ .$$

THEOREM 1. *Let* $1 < p < 2$ *and* $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *Then we have*

$$\mathscr{I}^p(G) \subset \mathscr{J}^q(G) \subset I^p(G) \cap J^q(G)$$

and each inclusion map is continuous.

PROOF: Let $f \in \mathscr{J}^p(G)$, $u \in U(\mathfrak{g}_C)$ and $m \geq 0$ integer. For any $x \in G$ there exist $k_1, k_2 \in K$ and $a \in \mathrm{Cl}(A^+)$ such that $x = k_1 a k_2$. If $a \in \mathrm{Cl}(A^+) \backslash A^+$, then $\Omega(a) = 0$. So we assume that $a \in A^+$. Then, by Lemma 5, we have

$$\begin{aligned}
&(1 + \sigma(x))^m |(uf)(x)| \sigma(x)^{l(1-2/q)} \Omega(x)^{1-2/q} \Xi(x)^{-2/q} \\
&= (1 + \sigma(a))^m |(uf)(x)| \sigma(a)^{l(1-2/q)} \Omega(a)^{1-2/q} \Xi(a)^{-2/q} \\
&\leq c_1^{1-2/q} (1 + \sigma(a))^{m + l(1-2/q)} e^{2(1-2/q)\rho(\log a)} e^{(2/q)\rho(\log a)} |(uf)(x)| \\
&\leq c_1^{1-2/q} c_2^{2/p} (1 + \sigma(a))^{m + l(1-2/q) + 2d/p} |(uf)(x)| \, \Xi(a)^{-2/p} \\
&= c_1^{1-2/q} c_2^{2/p} (1 + \sigma(x))^{m + l(1-2/q) + 2d/p} |(uf)(x)| \, \Xi(x)^{-2/p} \\
&\leq c_1^{1-2/q} c_2^{2/p} \mu^p_{u, m + [l(1-2/q) + 2d/p] + 1}(f) < \infty .
\end{aligned}$$

Hence $f \in \mathscr{J}^q(G)$ and $\nu^q_{u,m}(f) \leq c_1^{1-2/q} c_2^{2/p} \mu^p_{u, m + [l(1-2/q) + 2d/p] + 1}(f)$.

Now let $f \in \mathscr{J}^q(G)$ and m be an integer satisfying $m > \dfrac{1}{p}\left(l(p-1) + \dfrac{2pd}{q}\right)$. Then

$$|f(x)| \leq c_3 (1 + \sigma(x))^{-m} \sigma(x)^{l(-1+2/q)} \Omega(x)^{-1+2/q} \Xi(x)^{2/q} ,$$

where $c_3 = \nu^q_{1,m}(f)$.

$$\begin{aligned}
\int_G |f(x)|^p dx &= |W| \int_{A^+} |f(a)|^p \Omega(a) da \\
&\leq c_3 |W| \int_{A^+} (1 + \sigma(a))^{-mp} \sigma(a)^{l(p-2)} \Omega(a)^{p-1} \Xi(a)^{2p/q} da \\
&\leq c_1^{p-1} c_2^{2p/q} c_3 |W| \int_{A^+} (1 + \sigma(a))^{-mp} \sigma(a)^{l(p-2)} (1 + \sigma(a))^{2pd/q} da \\
&\leq c_1^{p-1} c_2^{2p/q} c_3 \int_A (1 + \sigma(a))^{-mp + 2pd/q} \sigma(a)^{l(p-2)} da \\
&= c_4 \int_0^\infty (1 + t)^{-mp + 2pd/q} t^{l(p-2) + l - 1} dt < \infty ,
\end{aligned}$$

where

$$c_4 = c_1^{p-1} c_2^{2p/q} \nu^q_{1,m}(f) 2\pi^{l/2} \Gamma(l/2)^{-1} .$$

Thus we have proved that there exists a constant $c_5 > 0$ such that $\|f\|_p \leq c_5 \nu^q_{1,m}(f) < \infty$.

