HIROSHIMA MATH. J.
30 (2000), 167–178

L^p boundedness of rough Marcinkiewicz integral on product torus

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(Received January 5, 1999)

ABSTRACT. This paper is a continuation of our study [D] [CDF] on rough Marcinkiewicz integral operator on product space. Suppose that $\Omega(x', y') \in L^q(S^{n-1} \times S^{m-1})$ $(n \geq 2, m \geq 2, q \geq 1)$ is homogeneous of degree zero satisfying the mean zero properties (1.1)–(1.3). For C^∞ functions $\tilde{f}$ on the product torus $\mathbf{T}^n \times \mathbf{T}^m$, the Marcinkiewicz integral operator on $\mathbf{T}^n \times \mathbf{T}^m$ is defined by

$$\tilde{\mu}_\Omega \tilde{f}(x, y) = \left(\int_{\mathbf{R}}\int_{\mathbf{R}} |\tilde{\Phi}_{t,s} * \tilde{f}(x, y)|^2 dtds\right)^{1/2},$$

where $\tilde{\Phi}_{t,s}$ has the Fourier series

$$\tilde{\Phi}_{t,s}(x, y) \sim \sum_{k_1, k_2} \hat{\Phi}(2^t k_1, 2^s k_2) e^{2\pi i k_1 \cdot x} e^{2\pi i k_2 \cdot y}.$$

In this paper we show that if $q > 1$ then the operator $\tilde{\mu}_\Omega$ can be extended to a bounded operator on $L^p(\mathbf{T}^n \times \mathbf{T}^m)$ for $1 < p < \infty$.

§ 1. Introduction and results

Let $\mathbf{R}^n$ be n-dimensional Euclidean space and S^{n-1} be the unit sphere in $\mathbf{R}^n$ $(n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$, where $x' = x/|x|$ for $x \neq 0$. In [S], Stein introducted the Marcinkiewicz integral operator μ_Ω of higher dimension as follows.

$$\mu_\Omega f(x) = \left(\int_0^\infty |F_t(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t(x) = \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy,$$

$\Omega \in L^1(S^{n-1})$ is homogeneous of degree zero satisfying $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$.

2000 *Mathematics Subject Classification.* 42B20, 42B99.
Key words and phrases. Marcinkiewicz integral, rough kernel, product torus.
The first author was supported by NNSF of China (Grant No. 19971010)

In [S], Stein proved that if Ω is continuous and satisfies a Lip_α $(0 < \alpha \le 1)$ condition on S^{n-1}, then μ_Ω is of type (p, p) for $1 < p \le 2$ and of weak type $(1, 1)$. It was pointed out in our previous paper [CDF] that to assert the L^p boundedness of μ_Ω for $1 < p < \infty$, the smoothness condition assumed on Ω can be replaced by a weaker size condition $\Omega \in L^q$ (S^{n-1}) $(q > 1)$. In [CDF], we considered the Marcinkiewicz integral operator on product space $\mathbf{R}^n \times \mathbf{R}^m$ by

$$\mu_\Omega f(x, y) = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F_{t,s}(x, y)|^2 \frac{dtds}{2^{2t} 2^{2s}} \right)^{1/2},$$

where

$$F_{t,s}(x, y) = \iint_{\substack{|x-u| \le 2^t \\ |y-v| \le 2^s}} \frac{\Omega(x-u, y-v)}{|x-u|^{n-1} |y-v|^{m-1}} f(u, v) du dv,$$

$\Omega \in L^q$ $(S^{n-1} \times S^{m-1})$ $(n \ge 2, m \ge 2, q \ge 1)$ satisfying the following conditions:

$$\Omega(tx, sy) = \Omega(x, y) \qquad \text{for any } t, s > 0, \tag{1.1}$$

$$\int_{S^{n-1}} \Omega(x', y') d\sigma(x') = 0 \qquad \text{for any } y' \in S^{m-1}, \tag{1.2}$$

$$\int_{S^{m-1}} \Omega(x', y') d\sigma(y') = 0 \qquad \text{for any } x' \in S^{n-1}. \tag{1.3}$$

The following theorem can be found in [CDF].

