Hiroshima Math. J.
36 (2006), 339–351

Stable extendibility of the tangent bundles over lens spaces

Dedicated to Professor Takao Matumoto on his sixtieth birthday

Mitsunori Imaoka and Hironori Yamasaki

(Received Oct. 2005)
(Revised Jan. 11, 2006)

Abstract. The purpose of this paper is to study the stable extendibility of the tangent bundle $\tau_n(p)$ of the $(2n+1)$-dimensional standard lens space $L^n(p)$ for odd prime p. We investigate the value of integer m for which $\tau_n(p)$ is stably extendible to $L^m(p)$ but not stably extendible to $L^{m+1}(p)$, and in particular we completely determine m for $p = 5$ or 7. A stable splitting of $\tau_n(p)$ and the stable extendibility of a Whitney sum of $\tau_n(p)$ are also discussed.

1. Introduction

Let F be the real number field R, the complex number field C or the quaternion number field H. For a subspace A of a space X, a t-dimensional F-vector bundle ζ over A is said to be *extendible* to X, if there is a t-dimensional F-vector bundle over X whose restriction to A is equivalent to ζ, that is, if ζ is equivalent to the induced bundle $i^*\eta$ of a t-dimensional F-vector bundle η over X under the inclusion map $i : A \to X$. Instead, if $i^*\eta$ is stably equivalent to ζ, namely $i^*\eta + m$ is equivalent to $\zeta + m$ for a trivial F-vector bundle m of dimension $m \geq 0$, ζ is called *stably extendible* to X (cf. [10], p. 20 and [4], p. 273).

Let $L^n(p) = S^{2n+1}/Z_p$ denote the $(2n+1)$-dimensional standard lens space mod p. For an R-vector bundle ζ over $L^n(p)$, we define an integer $s(\zeta)$ by

$$s(\zeta) = \max\{m \mid m \geq n \text{ and } \zeta \text{ is stably extendible to } L^m(p)\},$$

where $s(\zeta) = \infty$ if ζ is stably extendible to $L^m(p)$ for every $m \geq n$.

Let $\tau_n(p) = \tau(L^n(p))$ be the tangent bundle of $L^n(p)$. Then, concerning $s(\tau_n(p))$, the following theorems have been obtained.

Theorem ([7], Theorem 5.3). *Let p be an integer > 1. Then, $s(\tau_n(p)) = \infty$ if $n = 0, 1$ or 3.*

2000 *Mathematics Subject Classification*. Primary 55R50; Secondary 55N15.
Key Words and Phrases. tangent bundle, lens space, stably extendible, *KO*-theory.

THEOREM ([8], Theorem 4.3). *Let p be an odd prime. Then,* $s(\tau_n(p)) < 2n+2$, *if* $n \geq 2p-2$.

THEOREM ([6], Theorem 1). $s(\tau_n(3)) = \infty$ *if and only if* $0 \leq n \leq 3$.

The purpose of this paper is to develop these results on the stable extendibility of the tangent bundle $\tau_n(p)$. Our main results are stated as follows.

THEOREM 1. *Let p be an odd prime. Then,* $s(\tau_n(p)) = 2n+1$ *if* $n \geq 2p-2$.

THEOREM 2. (1) *If* $0 \leq n \leq 5$, *then* $s(\tau_n(5)) = \infty$.
(2) *If* $n \geq 6$, *then* $s(\tau_n(5)) = 2n+1$.

THEOREM 3. (1) *If* $0 \leq n \leq 7$, *then* $s(\tau_n(7)) = \infty$.
(2) *If* $n \geq 8$, *then* $s(\tau_n(7)) = 2n+1$.

These theorems give support to our following conjecture.

CONJECTURE. For an odd prime p,

$$s(\tau_n(p)) = \infty \quad \text{for } 0 \leq n \leq p, \qquad \text{and} \qquad s(\tau_n(p)) = 2n+1 \quad \text{for } n \geq p+1.$$

We organize the paper as follows. In §2, we state some known results necessary to establish our results. In §3 we prove Theorem 1. In §4, we study $\tau_n(5)$ and $\tau_n(7)$ and prove Theorems 2 and 3. In §5, as a consequence of the preceding results, we give Theorem 4 concerning Schwarzenberger's property. In §6, we study the extendibility of the m-times Whitney sum $m\tau_n(p)$ of $\tau_n(p)$ for $m \geq 1$, and show in Proposition 6.1 the inequality

$$s(m\tau_n(p)) \geq m(2n+1) \qquad \text{or} \qquad s(m\tau_n(p)) \geq m(2n+1)-1$$

if m is an odd or even integer respectively. Then, in Theorem 5 we give some condition for

$$s(m\tau_n(p)) = m(2n+1) \qquad \text{or} \qquad m(2n+1)-1 \leq s(m\tau_n(p)) \leq m(2n+1)+1$$

to hold according as m is odd or even.

The authors would express their thanks to Professor T. Matumoto for his valuable suggestions.

2. Preliminary

For an odd prime p, the structures of the reduced K-ring $\tilde{K}(L^n(p))$ and the reduced KO-ring $\widetilde{KO}(L^n(p))$ are determined by Kambe [5].