Next we prove that $\mathscr{J}^q(G) \subset J^q(G)$ and the inclusion map is continuous. Let $f \in \mathscr{J}^q(G)$ and $m > \dfrac{d + l}{q}$. Then,

$$\begin{aligned}\|f\|_{(q)}^q &= \int_G |f(x)|^q \sigma(x)^{l(q-2)} \Omega(x)^{q-2} dx \\ &= |W| \int_{A^+} |f(a)|^q \sigma(a)^{l(q-2)} \Omega(a)^{q-1} da \\ &\le |W| \{v_{1,m}^q(f)\}^q \int_{A^+} (1+\sigma(a))^{-mq} \Xi(a)^2 \Omega(a) da \\ &\le c_1 c_2^2 \{v_{1,m}^q(f)\}^q \int_A (1+\sigma(a))^{-mq+2d} da \\ &= c_6 \int_0^\infty (1+t)^{-mq+d+l-1} dt < \infty ,\end{aligned}$$

where

$$c_6 = c_1 c_2^2 \{v_{1,m}^q(f)\}^q 2\pi^{l/2} \Gamma(l/2)^{-1} .$$

This completes the proof.

LEMMA 6. *The space $C_c^\infty(K\backslash G/K)$ is dense in $\mathscr{I}^q(G)$.*

PROOF: For any $t>0$, let G_t denote the set of those $x \in G$ satisfying $\sigma(x)<t$ and let χ_t denote the characteristic function of G_t. Fix $a>0$ and a K-biinvariant function $\alpha \in C_c^\infty(G_a)$ such that $\int_G \alpha(x)dx = 1$. We put $g_t = (1-\chi_t)*\alpha = 1-\chi_t*\alpha$, where the star denotes the convolution on G. Then, by Harish-Chandra [5] Lemma 20, $g_t \in C_c^\infty(K\backslash G/K)$ and

$$g_t(x) = \begin{cases} 0 & \text{if } \sigma(x) \le t-a \\ 1 & \text{if } \sigma(x) \ge t+a \end{cases}$$

and

$$|(ug_t)(x)| \le \int_G |(u\alpha)(y)| dy \qquad (x \in G)$$

for $u \in U(\mathfrak{g}_C)$.

For any $f \in \mathscr{I}^q(G)$ we put

$$f_t = (1-g_t)f = (\chi_t * \alpha)f .$$

Then it is obvious that $f_t \in C_c^\infty(G) \cap \mathscr{I}^q(G)$. Fix $u \in U(\mathfrak{g}_C)$. Then there exist finite elements $u_i, u_i' \in U(\mathfrak{g}_C)$ such that

$$u(f-f_t) = u(g_t f) = \sum_i u_i' g_t \cdot u_i f .$$

If $\sigma(x) \geq t + a$, then

$$f(x) - f_t(x) = g_t(x) f(x) = f(x)$$

and if $\sigma(x) \geq t$, then, for any integer $m \geq 0$,

$$\begin{aligned}(1 + \sigma(x))^m |(uf)(x)| \sigma(x)^{l(1-2/q)} \Omega(x)^{1-2/q}(x) \Xi(x)^{-2/q} v^q_{a,m}(f) \\ \leq (1+t)^{-1} v^q_{u,m+1}(f)\,.\end{aligned}$$

Hence, if $\sigma(x) \geq t + a$, then

$$\begin{aligned}(1 + \sigma(x))^m |(uf)(x) - (uf_t)(x)| \sigma(x)^{l(1-2/q)} \Omega(x)^{1-2/q}(x) \Xi(x)^{-2/q} \\ \leq (1+t)^{-1} v^q_{u,m+1}(f)\,.\end{aligned}$$