THEOREM A. *Suppose that $\Omega \in L^q$ $(S^{n-1} \times S^{m-1})$ $(q > 1)$ satisfying* (1.1)–(1.3). *Then for $1 < p < \infty$, there is an $A_p > 0$, independent of f, such that*

$$\|\mu_\Omega f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)} \le A_p \|f\|_{L^p(\mathbf{R}^n \times \mathbf{R}^m)}.$$

Let $\Phi_{t,s}(x, y) = 2^{-nt} 2^{-ms} \Phi\left(\frac{x}{2^t}, \frac{y}{2^s}\right)$ with

$$\Phi(x, y) = |x|^{-n+1} |y|^{-m+1} \Omega(x', y') \chi_B(|x|) \chi_B(|y|),$$

where $\chi_B(z)$ is the characteristic function of the set $\{z : |z| < 1\}$. It is easy to see that

$$\mu_\Omega f(x, y) = \left(\int_{\mathbf{R}} \int_{\mathbf{R}} |\Phi_{t,s} * f(x, y)|^2 dtds \right)^{1/2}.$$

This suggests that we can define the Marcinkiewicz integral operator on product torus $\mathbf{T}^n \times \mathbf{T}^m$ by

$$\tilde{\mu}_\Omega \tilde{f}(x, y) = \left(\int_{\mathbf{R}} \int_{\mathbf{R}} |\tilde{\Phi}_{t,s} * \tilde{f}(x, y)|^2 dtds \right)^{1/2},$$

initially for $\tilde{f} \in C^\infty$ $(\mathbf{T}^n \times \mathbf{T}^m)$, where $\tilde{\Phi}_{t,s}$ has the Fourier series

$$\tilde{\Phi}_{t,s}(x, y) = \sum_{k_1, k_2} \hat{\Phi}(2^t k_1, 2^s k_2) e^{2\pi i k_1 \cdot x} e^{2\pi i k_2 \cdot y}.$$

Let us describe our definition more precisely in the following. For $N = n$ or m, the N-torus $\mathbf{T}^N$ can be identified with $\mathbf{R}^N / \Lambda_N$, where Λ_N is the unit lattice which is an additive group of points in $\mathbf{R}^N$ having integer coordinates. Let $\Lambda = \Lambda_n \times \Lambda_m$. Any $\tilde{f} \in C^\infty$ $(\mathbf{T}^n \times \mathbf{T}^m)$ has the Fourier series

$$\tilde{f}(x, y) = \sum_{(k_1, k_2) \in \Lambda} C_{k_1, k_2} e^{2\pi i k_1 \cdot x} e^{2\pi i k_2 \cdot y},$$

where

$$C_{k_1, k_2} = \iint_{Q_n \times Q_m} \tilde{f}(x, y) e^{-2\pi i k_1 \cdot x} e^{-2\pi i k_2 \cdot y} dx dy$$

and Q_N $(N = n, m)$ is the fundamental cube of $\mathbf{T}^N$ which is the set

$$Q_N = \{x = (x_1, x_2, \dots, x_N) \in \mathbf{R}^N : -1/2 \le x_j < 1/2, \;\; j = 1, 2, \dots, N\}.$$

Therefore noting $\hat{\Phi}(0, \eta) = \hat{\Phi}(\xi, 0) = 0$ for any η, ξ, we have

$$\tilde{\Phi}_{t,s} * \tilde{f}(x, y) = \sum_{\substack{(k_1, k_2) \in \Lambda \\ k_1 \neq 0, k_2 \neq 0}} \hat{\Phi}(2^t k_1, 2^s k_2) C_{k_1, k_2} e^{2\pi i k_1 \cdot x} e^{2\pi i k_2 \cdot y}.$$

The main purpose of this paper is to establish the following

THEOREM 1. *Suppose that $\Omega \in L^q$ $(S^{n-1} \times S^{m-1})$ $(q > 1)$ satisfying* (1.1)–(1.3). *Then for $1 < p < \infty$, there is a $B_p > 0$, independent of f, such that $B_p \le A_p$ and*

$$\|\tilde{\mu}_\Omega \tilde{f}\|_{L^p(\mathbf{T}^n \times \mathbf{T}^m)} \le B_p \|\tilde{f}\|_{L^p(\mathbf{T}^n \times \mathbf{T}^m)},$$

where A_p is the constant in Theorem A.

§ 2. Proof of Theorem 1

The proof of Theorem 1 will use some ideas in [F]. Let δ_ε be the dilation operator such that $\delta_\varepsilon f(x) = f(\varepsilon x)$. For any fixed integer $L > 0$, we choose a function $\psi \in \mathscr{S}(\mathbf{R}^n)$ that satisfies $\psi(x) \equiv 1$ on Q_n and

$$\operatorname{supp} \psi \subset \{x \in \mathbf{R}^n : -1/2 - 1/L < x_j \le 1/2 + 1/L, \;\; j = 1, 2, \dots, n\}.$$

We also choose a function $\Gamma \in \mathscr{S}(\mathbf{R}^m)$ that satisfies $\Gamma(y) \equiv 1$ on Q_m and

$$\operatorname{supp} \Gamma \subset \{y \in \mathbf{R}^m : -1/2 - 1/L < y_j \le 1/2 + 1/L, \;\; j = 1, 2, \dots, m\}.$$