Let η be the canonical C-line bundle over $L^n(p)$, the induced bundle from the canonical C-line bundle over the complex projective space CP^n under the projection $\pi : L^n(p) \to CP^n$, and $\sigma = \eta - 1$ its stable class in $\tilde{K}(L^n(p))$. Sometimes, we denote η (resp. σ) by η_n (resp. σ_n) to make it clear that η (resp. σ) is over $L^n(p)$.

Let $r : \tilde{K}(X) \to \widetilde{KO}(X)$ and $c : \widetilde{KO}(X) \to \tilde{K}(X)$ be the homomorphisms induced by the real restriction and the complexification of the vector bundles, respectively. We set $\bar{\sigma} = r(\sigma)$ in $\widetilde{KO}(L^n(p))$. Also, let $L_0^n(p)$ denote the $2n$-skeleton of $L^n(p)$ as in [5].

Then, we shall use the following result, where $[x]$ denotes the largest integer m with $m \le x$ for a real number x.

Theorem 2.1 ([5], Theorem 2, Lemma 3.4).
(1) *We have the following isomorphism of abelian groups:*

$$\widetilde{KO}(L^n(p)) \cong \begin{cases} \widetilde{KO}(L_0^n(p)) & \text{if } n \not\equiv 0 \mod 4, \\ Z_2 + \widetilde{KO}(L_0^n(p)) & \text{if } n \equiv 0 \mod 4. \end{cases}$$

(2) *Let $q = (p-1)/2$ and $n = s(p-1) + r$ $(0 \le r < p-1)$. Then,*

$$\widetilde{KO}(L_0^n(p)) = (Z_{p^{s+1}})^{[r/2]} + (Z_{p^s})^{q-[r/2]},$$

and the direct summand $(Z_{p^{s+1}})^{[r/2]}$ and $(Z_{p^s})^{q-[r/2]}$ are generated additively by $\bar{\sigma}^1, \dots, \bar{\sigma}^{[r/2]}$ and $\bar{\sigma}^{[r/2]+1}, \dots, \bar{\sigma}^q$ respectively. Moreover, the ring structure is given by

$$\bar{\sigma}^{q+1} = \sum_{i=1}^{q} \frac{-(2q+1)}{2i-1} \binom{q+i-1}{2i-2} \bar{\sigma}^i, \qquad \bar{\sigma}^{[n/2]+1} = 0,$$

where $\binom{a}{b}$ denotes a binomial coefficient.

We also apply the following property.

Lemma 2.2 ([5], Lemma 3.5(2)). *The homomorphism $c : \widetilde{KO}(L_0^n(p)) \to \tilde{K}(L_0^n(p))$ is a monomorphism.*

The following theorem due to Sjerve [11] is crucial in our proof, where $\pi_m : S^{2m+1} \to L^m(p)$ is the natural projection.

Theorem 2.3 ([11], Theorem A). *If $\zeta \in \widetilde{KO}(L^m(p)) \cap \ker \pi_m^*$, then the geometrical dimension of ζ satisfies* g.dim $\zeta \le 2\left[\frac{m}{2}\right] + 1$.

3. Proof of Theorem 1

By Theorem 2.3, we have the following.

Proposition 3.1. *For any $n \ge 1$, $s(\tau_n(p)) \ge 2n + 1$.*

PROOF. Let $m \geq n$ be an integer. Since $r(\eta_m) - 2 \in \ker \pi_m^* \subset \widetilde{KO}(L^m(p))$ for the projection $\pi_m : S^{2m+1} \to L^m(p)$, where $r : \tilde{K}(L^m(p)) \to \widetilde{KO}(L^m(p))$ is the homomorphism mentioned in §2, we have

$$\mathrm{g.dim}((n+1)(r(\eta_m) - 2)) \leq 2\left[\frac{m}{2}\right] + 1$$

by Theorem 2.3. Thus, there is a $\left(2\left[\frac{m}{2}\right] + 1\right)$-dimensional vector bundle β over $L^m(p)$ satisfying that $(n+1)r(\eta_m)$ is stably equivalent to β. When $m = 2n+1$, we have $2\left[\frac{m}{2}\right] + 1 = 2n+1$ and thus β is of dimension $2n+1$. Hence, $(n+1)r(\eta_{2n+1})$ is stably equivalent to $\beta + 1$, and $\tau_n(p)$ is stably equivalent to $i^*\beta$ since $\tau_n(p) + 1 = (n+1)r(\eta_n)$ is stably equivalent to $i^*\beta + 1$. Therefore, $\tau_n(p)$ is stably extendible to $L^{2n+1}(p)$, and we have the required inequality $s(\tau_n(p)) \geq 2n+1$. □

PROOF OF THEOREM 1. By Theorem 4.3 in [8], we have $s(\tau_n(p)) < 2n+2$ as we described in §1. Thus, by Proposition 3.1, we obtain the required result. □

REMARK 3.2. Proposition 3.1 is a special case of Theorem 4.2 in [7]. Therefore, Theorem 1 is originally due to Kobayashi-Maki-Yoshida ([7], [8]).

4. Stable extendibility of $\tau_n(5)$ and $\tau_n(7)$

Let p be an odd prime. Hereafter, we use the same notation α to denote the stable class of α in $\widetilde{KO}(L^n(p))$ (resp. $\tilde{K}(L^n(p))$) for a real (resp. complex) vector bundle α over $L^n(p)$. Also, we simply denote by $\alpha = \beta$ that vector bundle α and β are stably equivalent.