Now suppose that $\sigma(x) < t + a$. Since $f(x) - f_t(x) = 0$ for $\sigma(x) \leq t - a$, we assume that $t - a < \sigma(x) < t + a$. Let $t > a$. Then

$$\begin{aligned}&(1 + \sigma(x))^m |(uf)(x) - (uf_t)(x)| \sigma(x)^{l(1-2/q)} \Omega(x)^{1-2/q}(x) \Xi(x)^{-2/q} \\ &\leq \sum_i c_i (1 + \sigma(x))^m |(u_i f)(x)| \sigma(x)^{l(1-2/q)} \Omega(x)^{1-2/q}(x) \Xi(x)^{-2/q} \\ &\leq \sum_i c_i (1 + t - a)^{-1} v^q_{u_i,m+1}(f)\,,\end{aligned}$$

where

$$c_i = \int_G |(u_i'\alpha)(y)| dy\,.$$

This shows that $v^q_{u,m}(f - f_t) \to 0$ as $t \to \infty$ and f_t converges to f in $\mathscr{J}^q(G)$. Thus $C_c^\infty(K \backslash G/K)$ is dense in $\mathscr{J}^p(G)$.

THEOREM 2. *Let* $q > 2$ *and* $\dfrac{1}{p} + \dfrac{1}{q} = 1$. *If* $\dfrac{1}{p} - \dfrac{1}{q} < \dfrac{1}{r} \leq \dfrac{1}{p}$, *then*

$$\mathscr{J}^q(G) \subset I^r(G)\,,$$

and the inclusion map is continuous.

PROOF: Let $f \in \mathscr{J}^q(G)$ and assume that $m > \dfrac{2d}{q} + l\left(\dfrac{2}{q} - 1\right) + \dfrac{l}{r}$. Let c_1 and c_2 be the constants in Lemma 5. First we have

$$|f(x)| \leq c_3 (1 + \sigma(x))^{-m} \sigma(x)^{-l(1-2/q)} \Omega(x)^{-1+2/q} \Xi(x)^{2/q}$$

for all $x \in G$, where $c_3 = v^q_{1,m}(f)$. Then

$$\int_G |f(x)|^r dx$$

$$\leq c_3^r \int_G (1+\sigma(x))^{-rm}\sigma(x)^{-rl(1-2/q)}\Omega(x)^{-r(1-2/q)}\Xi(x)^{2r/q}dx$$

$$= c_3^r|W| \int_{A^+} (1+\sigma(a))^{-rm}\sigma(a)^{-rl(1-2/q)}\Omega(a)^{1-r(1-2/q)}\Xi(a)^{2r/q}da$$

$$\leq c_1^{1-r(1-2/q)}c_2^{2r/q}c_3^r|W| \int_{A^+} (1+\sigma(a))^{-rm+2rd/q}\sigma(a)^{rl(2/q-1)}e^{(2r/q-2r+2)\rho(\log a)}da$$

$$\leq c_1^{1-r(1-2/q)}c_2^{2r/q}c_3^r|W| \int_{A^+} (1+\sigma(a))^{-rm+2rd/q}\sigma(a)^{rl(2/q-1)}da$$

$$\leq c_4 \int_0^\infty (1+t)^{-rm+2rd/q}t^{rl(2/q-1)+l-1}dt < \infty ,$$

where

$$c_4 = c_1^{1-r(1-2/q)}c_2^{2r/q}c_3^r 2\pi^{l/2}\Gamma(l/2)^{-1} .$$

If we put

$$c_5 = \{c_1^{1-r(1-2/q)}c_2^{2r/q}2\pi^{l/2}\Gamma(l/2)^{-1}$$
$$\times \text{ the value of the integral in the last term}\}^{1/r} ,$$

we have $\|f\|_r \leq c_5 \nu_{1,m}^q(f)$.

If we choose $r=2$, then we obtain the following corollary.

CROROLLARY. *If $2 \leq q < 4$, then $\mathscr{I}^q(G) \subset I^2(G)$ and the inclusion map is continuous.*

The condition $\frac{1}{p} - \frac{1}{q} < \frac{1}{r}$ in Theorem 2 is necessary for the regularity of the function $\Omega(a)^{r(2/q-1)+1}$ on the walls of the Weyl chamber A^+ except for the origin. Hence if the rank $l=1$, Theorem 2 holds for $r \geq p$ and Corollary holds for $q \geq 2$. In fact, we have the following proposition.