In addition, we require $0 \le \psi \le 1$, $0 \le \Gamma \le 1$. For any $\tilde{f} \in C^\infty$ $(\mathbf{T}^n \times \mathbf{T}^m)$, without loss of generality, we may assume that $\tilde{f}$ has the Fourier series

$$\tilde{f}(x, y) = \sum_{\substack{(k_1, k_2) \in \Lambda \\ k_1 \neq 0, k_2 \neq 0}} C_{k_1, k_2} e^{2\pi i k_1 \cdot x} e^{2\pi i k_2 \cdot y}.$$

So we can view $\tilde{f}$ as a periodic function on $\mathbf{R}^n \times \mathbf{R}^m$. Let M be an integer larger than L. We consider the difference

$$E_M(x, y, t, s) = \psi\left(\frac{x}{M}\right)\Gamma\left(\frac{y}{M}\right)\tilde{\Phi}_{t,s} * \tilde{f}(x, y) - \Phi_{t,s} * (\tilde{f}(\delta_{1/M}\psi) \otimes (\delta_{1/M}\Gamma))(x, y).$$

We need the following lemma.

LEMMA 1. *Under the conditions of Theorem* 1, *with the choices of* ψ *and* Γ, *we have*

$$\lim_{M \to \infty}\left(\int_{\mathbf{R}}\int_{\mathbf{R}} |E_M(x, y, t, s)|^2 dt ds\right)^{1/2} = 0$$

uniformly for $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$.

Using Lemma 1, we may prove Theorem 1. In fact, since $\tilde{\mu}_\Omega \tilde{f}$ is a periodic function, for any integer $M > 0$,

$$\text{(2.1)} \qquad \|\tilde{\mu}_\Omega \tilde{f}\|_{L^p(\mathbf{T}^n \times \mathbf{T}^m)} = \left(\int_{Q_n}\int_{Q_m} |\tilde{\mu}_\Omega \tilde{f}(x, y)|^p dx dy\right)^{1/p}$$

$$= \left(M^{-(n+m)}\int_{MQ_n}\int_{MQ_m} |\tilde{\mu}_\Omega \tilde{f}(x, y)|^p dx dy\right)^{1/p}.$$

Noting $\psi\left(\frac{x}{M}\right) \equiv 1$ on MQ_n and $\Gamma\left(\frac{y}{M}\right) \equiv 1$ on MQ_m, by (2.1) we have

$$\|\tilde{\mu}_\Omega \tilde{f}\|_{L^p(\mathbf{T}^n \times \mathbf{T}^m)}$$

$$= \left(M^{-(n+m)}\int_{MQ_n}\int_{MQ_m} \left|\psi\left(\frac{x}{M}\right)\Gamma\left(\frac{y}{M}\right)\tilde{\mu}_\Omega \tilde{f}(x, y)\right|^p dx dy\right)^{1/p}$$

$$= \left(M^{-(n+m)}\int_{MQ_n}\int_{MQ_m}\left(\int_{\mathbf{R}}\int_{\mathbf{R}}\left|\psi\left(\frac{x}{M}\right)\Gamma\left(\frac{y}{M}\right)\tilde{\Phi}_{t,s} * \tilde{f}(x, y)\right|^2 dt ds\right)^{p/2} dx dy\right)^{1/p}.$$

From this and Lemma 1, we get

(2.2) $\|\tilde{\mu}_\Omega \tilde{f}\|_{L^p(\mathbf{T}^n \times \mathbf{T}^m)}$

$$\leq \lim_{M\to\infty} \left(M^{-(n+m)} \int_{MQ_n} \int_{MQ_m} \left(\int_{\mathbf{R}} \int_{\mathbf{R}} |E_M(x, y, t, s)|^2 dtds \right)^{p/2} dxdy \right)^{1/p}$$

$$+ \lim_{M\to\infty} \left(M^{-(n+m)} \int_{MQ_n} \int_{MQ_m} \left(\int_{\mathbf{R}} \int_{\mathbf{R}} |\Phi_{t,s} * [\tilde{f}(\delta_{1/M}\psi) \otimes (\delta_{1/M}\Gamma)](x, y)|^2 dtds \right)^{p/2} dxdy \right)^{1/p}$$

$$\leq \lim_{M\to\infty} \left(M^{-(n+m)} \int_{\mathbf{R}^n} \int_{\mathbf{R}^m} \left(\int_{\mathbf{R}} \int_{\mathbf{R}} |\Phi_{t,s} * [\tilde{f}(\delta_{1/M}\psi) \otimes (\delta_{1/M}\Gamma)](x, y)|^2 dtds \right)^{p/2} dxdy \right)^{1/p}.$$

Let $G(x, y) = \tilde{f}(x, y)\psi\left(\frac{x}{M}\right)\Gamma\left(\frac{y}{M}\right)$. Then the last integral in (2.2) is

$$\left(M^{-(n+m)} \int_{\mathbf{R}^n} \int_{\mathbf{R}^m} |\mu_\Omega G(x, y)|^p dxdy \right)^{1/p}.$$