Using ring structures of $KO(L^n(p))$ and $K(L^n(p))$ for an odd prime p, we have the following lemma, where and hereafter we denote $r(\eta)$ or $c(r(\eta))$ simply by $r\eta$ or $cr\eta$ for the homomorphisms $r : K(L^n(p)) \to KO(L^n(p))$ and $c : KO(L^n(p)) \to K(L^n(p))$.

LEMMA 4.1. *In $KO(L^n(p))$,*

$$(r\eta)^2 = r(\eta^2) + 2, \qquad (r\eta)^3 = r(\eta^3) + 3r\eta.$$

PROOF. Recall that $cr\eta = \eta + \eta^{-1}$ (cf. [3], Proposition 11.3, p. 191). Since $c : KO(L^n(p)) \to K(L^n(p))$ is a ring homomorphism, we have $c(r(\eta^2)) = \eta^2 + \eta^{-2}$ and $c((r\eta)^2) = (cr\eta)^2 = (\eta + \eta^{-1})^2 = \eta^2 + \eta^{-2} + 2$. Then, by Lemma 2.2, $(r\eta)^2 = r(\eta^2) + 2$ in $KO(L^n(p))$. In the same way, $c(r(\eta^3)) = \eta^3 + \eta^{-3}$ and $c((r\eta)^3) = (cr\eta)^3 = (\eta + \eta^{-1})^3 = \eta^3 + \eta^{-3} + 3(\eta + \eta^{-1})$. Thus, we have $(r\eta)^3 = r(\eta^3) + 3r\eta$, and complete the proofs. □

Since $\tau_n(p)$ is stably trivial for $n = 0$ or 1 (cf. [7]), we have

Lemma 4.2.

$$s(\tau_n(p)) = \infty \qquad \textit{for } n = 0 \textit{ or } 1.$$

Concerning $\tau_n(5)$ for $2 \le n \le 5$, we have the following.

Proposition 4.3. *The following stable equivalences hold:*

$$\tau_2(5) = 2r(\eta^2) + 1, \quad \tau_3(5) = r(\eta^2) + 5, \quad \tau_4(5) = 9 \quad \textit{and} \quad \tau_5(5) = r\eta + 9.$$

Hence, $s(\tau_n(5)) = \infty$ *for* $2 \le n \le 5$.

Proof. Let $n = 2$ or 3. Then, by Theorem 2.1, $\widetilde{KO}(L^n(5)) = Z_5\{\bar{\sigma}\}$ and $\bar{\sigma}^2 = 0$. Thus, we have $5r\eta - 10 = 0$ and $(r\eta)^2 - 4r\eta + 4 = 0$. Then, using Lemma 4.1, we obtain $r(\eta^2) + r\eta - 4 = 0$. Since $\tau_n(5) = (n+1)r\eta - 1$, we have

$$\tau_2(5) = 3r\eta - 1 = -2r\eta + 9 = 2r(\eta^2) + 1;$$

$$\tau_3(5) = 4r\eta - 1 = -r\eta + 9 = r(\eta^2) + 5.$$

Similarly, for $n = 4$ or 5, $\widetilde{KO}(L_0^n(5)) = Z_5\{\bar{\sigma}, \bar{\sigma}^2\}$ and thus $5r\eta - 10 = 0$. Then, we have $\tau_4(5) = 5r\eta - 1 = 9$, $\tau_5(5) = 6r\eta - 1 = r\eta + 9$. Thus, we have $s(\tau_n(5)) = \infty$ for $n = 2, 3, 4$ or 5 as is required, since $r(\eta^2)$ and $r\eta$ over $L^n(5)$ are extendible to $L^m(5)$ for every $m \ge n$. $\square$

Remark 4.4. According to Yoshida [12], $L^3(p)$ has a tangent 5-field. Hence, $\tau_3(p) = \beta \oplus 5$ for a 2-plane bundle β in general.

Proposition 4.5.

$$s(\tau_n(5)) = 2n + 1 \qquad \textit{for } n = 6 \textit{ or } 7.$$

Proof. Let $n = 6$ or 7. By Proposition 3.1, we have $s(\tau_n(5)) \ge 2n + 1$. To establish the opposite inequality, we suppose that $\tau_n(5)$ is stably extendible to $L^{2n+2}(5)$, and derive a contradiction from the hypothesis. Thus, there is a $(2n+1)$-dimensional vector bundle α over $L^{2n+2}(5)$ satisfying that $\tau_n(5)$ is stably equivalent to $i^*\alpha$ for the inclusion map $i : L^n(5) \to L^{2n+2}(5)$. By Theorem 2.1, $\widetilde{KO}(L^{2n+2}(5))$ is generated additively by $\bar{\sigma}$ and $\bar{\sigma}^2$ modulo a 2-torsion. Thus, we can put $\alpha - (2n+1) = a\bar{\sigma} + b\bar{\sigma}^2 + \delta$ in $\widetilde{KO}(L^{2n+2}(5))$, where δ is zero or a 2-torsion element. Then, since $i^*\delta = 0$ in $\widetilde{KO}(L^n(5)) = Z_{5^2}\{\bar{\sigma}\} + Z_5\{\bar{\sigma}^2\}$, we have $i^*\alpha - (2n+1) = a\bar{\sigma} + b\bar{\sigma}^2$ in $\widetilde{KO}(L^n(5))$.