PROPOSITION. *We assume that the rank l of G/K is one. Let $q \geq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then we have $\mathscr{I}^q(G) \subset I^r(G)$ for all $r \geq p$ and, especially, $\mathscr{I}^q(G) \subset I^2(G)$.*

PROOF: Suppose that $\frac{1}{p} - \frac{1}{q} \geq \frac{1}{r}$. Let $G_t = \{x \in G; \sigma(x) < t\}$ be defined as in the proof of Lemma 6. We denote by m_t the supremum of the absolute value of $f \in \mathscr{J}^q(G)$ on G_t. The function $\Omega(a)$ takes the minimal value at $\sigma(a) = 1$ in $(G \backslash G_1) \cap A_+$. Hence

$$\begin{aligned}
\|f\|_r^r &\leq m_1^r \operatorname{vol}(G_1) + \{v_{1,m}^q(f)\}^r \int_{G_1} (1 + \sigma(x))^{-mr} \Omega(x)^{r(2/q-1)} \Xi(x)^{2r/q} dx \\
&\leq m_1^r \operatorname{vol}(G_1) + c' \int_{(G \backslash G_1) \cap A_+} (1 + \sigma(a))^{-mr} \Omega(a)^{1+r(2/q-1)} \Xi(a)^{2r/q} da \\
&\leq m_1^r \operatorname{vol}(G_1) + c'' \int_1^\infty (1 + t)^{-rm+rd} dt < \infty
\end{aligned}$$

for $m > d + 1/r$.

6. Fourier transforms of $\mathscr{J}^q(G)$

LEMMA 7. *We assume that $q \geq 2$. Let φ be a measurable function on G such that there exist a constant $C > 0$ and an integer $m \geq 0$ satisfying*

$$|\varphi(x)| \leq C \Xi(x)^{2/q} (1 + \sigma(x))^m \qquad (x \in G). \tag{4.1}$$

Then

$$L(f) = \int_G (uf)(x) \varphi(x) dx$$

converges absolutely for all $f \in \mathscr{J}^q(G)$ and $u \in U(\mathfrak{g}_C)$, and L is a continuous linear functional on $\mathscr{J}^q(G)$. If φ and $u^*\varphi$ satisfy an inequality of the same type as (4.1), *then*

$$\int_G uf \cdot \varphi dx = \int_G f \cdot u^* \varphi dx,$$

where u^* is the adjoint differential operator of u.

PROOF: By the inequality

$$|(uf)(x)| \leq v_{u,n}^q(f)(1 + \sigma(x))^{-n} \sigma(x)^{-l(1-2/q)} \Omega(x)^{-1+2/q} \Xi(x)^{2/q}$$

$(x \in G)$,

$$\int_G |(uf)(x)||\varphi(x)|dx \leq c_1 \int_G (1+\sigma(x))^{m-n}\sigma(x)^{-l(1-2/q)}\Omega(x)^{-1+2/q}\Xi(x)^{4/q}dx$$

$$\leq c_2 \int_{A^+} (1+\sigma(a))^{m+4d/q-n}\sigma(a)^{l(2/q-1)}da$$

$$= c_3 \int_0^\infty (1+t)^{m+4d/q-n}t^{2l/q-1}da < \infty$$

for $n > m + \frac{4}{q}d + 2lq$.

The second part of Lemma 7 is already clear.

Let $Z_K(U(\mathfrak{g}_C))$ the centralizer of K in $U(\mathfrak{g}_C)$. For any $u \in U(\mathfrak{g}_C)$ we can find a unique element $a_u \in U(\mathfrak{a}_C)$ such that $u - a_u \in \mathfrak{k}U(\mathfrak{g}_C) + U(\mathfrak{g}_C)\mathfrak{n}$. For any $z \in Z_K(U(\mathfrak{g}_C))$, we put $\tau(z) = e^\rho \circ a_z \circ e^{-\rho}$. Then $\tau(z) \in U(\mathfrak{a}_C)$. The following lemma is due to Trombi–Varadarajan [7, Lemma 3.5.3].