By Theorem A, we have

$$\left(M^{-(n+m)} \int_{\mathbf{R}^n} \int_{\mathbf{R}^m} |\mu_\Omega G(x, y)|^p dxdy \right)^{1/p}$$

$$\leq A_p M^{-(np+mp)} \left(\int_{\mathbf{R}^n} \int_{\mathbf{R}^m} |G(x, y)|^p dxdy \right)^{1/p}$$

$$= A_p M^{-(np+mp)} \left(\int_{\mathbf{R}^n} \int_{\mathbf{R}^m} \left| \tilde{f}(x, y)\psi\left(\frac{x}{M}\right)\Gamma\left(\frac{y}{M}\right) \right|^p dxdy \right)^{1/p}.$$

By the choices of ψ and Γ we have

$$\left(M^{-(n+m)} \int_{\mathbf{R}^n} \int_{\mathbf{R}^m} |\mu_\Omega G(x, y)|^p dxdy \right)^{1/p} \tag{2.3}$$

$$\leq A_p \left(M^{-(n+m)} \int_{\Delta_n} \int_{\Delta_m} |\tilde{f}(x, y)|^p dxdy \right)^{1/p},$$

where

$$\Delta_n = \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n : -\frac{M}{2} - \frac{M}{L} < x_j \le \frac{M}{2} + \frac{M}{L},\ j = 1, 2, \ldots, n \right\},$$

$$\Delta_m = \left\{ y = (y_1, y_2, \ldots, y_m) \in \mathbf{R}^m : -\frac{M}{2} - \frac{M}{L} < y_j \le \frac{M}{2} + \frac{M}{L},\ j = 1, 2, \ldots, m \right\}.$$

Therefore if $M > L$, since $\tilde{f}(x, y)$ is a periodic function satisfying

$$\tilde{f}(x+1, y) = \tilde{f}(x, y+1) = \tilde{f}(x, y) \qquad \text{for any } (x, y) \in \mathbf{R}^n \times \mathbf{R}^m,$$

by (2.3) we have

$$\begin{aligned} (2.4)\qquad & \left(M^{-(n+m)} \int_{\mathbf{R}^n} \int_{\mathbf{R}^m} |\mu_\Omega G(x, y)|^p dxdy \right)^{1/p} \\ & \le A_p \left(M^{-(n+m)} \left[M + \frac{2M}{L} \right]^{n+m} \int_{Q_n} \int_{Q_m} |\tilde{f}(x, y)|^p dxdy \right)^{1/p} \\ & = A_p \left(\left[1 + \frac{2}{L} \right]^{n+m} \int_{Q_n} \int_{Q_m} |\tilde{f}(x, y)|^p dxdy \right)^{1/p}. \end{aligned}$$

Thus by (2.2)–(2.4) we obtain

$$\|\tilde{\mu}_\Omega \tilde{f}\|_{L^p(\mathbf{T}^n \times \mathbf{T}^m)} \le A_p \left[1 + \frac{2}{L} \right]^{(n+m)/p} \|\tilde{f}\|_{L^p(\mathbf{T}^n \times \mathbf{T}^m)}.$$

Since $L > 0$ is arbitrary, we have

$$\|\tilde{\mu}_\Omega \tilde{f}\|_{L^p(\mathbf{T}^n \times \mathbf{T}^m)} \le A_p \|\tilde{f}\|_{L^p(\mathbf{T}^n \times \mathbf{T}^m)}.$$

The proof of Theorem 1 is complete.

Thus the proof of Theorem 1 is reduced to proving Lemma 1. However, the proof of Lemma 1 will depend strongly on the folowing Lemma 2.

LEMMA 2. *Suppose that $\Omega \in L^q\ (S^{n-1} \times S^{m-1})\ (q > 1)$ satisfying* (1.1)–(1.3), *then there are $\delta, \alpha, \beta, \alpha', \beta' > 0$ and constants $C_1, C_2 > 0$, independent of $|\xi|, |\eta|$ and γ, such that*
(i) $|\hat{\Phi}(\xi, \eta)|^2 \le C_1 \min\{|\xi|^{1/2}|\eta|^{1/2}, |\xi|^{-\delta}|\eta|^{-\delta}, |\xi|^{\alpha}|\eta|^{-\beta}, |\xi|^{-\beta}|\eta|^{\alpha}\}$;
(ii) $|\hat{\Phi}(\gamma + \xi, \eta) - \hat{\Phi}(\gamma, \eta)| \le C_2 |\xi| \min\{|\eta|^{\alpha'}, |\eta|^{-\beta'}\}$.