Since $i^*\alpha = \tau_n(5)$ and $\tau_n(5) - (2n+1) = (n+1)\bar{\sigma}$, we have

$$\begin{cases} a \equiv n+1 \mod 5^2, \\ b \equiv 0 \mod 5. \end{cases}$$

Hence, we can put

$$\begin{cases} a = 5k + a_1 & \text{with } k \equiv 1 \mod 5, \\ b = 5l \end{cases}$$

for some integers k and l, where $a_1 = 2$ and 3 when $n = 6$ and 7 respectively.

Since $K(L^{2n+2}(5))$ has no 2-torsion (cf. [5]), $c\delta = 0$. Then, we have

$$\begin{aligned} c\alpha - (2n+1) = ac\bar{\sigma} + bc\bar{\sigma}^2 &= a((\eta + \eta^{-1}) - 2) + b((\eta + \eta^{-1})^2 - 4(\eta + \eta^{-1}) + 4) \\ &= (a - 4b)(\eta + \eta^{-1}) + b(\eta^2 + \eta^{-2}) - (2a - 6b). \end{aligned}$$

Let $C_i(\gamma)$ denote the i-th Chern class of a complex vector bundle γ, and $C(\gamma) = 1 + C_1(\gamma) + \cdots$ the total Chern class. We denote $C_i(\gamma)$ and $C(\gamma)$ in the Z_5-coefficient cohomology group by the same letters. Then, for $x = C_1(\eta)$,

$$\bigoplus_{i \geq 0} H^{2i}(L^{2n+2}(5); Z_5) \cong Z_5[x]/(x^{2n+3})$$

as graded algebras (cf. [11]), and we have

$$C(c\alpha) = C(\eta + \eta^{-1})^{a-4b} C(\eta^2 + \eta^{-2})^b = (1 - x^2)^{a-4b}(1 - 4x^2)^b.$$

Since $a - 4b = 5(k - 4l) + a_1$ with $k \equiv 1 \mod 5$ and $b = 5l$, and since $n = 6$ or 7,

$$\begin{aligned} C(c\alpha) &= (1 - x^2)^{a_1}((1 - x^2)^5)^{k-4l}((1 - 4x^2)^5)^l \\ &= (1 - x^2)^{a_1}(1 - x^{10})^{k-4l}(1 - 4^5x^{10})^l \\ &= (1 - x^2)^{a_1}(1 - (k - 4l)x^{10})(1 - 4^5lx^{10}) \\ &= (1 - x^2)^{a_1}(1 - kx^{10}) \\ &= (1 - x^2)^{a_1}(1 - x^{10}) \\ &= 1 - a_1x^2 + \cdots + (-1)^{a_1+1}x^{10+2a_1}. \end{aligned}$$

Since $10 + 2a_1 = 2n + 2$, we have $C_{2n+2}(c\alpha) \neq 0$ which contradicts that α is $(2n+1)$-dimensional. Thus, we have completed the proof. □

PROOF OF THEOREM 2. We obtain (1) by Lemma 4.2 and Proposition 4.3, and (2) by Theorem 1 and Proposition 4.5. □

Next, we consider the proof of Theorem 3, but we can proceed similarly to Theorem 2.

PROPOSITION 4.6. *We have the following stable equivalences:*

$$\tau_2(7) = r(\eta^3) + r\eta + 1, \qquad \tau_3(7) = r(\eta^3) + 2r\eta + 1,$$

$$\tau_4(7) = 2r(\eta^3) + 2r(\eta^2) + 1, \qquad \tau_5(7) = 2r(\eta^3) + 2r(\eta^2) + r\eta + 1,$$

$$\tau_6(7) = 13 \qquad \textit{and} \qquad \tau_7(7) = r\eta + 13.$$

Hence, $s(\tau_n(7)) = \infty$ *for* $2 \le n \le 7$.

PROOF. First, let $n = 2$ or 3. Then, $\widetilde{KO}(L^n(7)) = Z_7\{\bar{\sigma}\}$, $\bar{\sigma}^2 = 0$ and $\bar{\sigma}^3 = 0$ by Theorem 2.1. Thus, we have $7r\eta - 14 = 0$, $(r\eta)^2 - 4r\eta + 4 = 0$ and $(r\eta)^3 - 6(r\eta)^2 + 12r\eta - 8 = 0$. Then, using Lemma 4.1 and these three equations, we obtain $r(\eta^3) + 5r\eta - 12 = 0$. Since $\tau_n(7) = (n+1)r\eta - 1$ in $KO(L^n(7))$, we have

$$\tau_2(7) = 3r\eta - 1 = -4r\eta + 13 = r(\eta^3) + r\eta + 1;$$

$$\tau_3(7) = 4r\eta - 1 = -3r\eta + 13 = r(\eta^3) + 2r\eta + 1.$$

Next, let $n = 4$ or 5. Then, $\widetilde{KO}(L_0^n(7)) = Z_7\{\bar{\sigma}, \bar{\sigma}^2\}$ and $\bar{\sigma}^3 = 0$ by Theorem 2.1. Thus, we have $7r\eta - 14 = 0$, $7(r\eta)^2 - 28r\eta + 28 = 0$ and $(r\eta)^3 - 6(r\eta)^2 + 12r\eta - 8 = 0$. Then, using Lemma 4.1 and these three equations, we obtain $r(\eta^3) + r(\eta^2) + r\eta - 6 = 0$. Since $\tau_n(7) = (n+1)r\eta - 1$ in $KO(L^n(7))$, we have