LEMMA 8. *Let $s \in S(\mathfrak{a}_C^*)$ and $d_s = \deg(s)$. Then, if $z \in Z_K(U(\mathfrak{g}_C))$, $(z - \tau(z)(\lambda))^{d_s+1}\partial(s)(\varphi_\lambda(x)) = 0$ for all $\lambda \in \mathfrak{a}_C^*$ and $x \in G$. Furthermore, given $u \in U(\mathfrak{g}_C)$, there exist constants $c_{u,s} > 0$ and $m_{u,s} \geq 0$ such that for all $x \in G$, $\lambda \in T_p$,*

$$|\partial(s)u\varphi_\lambda(x)| \leq c_{u,s}\{(1+|\lambda|)(1+\sigma(x))\}^{m_{u,s}}\Xi(x)^{2(1-1/p)}.$$

THEOREM 3. *Let $1 < p < 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in \mathscr{I}^q(G)$, then the integral*

$$\tilde{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx$$

converges absolutely for any $\lambda \in T_p$. Moreover, the map $f \mapsto \tilde{f}$ is continuous from $\mathscr{I}^q(G)$ to $\overline{\mathscr{Z}}(T_p)$.

PROOF: The first part follows from Lemma 7 and Lemma 8. Let $\lambda \in \operatorname{Int} T_p$ and $f \in \mathscr{I}^q(G)$. Then for $s \in S(\mathfrak{a}_C^*)$

$$\int_G |f(x)\partial(s)\varphi_{-\lambda}(x)|dx \leq c_1 \int_G |f(x)|(1+|\lambda|)^m(1+\sigma(x))^m\Xi(x)^{2(1-1/p)}dx$$

$$\leq c_2(1+|\lambda|)^m\nu_{1,s}(f).$$

We can prove the latter part in the same way as in Trombi-Varadarajan [7] Theorem 3.5.5. For λ, f as above, we have, for any $z \in Z_K(U(\mathfrak{g}_C))$,

$$\int_G (((z^* - \tau(z)(-\lambda))^{d_s+1})f)(x)\partial(s^*)(\varphi_{-\lambda}(x))dx = 0\,.$$

Then,

$$|\tau(z)(-\lambda)|^{d_s+1}\,|(\partial(s)\tilde{f})(\lambda)| \leq 2^{d_s+1}(1+|\lambda|)^{m_s} \sum_{1\leq i\leq d_s+1} |\tau(z)(-\lambda)|^{d_s+1-i}\mu_{1,s}(z^{*i}f)\,.$$

Since $U(\mathfrak{a}_C)$ is a finite module over $\tau(Z_K(U(\mathfrak{g}_C))$, we have the following. Given $s \in S(\mathfrak{a}_C^*)$, there exists $m_s \geq 0$, and for each $v \in U(\mathfrak{a}_C)$, a continuous seminorm $v_{v,s}$ on $\mathscr{I}^q(G)$ such that

$$|v(\lambda)|\,|(\partial(s)\tilde{f})(\lambda)| \leq (1+|\lambda|)^{m_s} v_{v,s}(f)$$

for all $\mathscr{I}^q(G)$, $\lambda \in \operatorname{Int} T_p$. Since m_s does not depend on v, $\tilde{f} \in \overline{\mathscr{Z}}(T_p)$ and the map $f \mapsto \tilde{f}$ is continuous.

7. Coincidence theorem

THEOREM 4. *If* $2 \leq q < 4$ *and* $\dfrac{1}{p} + \dfrac{1}{q} = 1$, *then* $\mathscr{I}^q(G) = \mathscr{I}^p(G)$. *If the rank of* G/K *is one, then* $\mathscr{I}^q(G) = \mathscr{I}^p(G)$ *for all* $q \geq 2$.

PROOF: By Theorem 3 and Proposition, if $f \in \mathscr{I}^q(G)$, then $\tilde{f} \in \overline{\mathscr{Z}}(T_p)$. There exists a function $f_1 \in \mathscr{I}^p(G)$ such that $\tilde{f}_1 = \tilde{f}$ by Lemma 3. Then by the Corollary of Theorem 2 and Lemma 1 $f_1(x) = f(x)$ for almost all x. Since f_1, $f_1 \in C^\infty(K\backslash G/K)$, $f = f_1$. Thus we have $f \in \mathscr{I}^p(G)$.

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