PROOF. The conclusion (i) is just Lemma 2.2 in [CDF]. Below we only give the proof of (ii). Denote $I = |\hat{\Phi}(\gamma + \xi, \eta) - \hat{\Phi}(\gamma, \eta)|$. Recalling that

$$\Phi(x, y) = |x|^{-n+1} |y|^{-m+1} \Omega(x', y') \chi_B(|x|) \chi_B(|y|),$$

we have

$$I = \left| \int_{|x|\le 1} \int_{|y|\le 1} \frac{\Omega(x', y')}{|x|^{n-1}|y|^{m-1}} e^{-2\pi i(\gamma\cdot x+\eta\cdot y)}[e^{-2\pi i\xi\cdot x} - 1]dxdy \right|.$$

By (1.2) we have

$$(2.5) \qquad I = \left| \iint_{S^{n-1}\times S^{m-1}} \Omega(x', y') \int_0^1 \int_0^1 [e^{-2\pi i h\eta\cdot y'} - 1] \right.$$

$$\left. \times\, e^{-2\pi i\gamma r\cdot x'}[e^{-2\pi i r\xi\cdot x'} - 1]drdhd\sigma(x')d\sigma(y') \right| = O(|\xi|\,|\eta|).$$

Let $S(r, x', \xi, \gamma) = e^{-2\pi i\gamma r\cdot x'}[e^{-2\pi i r\xi\cdot x'} - 1]$, then $|S(r, x', \xi, \gamma)| \le C|r\xi|$. On the other hand, we have

$$I^2 \le \int_0^1 \int_0^1 \left| \iint_{S^{n-1}\times S^{m-1}} \Omega(x', y')e^{-2\pi i h\eta\cdot y'} S(r, x', \xi, \gamma)d\sigma(x')d\sigma(y') \right|^2 drdh$$

$$= \int_0^1 \int_0^1 \iint_{(S^{n-1}\times S^{m-1})^2} \Omega(x', y')\overline{\Omega(u', v')}e^{-2\pi i h\eta\cdot(y'-v')}$$

$$\times\, S(r, x', \xi, \gamma)S_1(r, u', \xi, \gamma)d\sigma(x')d\sigma(y')d\sigma(u')d\sigma(v')drdh$$

$$= \iint_{(S^{n-1}\times S^{m-1})^2} \Omega(x', y')\overline{\Omega(u', v')} \int_0^1 \int_0^1 e^{-2\pi i h\eta\cdot(y'-v')}$$

$$\times\, S(r, x', \xi, \gamma)S_1(r, u', \xi, \gamma)drdhd\sigma(x')d\sigma(y')d\sigma(u')d\sigma(v'),$$

where $S_1(r, u', \xi, \gamma) = e^{-2\pi i\gamma r\cdot(-u')}[e^{2\pi i r\xi\cdot u'} - 1]$. Clearly we have

$$(2.6) \qquad \left| \int_0^1 \int_0^1 e^{-2\pi i h\eta\cdot(y'-v')} S(r, x', \xi, \gamma)S_1(r, u', \xi, \gamma)drdh \right| \le C|\xi|^2.$$

On the other hand,

$$(2.7) \qquad \left| \int_0^1 \int_0^1 e^{-2\pi i h\eta\cdot(y'-v')} S(r, x', \xi, \gamma)S_1(r, u', \xi, \gamma)drdh \right|$$

$$\le \int_0^1 \left| \int_0^1 e^{-2\pi i h\eta\cdot(y'-v')}dh \right| |S(r, x', \xi, \gamma)S_1(r, u', \xi, \gamma)|dr$$

$$\le C|\xi|^2|\eta\cdot(y' - v')|^{-1}.$$

By (2.6) and (2.7), we may take an $\varepsilon > 0$ satisfying $\varepsilon < 1/q'$ such that

$$(2.8) \qquad \left| \int_0^1 \int_0^1 e^{-2\pi i h\eta\cdot(y'-v')} S(r, x', \xi, \gamma)S_1(r, u', \xi, \gamma)drdh \right| \le C|\xi|^2|\eta\cdot(y' - v')|^{-\varepsilon}.$$

By [DR] and using (2.8), we have

$$I^2 \le C|\xi|^2 \iint_{(S^{n-1}\times S^{m-1})^2} |\Omega(x',y')\overline{\Omega(u',v')}|$$
$$\times |\eta\cdot(y'-v')|^{-\varepsilon} d\sigma(x')d\sigma(y')d\sigma(u')d\sigma(v')$$
$$\le C|\xi|^2|\eta|^{-\varepsilon}\|\Omega\|^2_{L^q(S^{n-1}\times S^{m-1})}\left(\iint_{(S^{m-1}\times S^{m-1})^2}|y'-v'|^{-q'\varepsilon}d\sigma(y')d\sigma(v')\right)^{1/q'}$$
$$\le C|\xi|^2|\eta|^{-\varepsilon}.$$

Thus we get

$$|\hat{\Phi}(\gamma+\xi,\eta)-\hat{\Phi}(\gamma,\eta)| \le C|\xi|\,|\eta|^{-\varepsilon/2}. \tag{2.9}$$

Finally by (2.5) and (2.9) we complete indeed the proof of Lemma 2 (ii) if taking $\alpha'=1$ and $\beta'=\varepsilon/2$.