$$\tau_4(7) = 5r\eta - 1 = -2r\eta + 13 = 2r(\eta^3) + 2r(\eta^2) + 1;$$

$$\tau_5(7) = 6r\eta - 1 = -r\eta + 13 = 2r(\eta^3) + 2r(\eta^2) + r\eta + 1.$$

Similarly, for $n = 6$ or 7, we also have $7r\eta - 14 = 0$ by Theorem 2.1. Thus, we have $\tau_6(7) = 7r\eta - 1 = 13$ and $\tau_7(7) = 8r\eta - 1 = r\eta + 13$. Hence, $s(\tau_n(7)) = \infty$ for $2 \le n \le 7$ as is required, since $r(\eta^3)$, $r(\eta^2)$ and $r\eta$ over $L^n(7)$ are extendible to $L^m(7)$ for every $m \ge n$. □

PROPOSITION 4.7.

$$s(\tau_n(7)) = 2n + 1 \qquad \textit{for } 8 \le n \le 11.$$

PROOF. Let $n = 8, 9, 10$ or 11. By Proposition 3.1, we have $s(\tau_n(7)) \ge 2n + 1$. We suppose that $\tau_n(7)$ is stably extendible to $L^{2n+2}(7)$, and derive a contradiction from the hypothesis. Thus, there is a $(2n+1)$-dimensional vector bundle α over $L^{2n+2}(7)$ satisfying that $\tau_n(7)$ is stably equivalent to $i^*\alpha$. By Theorem 2.1, $\widetilde{KO}(L^n(7))$ and $\widetilde{KO}(L^{2n+2}(7))$ are both generated additively by $\bar{\sigma}$, $\bar{\sigma}^2$ and $\bar{\sigma}^3$ modulo a 2-torsion. Thus, we can put $\alpha - (2n+1) = a\bar{\sigma} + b\bar{\sigma}^2 + d\bar{\sigma}^3 + \delta$, where δ is zero or a 2-torsion element. Then, since $i^*\delta = 0$ in $\widetilde{KO}(L^n(7))$, we have $i^*\alpha - (2n+1) = a\bar{\sigma} + b\bar{\sigma}^2 + d\bar{\sigma}^3$ in

$$\widetilde{KO}(L_0^n(7)) = \begin{cases} Z_{7^2}\{\bar{\sigma}\} + Z_7\{\bar{\sigma}^2, \bar{\sigma}^3\} & n = 8 \text{ or } 9, \\ Z_{7^2}\{\bar{\sigma}, \bar{\sigma}^2\} + Z_7\{\bar{\sigma}^3\} & n = 10 \text{ or } 11. \end{cases}$$

Since $i^*\alpha = \tau_n(7)$ and $\tau_n(7) - (2n+1) = (n+1)\bar{\sigma}$, we have

$$\begin{cases} a \equiv n+1 \mod 7^2, \\ b \equiv 0 \mod 7 \ (n = 8, 9), \ \bmod 7^2 \ (n = 10, 11), \\ d \equiv 0 \mod 7. \end{cases}$$

Hence, we can put

$$\begin{cases} a = 7k + a_1 \quad \text{with } k \equiv 1 \mod 7, \\ b = 7l, \\ d = 7h \end{cases}$$

for some integers k, l and h, where $a_1 = 2, 3, 4$ or 5 according as $n = 8, 9, 10$ or 11. Consider the complexification of α. Then,

$$\begin{aligned} c\alpha - (2n+1) &= ac\bar{\sigma} + bc\bar{\sigma}^2 + dc\bar{\sigma}^3 \\ &= a((\eta + \eta^{-1}) - 2) + b((\eta + \eta^{-1})^2 - 4(\eta + \eta^{-1}) + 4) \\ &\quad + d((\eta + \eta^{-1})^3 - 6(\eta + \eta^{-1})^2 + 12(\eta + \eta^{-1}) - 8) \\ &= (a - 4b + 15d)(\eta + \eta^{-1}) + (b - 6d)(\eta^2 + \eta^{-2}) + d(\eta^3 + \eta^{-3}) \\ &\quad - (2a - 6b + 20d). \end{aligned}$$

Recall that $\bigoplus_{i \geq 0} H^{2i}(L^{2n+2}(7); Z_7) \simeq Z_7[x]/(x^{2n+3})$ as graded algebras, where $x = C_1(\eta)$. Then, we have

$$\begin{aligned} C(c\alpha) &= C(\eta + \eta^{-1})^{a-4b+15d} C(\eta^2 + \eta^{-2})^{b-6d} C(\eta^3 + \eta^{-3})^d \\ &= (1 - x^2)^{a-4b+15d}(1 - 4x^2)^{b-6d}(1 - 9x^2)^d. \end{aligned}$$