Now let us turn to the proof of Lemma 1. Denote the Fourier transform of E_M on (x,y)-variable by $\hat{E}_M$, then

$$E_M(x,y,t,s)=\iint_{\mathbf{R}^n\times\mathbf{R}^m}\hat{E}_M(\xi,\eta,t,s)e^{2\pi ix\cdot\xi}e^{2\pi iy\cdot\eta}d\xi d\eta.$$

Recall

$$\tilde{f}(x,y)=\sum_{\substack{(k_1,k_2)\in\Lambda\\ k_1\neq 0,k_2\neq 0}} C_{k_1,k_2}e^{2\pi ik_1\cdot x}e^{2\pi ik_2\cdot y}$$

with rapidly decay cofficients C_{k_1,k_2}. If we denote

$$H_{k_1,k_2,M}(x,y,t,s)=\hat{\Phi}(2^tk_1,2^sk_2)\iint_{\mathbf{R}^n\times\mathbf{R}^m}M^{n+m}\hat{\psi}(M\xi)\hat{\Gamma}(M\eta)e^{2\pi ix\cdot\xi}e^{2\pi iy\cdot\eta}d\xi d\eta,$$

$$J_{k_1,k_2,M}(x,y,t,s)=\iint_{\mathbf{R}^n\times\mathbf{R}^m}M^{n+m}\hat{\psi}(M\xi)\hat{\Gamma}(M\eta)$$
$$\times e^{2\pi ix\cdot\xi}e^{2\pi iy\cdot\eta}\hat{\Phi}(2^tk_1+2^t\xi,2^sk_2+2^s\eta)d\xi d\eta$$

and

$$P_{k_1,k_2,M}(t,s)=\iint_{\mathbf{R}^n\times\mathbf{R}^m}|\hat{\psi}(\xi)|\,|\hat{\Gamma}(\eta)|$$
$$\times\left|\hat{\Phi}\left(2^tk_1+\frac{2^t\xi}{M},2^sk_2+\frac{2^s\eta}{M}\right)-\hat{\Phi}(2^tk_1,2^sk_2)\right|d\xi d\eta,$$

then we have

$$\begin{aligned}
&|E_M(x,y,t,s)| \\
&= \left| \sum_{\substack{(k_1,k_2)\in\Lambda \\ k_1\neq 0, k_2\neq 0}} C_{k_1,k_2} e^{2\pi i k_1\cdot x} e^{2\pi i k_2\cdot y} [H_{k_1,k_2,M}(x,y,t,s) - J_{k_1,k_2,M}(x,y,t,s)] \right| \\
&\leq \sum_{\substack{(k_1,k_2)\in\Lambda \\ k_1\neq 0, k_2\neq 0}} |C_{k_1,k_2}| P_{k_1,k_2,M}(t,s).
\end{aligned}$$

Thus

$$\begin{aligned}
&\left(\int_{\mathbf{R}}\int_{\mathbf{R}} |E_M(x,y,t,s)|^2 dtds\right)^{1/2} \\
&\qquad \leq \sum_{\substack{(k_1,k_2)\in\Lambda \\ k_1\neq 0, k_2\neq 0}} |C_{k_1,k_2}| \left(\iint_{\mathbf{R}\times\mathbf{R}} P_{k_1,k_2,M}(t,s)^2 dtds\right)^{1/2}.
\end{aligned}$$

Since $\tilde{f} \in C^\infty$ $(\mathbf{T}^n \times \mathbf{T}^m)$, so for any $\varepsilon > 0$ there is a finite set $\Lambda^1 \subset \Lambda$ such that

$$\sum_{(k_1,k_2)\notin\Lambda^1} |C_{k_1,k_2}| < \varepsilon.$$

Write

$$\tag{2.10} \begin{aligned}
&{\sum}'(M) = \sum_{\substack{(k_1,k_2)\in\Lambda^1 \\ k_1\neq 0, k_2\neq 0}} |C_{k_1,k_2}| \left(\iint_{\mathbf{R}\times\mathbf{R}} P_{k_1,k_2,M}(t,s)^2 dtds\right)^{1/2}, \\
&{\sum}''(M) = \sum_{\substack{(k_1,k_2)\notin\Lambda^1 \\ k_1\neq 0, k_2\neq 0}} |C_{k_1,k_2}| \left(\iint_{\mathbf{R}\times\mathbf{R}} P_{k_1,k_2,M}(t,s)^2 dtds\right)^{1/2}.
\end{aligned}$$

Below we will estimate ${\sum}'(M)$ and ${\sum}''(M)$, respectively. Let us first consider ${\sum}''(M)$. By Hölder's inequality,

$$\tag{2.11} \begin{aligned}
{\sum}''(M) \leq &\sum_{(k_1,k_2)\notin\Lambda^1} 2|C_{k_1,k_2}| \left(\iint_{\mathbf{R}\times\mathbf{R}} \iint_{\mathbf{R}^n\times\mathbf{R}^m} |\hat{\psi}(\xi)\hat{\Gamma}(\eta)|^2 \right. \\
&\left. \times \left|\hat{\Phi}\left(2^t k_1 + \frac{2^t\xi}{M}, 2^s k_2 + \frac{2^s\eta}{M}\right)\right|^2 d\xi d\eta dtds\right)^{1/2} \\
&+ \sum_{(k_1,k_2)\notin\Lambda^1} 2|C_{k_1,k_2}| \\
&\times \left(\iint_{\mathbf{R}\times\mathbf{R}} \iint_{\mathbf{R}^n\times\mathbf{R}^m} |\hat{\psi}(\xi)\hat{\Gamma}(\eta)|^2 |\hat{\Phi}(2^t k_1, 2^s k_2)|^2 d\xi d\eta dtds\right)^{1/2}.
\end{aligned}$$

Note that there exists an $A > 0$ such that

$$\iint_{\mathbf{R}\times\mathbf{R}} |\hat{\Phi}(2^t\xi, 2^s\eta)|^2 dtds \le A \tag{2.12}$$

uniformly for $(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^m$. In fact, by Lemma 2 (i), it is easy to see that

$$\begin{aligned}
&\iint_{\mathbf{R}\times\mathbf{R}} |\hat{\Phi}(2^t\xi, 2^s\eta)|^2 dtds \\
&\le \int_{|2^t\xi|\le 1}\int_{|2^s\eta|\le 1} |2^t\xi|^{\alpha}|2^s\eta|^{\beta} dtds + \int_{|2^t\xi|\le 1}\int_{|2^s\eta|\ge 1} |2^t\xi|^{\alpha}|2^s\eta|^{-\beta} dtds \\
&\quad + \int_{|2^t\xi|\ge 1}\int_{|2^s\eta|\le 1} |2^t\xi|^{-\alpha}|2^s\eta|^{\beta} dtds + \int_{|2^t\xi|\ge 1}\int_{|2^s\eta|\ge 1} |2^t\xi|^{-\alpha}|2^s\eta|^{-\beta} dtds \le A.
\end{aligned}$$

Clearly $A > 0$ is independent of $(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^m$. With the choices of ψ and Γ, by (2.11), (2.12) and the Plancherel theorem we have

$$\sum{}''(M) \le A(L) \sum_{(k_1,k_2)\notin \Lambda^1} |C_{k_1,k_2}| \le \varepsilon A(L), \tag{2.13}$$

where $A(L)$ is independent of ε.

Finally, let us handle the term $\sum'(M)$. Since Λ^1 is finite, we need only to check

$$\lim_{M\to\infty} \iint_{\mathbf{R}\times\mathbf{R}} P_{k_1,k_2,M}(t,s)^2 dtds = 0$$

for any fixed $(k_1,k_2) \in \Lambda^1$ with $k_1 \ne 0, k_2 \ne 0$. Since $\hat{\psi} \in \mathscr{S}(\mathbf{R}^n), \hat{\Gamma} \in \mathscr{S}(\mathbf{R}^m)$, by (2.12) if we denote

$$\begin{aligned}
\mathscr{B}_M(t,s) = \iint_{B^n\times B^m} |\hat{\psi}(\xi)|\,|\hat{\Gamma}(\eta)| \Bigg| &\hat{\Phi}\left(2^t k_1 + \frac{2^t\xi}{M}, 2^s k_2 + \frac{2^s\eta}{M}\right) \\
&- \hat{\Phi}(2^t k_1, 2^s k_2)\Bigg| d\xi d\eta,
\end{aligned}$$

where B^n and B^m are bounded sets in $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, then it suffices to show

$$\lim_{M\to\infty} \iint_{\mathbf{R}\times\mathbf{R}} \mathscr{B}_M(t,s)^2 dtds = 0. \tag{2.14}$$