Since $a - 4b + 15d = 7(k - 4l + 15h) + a_1$ with $k \equiv 1 \mod 7$, $b - 6d = 7(l - 6h)$ and $d = 7h$, we have

$$\begin{aligned} C(c\alpha) &= (1 - x^2)^{a_1}((1 - x^2)^7)^{k-4l+15h}((1 - 4x^2)^7)^{l-6h}((1 - 9x^2)^7)^h \\ &= (1 - x^2)^{a_1}(1 - x^{14})^{k-4l+15h}(1 - 4^7x^{14})^{l-6h}(1 - 9^7x^{14})^h \\ &= (1 - x^2)^{a_1}(1 - (k - 4l + 15h)x^{14})(1 - 4(l - 6h)x^{14})(1 - 2hx^{14}) \\ &= (1 - x^2)^{a_1}(1 - (k - 9h)x^{14})(1 - 2hx^{14}) \\ &= (1 - x^2)^{a_1}(1 - (k - 7h)x^{14}) \\ &= (1 - x^2)^{a_1}(1 - x^{14}) \\ &= 1 - a_1x^2 + \cdots + (-1)^{a_1+1}x^{14+2a_1}. \end{aligned}$$

Since $14 + 2a_1 = 2n + 2$, we have $C_{2n+2}(c\alpha) \neq 0$, which contradicts that α is $(2n+1)$-dimensional. Thus, we obtain the required result. $\square$

Proof of Theorem 3. We have (1) by Lemma 4.2 and Proposition 4.6, and (2) by Theorem 1 and Proposition 4.7. $\square$

5. Application to stably splitting problem

A splitting (resp. stably splitting) problem of vector bundles can be stated: When is a given k-plane bundle equivalent (resp. stably equivalent) to a sum of k line bundles? Concerning this, the following result is called Schwarzenberger's property.

Theorem ([1], [2], [9], [10]). *Let $F = C$ or R. If a k-dimensional F-vector bundle ζ over FP^n is extendible to FP^m for every $m > n$, then ζ is stably equivalent to the Whitney sum of k numbers of F-line bundles.*

We remark that the theorem is also valid if the condition for extendibility is changed to that for stably extendibility (cf. [8], [4]). Then, some related results are shown as follows:

Theorem ([4], Theorem B). *If a k-dimensional H-vector bundle ζ over HP^n is stably extendible to HP^m for every $m > n$ and its top non-zero Pontrjagin class is not zero* mod 2*, then ζ is stably equivalent to the Whitney sum of k numbers of H-line bundles provided $k \leq n$.*

Theorem ([8], Theorem B). *If a k-dimensional vector bundle ζ over $L^n(3)$ is stably extendible to $L^m(3)$ for every $m > n$, then ζ is stably equivalent to the Whitney sum of $\left[\frac{k}{2}\right]$ numbers of 2-plane bundles.*

We have another answer from Lemma 5.2 in [7], Theorems 2 and 3 and Propositions 4.3 and 4.6.

Theorem 4. *Let $p = 5$ or 7 and $n \geq 1$. Then, $\tau_n(p)$ is stably equivalent to the Whitney sum of $\left[\frac{2n+1}{2}\right]$ numbers of 2-plane bundles if and only if $s(\tau_n(p)) = \infty$ holds.*

6. Study on $m\tau_n(p)$

Let $m\tau_n(p)$ be the m-times Whitney sum of the tangent bundle $\tau_n(p)$. We have the following in the similar way to the proof of Proposition 3.1.

Proposition 6.1. *Let $m \geq 1$. Then, for any $n \geq 1$, we have*

$$s(m\tau_n(p)) \geq m(2n+1) \qquad \text{or} \qquad s(m\tau_n(p)) \geq m(2n+1) - 1$$

if m is an odd or even integer respectively.

Proof. For any integer $k \geq 1$, we have

$$\text{g.dim}(m(n+1)(r\eta_k - 2)) \leq 2\left[\frac{k}{2}\right] + 1$$

by Theorem 2.3. Thus, there is a $\left(2\left[\frac{k}{2}\right]+1\right)$-dimensional vector bundle β satisfying that $m(n+1)r\eta_k$ is stably equivalent to β. Let m be an odd (resp. even) integer. When $k = m(2n+1)$ (resp. $k = m(2n+1)-1$), we have $2\left[\frac{k}{2}\right]+1 = m(2n+1)$ (resp. $= m(2n+1)-1$). Thus, $m(n+1)r\eta_{m(2n+1)}$ (resp. $m(n+1)r\eta_{m(2n+1)-1}$) is stably equivalent to $\gamma + m$ for the $m(2n+1)$-dimensional vector bundle $\gamma = \beta$ (resp. $= \beta + 1$). Then, $m\tau_n(p)$ is stably equivalent to $i^*(\gamma)$ since $m\tau_n(p) + m = m(n+1)r\eta_n$, and thus we have the required inequality $s(m\tau_n(p)) \geq m(2n+1)$ (resp. $s(m\tau_n(p)) \geq m(2n+1)-1$). □

Now, in order to consider the case when $s(m\tau_n(p)) = m(2n+1)$ or $s(m\tau_n(p)) \leq m(2n+1)+1$ holds in Proposition 6.1, we first define an integer $\varepsilon_p(t,l)$.

Definition. For a non-negative integer t and a positive integer l, define an integer $\varepsilon_p(t,l)$ as follows.

$$\varepsilon_p(t,l) = \min\left\{ 2j \;\middle|\; 2\left[\frac{t}{2}\right] + 1 < 2j \text{ and } \binom{\left[\frac{t}{2}\right]+l}{j} \not\equiv 0 \bmod p \right\}.$$

Then, we have $t < \varepsilon_p(t,l) \leq 2\left[\frac{t}{2}\right] + 2l$ and $\varepsilon_p(t,1) = 2\left[\frac{t}{2}\right]+2$, and the following lemma.

Lemma 6.2. *Let p be an odd prime and ζ a t-dimensional vector bundle over $L^n(p)$. If there is a positive integer l with $\varepsilon_p(t,l) \leq n$, then ζ is not stably equivalent to $\left(\left[\frac{t}{2}\right]+l\right)r\eta$.*

Proof. We write simply $\varepsilon(t,l)$ instead of $\varepsilon_p(t,l)$. For the Pontrjagin class of $\left(\left[\frac{t}{2}\right]+l\right)r\eta$, we have

$$P_{\varepsilon(t,l)/2}\left(\left(\left[\frac{t}{2}\right]+l\right)r\eta\right) = \binom{\left[\frac{t}{2}\right]+l}{\frac{\varepsilon(t,l)}{2}} x^{\varepsilon(t,l)} \in H^{2\varepsilon(t,l)}(L^n(p); Z),$$

which is not zero by the definition of $\varepsilon(t,l)$ and the assumption $\varepsilon(t,l) \leq n$. However, since ζ is of dimension t and $\left[\frac{t}{2}\right] < \frac{\varepsilon(t,l)}{2}$, we have $P_{\varepsilon(t,l)/2}(\zeta) = 0$. Thus, ζ is not stably equivalent to $\left(\left[\frac{t}{2}\right]+l\right)r\eta$, as is required. □

The following is also obtained using the calculation in the proof of Theorem 1.1 in [7].

Proposition 6.3. *Let p be an odd prime, and ζ a t-dimensional vector bundle over $L^n(p)$. Assume that there is a positive integer l satisfying*

(1) *ζ is stably equivalent to* $\left(\left[\frac{t}{2}\right]+l\right)r\eta$, *and*
(2) $p^{[n/(p-1)]} > \left[\frac{t}{2}\right]+l$.
Then, $s(\zeta) < \varepsilon_p(t,l)$.

Proof. Here, we put $h=\left[\frac{t}{2}\right]+l$, and write $\varepsilon(t,l)$ instead of $\varepsilon_p(t,l)$. Then, by Lemma 6.2, $n<\varepsilon(t,l)$. Now, we suppose that ζ is stably extendible to $L^{\varepsilon(t,l)}(p)$, and derive a contradiction from the hypothesis. Thus, there exists a t-dimensional vector bundle α over $L^{\varepsilon(t,l)}(p)$ satisfying that $i^*\alpha$ is stably equivalent to $hr\eta$.

Now, we apply the same methods used in the proof of Theorem 1.1 in [7]. The integers c_i used there are $c_1=h$ and $c_i=0$ for $2\le i\le p-1$ in our case. Then, the total Pontrjagin class of $j^*\alpha$, where j is the inclusion map $j: L_0^{\varepsilon(t,l)}(p)\to L^{\varepsilon(t,l)}(p)$, is given as

$$P(j^*\alpha)=(1+x^2)^h \qquad \text{in } H^*(L_0^{\varepsilon(t,l)}(p);Z).$$

Here, the following equality is used to calculate the above Pontrjagin class as in [7]:

$$(1+i^2x^2)^{p^{[n/(p-1)]}}=1+i^{2p^{[n/(p-1)]}}x^{2p^{[n/(p-1)]}}=1 \qquad \text{in } H^*(L_0^{\varepsilon(t,l)}(p);Z)$$

for $1\le i\le \frac{p-1}{2}$, and it holds because $p^{[n/(p-1)]}>h$ from the assumption (2) and $2h\ge\varepsilon(t,l)$ as mentioned above. Then, from the total Pontrjagin class of $j^*\alpha$ and by the definition of $\varepsilon(t,l)$, we have

$$P_{\varepsilon(t,l)/2}(j^*\alpha)=\binom{h}{\frac{\varepsilon(t,l)}{2}}x^{\varepsilon(t,l)}\ne 0 \qquad \text{in } H^{2\varepsilon(t,l)}(L_0^{\varepsilon(t,l)}(p);Z),$$

which contradicts that $j^*\alpha$ is of dimension t and $t<\varepsilon(t,l)$. Thus, we have completed the proof. $\square$

Then, we have the following.

Theorem 5. *Let $m\ge 1$ and $n\ge 1$ be integers.*
(1) *If m is odd,*

$$p^{[n/(p-1)]}>m(n+1) \qquad \text{and} \qquad \binom{m(n+1)}{m(n+1)-\frac{m-1}{2}}\not\equiv 0 \mod p,$$

then $s(m\tau_n(p))=m(2n+1)$.
(2) *If m is even,*

$$p^{[n/(p-1)]}>m(n+1) \qquad \text{and} \qquad \binom{m(n+1)}{mn+1+\frac{m}{2}}\not\equiv 0 \mod p,$$

then $s(m\tau_n(p))=m(2n+1)-1,\ m(2n+1)$ *or* $m(2n+1)+1$.