Set

$$I_M(t,s) = \iint_{B^n \times B^m} |\hat{\psi}(\xi)|\,|\hat{\Gamma}(\eta)| \times \left|\hat{\Phi}\left(2^t k_1 + \frac{2^t\xi}{M}, 2^s k_2 + \frac{2^s\eta}{M}\right) - \hat{\Phi}\left(2^t k_1, 2^s k_2 + \frac{2^s\eta}{M}\right)\right| d\xi d\eta$$

and

$$J_M(t,s) = \iint_{B^n \times B^m} |\hat{\psi}(\xi)|\,|\hat{\Gamma}(\eta)| \left|\hat{\Phi}\left(2^t k_1, 2^s k_2 + \frac{2^s\eta}{M}\right) - \hat{\Phi}(2^t k_1, 2^s k_2)\right| d\xi d\eta,$$

we have

$$\iint_{\mathbf{R}\times\mathbf{R}} \mathscr{B}_M(t,s)^2 dtds \le C\iint_{\mathbf{R}\times\mathbf{R}} I_M(t,s)^2 dtds + C\iint_{\mathbf{R}\times\mathbf{R}} J_M(t,s)^2 dtds$$
$$:= \mathscr{I}_M + \mathscr{J}_M.$$

Since the estimates of $\mathscr{I}_M$ and $\mathscr{J}_M$ are same, we will only prove that $\lim_{M\to\infty} \mathscr{I}_M = 0$. By Lemma 2 (ii) we have

$$\left|\hat{\Phi}\left(2^t k_1 + \frac{2^t\xi}{M}, 2^s k_2 + \frac{2^s\eta}{M}\right) - \hat{\Phi}\left(2^t k_1, 2^s k_2 + \frac{2^s\eta}{M}\right)\right| \le C\left|\frac{2^t\xi}{M}\right| \min\left\{\left(2^s\left|k_2 + \frac{\eta}{M}\right|\right)^{\alpha'}, \left(2^s\left|k_2 + \frac{\eta}{M}\right|\right)^{-\beta'}\right\}. \tag{2.15}$$

On the other hand, since $\xi \in B_n$ and B_n is bounded, we take M sufficiently large such that $\left|k_1 + \frac{\xi}{M}\right| \le 2|k_1|$ for all $\xi \in B_n$. Using the conclusion of Lemma 2 (i), we have

$$\left|\hat{\Phi}\left(2^t k_1 + \frac{2^t\xi}{M}, 2^s k_2 + \frac{2^s\eta}{M}\right) - \hat{\Phi}\left(2^t k_1, 2^s k_2 + \frac{2^s\eta}{M}\right)\right| \le C|2^t k_1|^{-\alpha} \min\left\{\left(2^s\left|k_2 + \frac{\eta}{M}\right|\right)^{\beta}, \left(2^s\left|k_2 + \frac{\eta}{M}\right|\right)^{-\beta}\right\}. \tag{2.16}$$

Hence by (2.15) and (2.16) we have

$$\mathscr{I}_M \le C\iint_{B_n \times B_m} |\xi|^2 |\hat{\psi}(\xi)\hat{\Gamma}(\eta)| \int_{\mathbf{R}} \min\left\{\left(2^s\left|k_2 + \frac{\eta}{M}\right|\right)^{2\alpha'}, \left(2^s\left|k_2 + \frac{\eta}{M}\right|\right)^{-2\beta'}\right\} ds \times \frac{1}{M^2}\int_{-\infty}^{(1/2)\log_2 M} 2^{2t} dt d\xi d\eta$$

$$+ C\iint_{B_n\times B_m} |\hat{\psi}(\xi)\hat{\Gamma}(\eta)| \int_{\mathbf{R}} \min\left\{\left(2^s\left|k_2+\frac{\eta}{M}\right|\right)^{2\beta}, \left(2^s\left|k_2+\frac{\eta}{M}\right|\right)^{-2\beta}\right\} ds$$
$$\times \int_{(1/2)\log_2 M}^{\infty} |2^t k_1|^{-2\alpha} dt d\xi d\eta$$
$$= o(1)$$

as $M \to \infty$. Thus (2.14) follows from this. Combining (2.10) with (2.13) and (2.14), we finish the proof of Lemma 1.

Acknowledgment

The authors would like to express their gratitude to the referee for his very valuable comments.

References

[CDF] J. Chen, Y. Ding and D. Fan, L^p boundedness of rough Marcinkiewicz integral on product domains, Chinese J. Comt. Math. (To appear).

[D] Y. Ding, L^2-boundedness of Marcinkiewicz integral with rough kernel, Hokkaido Math. J. **27** (1998), 105–115.

[DR] J. Duoandikoetxea and J. L. Rubio de Francia, Maximal and singular integral operators via Fourier transform estimates, Invent. Math., **84** (1986), 541–561.

[F] D. Fan, Multiplinear on certain Function spaces, Rend. Circ. Mat. Palermo II **XLIII** (1994), 449–463.

[S] E. M. Stein, On the function of Littlewood-Paley, Lusin and Marcinkiewicz, Trans. Amer. Math. Soc. **88** (1958), 430–466.

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