PROOF. First, we assume that m is odd, and prove (1). By Proposition 6.1, we have $s(m\tau_n(p)) \geq m(2n+1)$. Thus, we assume further that

$$p^{[n/(p-1)]} > m(n+1) \quad \text{and} \quad \binom{m(n+1)}{\frac{m(2n+1)+1}{2}} = \binom{m(n+1)}{m(n+1)-\frac{m-1}{2}} \not\equiv 0 \mod p,$$

and prove the inequality $s(m\tau_n(p)) \leq m(2n+1)$. Consider $\varepsilon_p\left(m(2n+1), \frac{m+1}{2}\right)$. Since $2\left[\frac{m(2n+1)}{2}\right] + 1 < m(2n+1) + 1$, and by the latter assumption above, we have $\varepsilon_p\left(m(2n+1), \frac{m+1}{2}\right) \leq m(2n+1) + 1$. Hence, by Proposition 6.3, we have $s(m\tau_n(p)) < \varepsilon_p\left(m(2n+1), \frac{m+1}{2}\right) \leq m(2n+1) + 1$, and thus we have proved (1).

Next, we assume that m is even, and prove (2). By Proposition 6.1, we have $s(m\tau_n(p)) \geq m(2n+1) - 1$. Thus, we further assume that

$$p^{[n/(p-1)]} > m(n+1) \qquad \text{and} \qquad \binom{m(n+1)}{\frac{m(2n+1)+2}{2}} = \binom{m(n+1)}{mn+1+\frac{m}{2}} \not\equiv 0 \mod p,$$

and prove $s(m\tau_n(p)) \leq m(2n+1) + 1$. Then, since $2\left[\frac{m(2n+1)}{2}\right] + 1 < m(2n+1) + 2$, and by the last assumption above, we have $\varepsilon_p\left(m(2n+1), \frac{m}{2}\right) \leq m(2n+1) + 2$. Hence, by Proposition 6.3, $s(m\tau_n(p)) < m(2n+1) + 2$, and thus we have proved (2) and completed the proof of Theorem 5. □

We illustrate the results of Theorems 5 for $p = 5$ or 7 and for $2 \leq m \leq 5$.

EXAMPLE. Let $n \geq 1$, and $p = 5$ or 7.

(1) If $n \geq 2p - 2$, then $s(2\tau_n(p)) = 4n+1, 4n+2$ or $4n+3$.

(2) Assume that

$$\begin{cases} n \geq 3p-3 \text{ and } n+1 \not\equiv 0 \mod p & \text{for } p = 5, \\ n = 12, 14, 15 \text{ or } n \geq 3p-3 \text{ and } n+1 \not\equiv 0 \mod p & \text{for } p = 7. \end{cases}$$

Then, $s(3\tau_n(p)) = 6n + 3$.

(3) Assume that $n \geq 3p - 3$ and $n + 1 \not\equiv 0 \mod p$. Then, $s(4\tau_n(p)) = 8n+3, 8n+4$ or $8n+5$.

(4) Assume that $n \geq 3p - 3$. For $p = 5$, we have no information on $s(5\tau_n(5))$ from Theorem 5. For $p = 7$, if $\frac{1}{2}(5n+4)(5n+5) \not\equiv 0$ mod 7, then $s(5\tau_n(7)) = 10n + 5$.

References

[1] J. F. Adams and Z. Mahmud, Maps between classifying spaces, Invent. Math. **35** (1976), 1–41.

[2] F. Hirzebruch, Topological methods in algebraic geometry, Springer-Verlag, Berlin-Heidelberg-New York, 1978.

[3] D. Husemoller, Fibre Bundles, Second Edition, Graduate texts in Mathematics **20**, Springer-Verlag, New York-Heidelberg-Berlin, 1975.
[4] M. Imaoka and K. Kuwana, Stably extendible vector bundles over the quaternionic projective spaces, Hiroshima Math. J. **29** (1999), 273–279.
[5] T. Kambe, The structure of K_Λ-rings of the lens space and their applications, J. Math. Soc. Japan, **18** (1966), 135–146.
[6] T. Kobayashi and K. Komatsu, Extendibility and stable extendibility of vector bundles over lens spaces mod 3, to appear in Hiroshima Math. J. **35** (2005).
[7] T. Kobayashi, H. Maki and T. Yoshida, Remarks on extendible vector bundles over lens spaces and real projective spaces, Hiroshima Math. J. **5** (1975), 487–497.
[8] T. Kobayashi, H. Maki and T. Yoshida, Stably extendible vector bundles over the real projective spaces and the lens spaces, Hiroshima Math. J. **29** (1999), 631–638.
[9] E. Rees, On submanifolds of projective space, J. London Math. Soc. **19** (1979), 159–162.
[10] R. L. E. Schwarzenberger, Extendible vector bundles over real projective space, Quart. J. Math. Oxford (2) **17** (1966), 19–21.
[11] D. Sjerve, Geometric dimension of vector bundles over lens spaces, Trans. Amer. Math. Soc. **134** (1968), 545–557.
[12] T. Yoshida, A remark on vector fields on lens spaces, J. Sci. Hiroshima Univ. Ser. A-I, **31** (1967), 13–15.

Mitsunori Imaoka
Department of Mathematics Education
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739-8524 *Japan*
imaoka@hiroshima-u.ac.jp

Hironori Yamasaki
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739-8526 *Japan*
hyamasaki@hi.enjoy.ne.jp