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On the rate of convergence for some linear operators

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Abstract. We consider certain linear operators L_n in polynomial weighted spaces of functions of one variable and study approximation properties of these operators, including theorems on the degree of approximation.

1. Introduction

Approximation properties of Szász-Mirakyan operators

(1)
$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \qquad x \in R_0 = [0,+\infty),\ n \in N := \{1,2\ldots\},$$

in polynomial weighted spaces C_p were examined in [1]. The space C_p, $p \in N_0 := \{0,1,2,\ldots\}$, is associated with the weighted function

$$w_0(x) := 1, \qquad w_p(x) := (1+x^p)^{-1} \qquad \text{if } \ p \geq 1, \tag{2}$$

and consists of all real-valued continuous functions f on R_0 for which $w_p f$ is uniformly continuous and bounded on R_0. The norm on C_p is defined by

$$\|f\|_p \equiv \|f(\cdot)\|_p := \sup_{x \in R_0} w_p(x)|f(x)|. \tag{3}$$

In [1] there were theorems on the degree of approximation of $f \in C_p$ by operators S_n defined by (1). From these theorems it was deduced that

$$\lim_{n\to\infty} S_n(f;x) = f(x)$$

for every $f \in C_p$, $p \in N_0$, and $x \in R_0$. Moreover, the above convergence is uniform on every interval $[x_1,x_2]$, $x_2 > x_1 \geq 0$.

In this paper by $M_k(\alpha,\beta)$, we shall denote suitable positive constants depending only on indicated parameters α and β.

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The Szász-Mirakyan operators are important in approximation theory. They have been studied intensively, in connection with different branches of analysis, such as numerical analysis. Recently in many papers various modifications of operators S_n were introduced. Approximation properties of modified Szász-Mirakyan operators

$$B_n(f;r;x) := \frac{1}{g((nx+1)^2;r)} \sum_{k=0}^{\infty} \frac{(nx+1)^{2k}}{(k+r)!} f\left(\frac{k+r}{n(nx+1)}\right), \qquad x \in R_0,\ n,r \in N,$$

where

$$g(t;r) = \sum_{k=0}^{\infty} \frac{t^k}{(k+r)!}, \qquad t \in R_0,$$

in polynomial weighted spaces were examined in [10].

In [10] it was proved that if $f \in C_p$, $p \in N_0$, then

$$\|B_n(f;r;\cdot) - f(\cdot)\|_p \le M_0 \omega_1\left(f; C_p; \frac{1}{n}\right), \qquad n,r \in N,$$

where $M_0 = const. > 0$ and $\omega_1(f;C_p;\cdot)$ is the modulus of continuity of f defined by

$$\omega_1(f;C_p;t) := \sup_{0 \le h \le t} \|\Delta_h f(\cdot)\|_p, \qquad t \in R_0,$$

where $\Delta_h f(x) = f(x+h) - f(x)$ for $h, x \in R_0$. In particular, if $f \in C_p^1$, $p \in N_0$, then

$$\|B_n(f;r;\cdot) - f(\cdot)\|_p \le \frac{M_1}{n}, \qquad n,r \in N,$$

where $M_1 = const. > 0$. The above inequalities estimate the rate of uniform convergence of $\{B_n(f;r;\cdot)\}$. Similar results in exponential weighted spaces can be found in [9, 11].

The Szász-Mirakyan operators S_n are defined in terms of a sample of the given function f on the points k/n, called knots, for $k \in N_0$, $n \in N$. For the operators introduced in [9–11] the knots are the numbers $(k+r)/(n(nx+1))$ for $k \in N_0$, $n \in N$ and $x \in R_0$ (r being fixed).

Thus the question arises, whether the knots $(k+r)/(n(nx+1))$ cannot be replaced by a given subset of points, which are independent of x, provided this will not change essentially the degree of convergence. In connection with this question we introduce the operators (4).

Let B_p, $p \in N$, be the set of all real-valued continuous functions $f(x)$ on R_0 for which $w_p(x)x^k f^{(k)}(x)$, $k = 0, 1, 2 \dots, p$, are continuous and bounded on R_0 and $f^{(p)}(x)$ is uniformly continuous on R_0. The norm on B_p is given by (3).

We introduce the following class of operators in B_p, $p \in N$.

Definition. *Let $r \in N$, $p \in N$, $s > 0$ be fixed numbers. For functions $f \in B_p$ we define the operators*

(4) $L_n(f; p, r, s; x)$

$$:= \frac{1}{g(n^s x; r)} \sum_{k=0}^{\infty} \frac{(n^s x)^k}{(k+r)!} \sum_{j=0}^{p} \frac{f^{(j)}\left(\frac{k+r}{n^s}\right)\left(x - \frac{k+r}{n^s}\right)^j}{j!}, \qquad x \in R_0,\ n \in N,$$

where

$$g(t; r) = \sum_{k=0}^{\infty} \frac{t^k}{(k+r)!}, \qquad t \in R_0. \tag{5}$$

Observe that

$$g(0; r) = \frac{1}{r!}, \qquad g(t; r) = \frac{1}{t^r}\left(e^t - \sum_{j=0}^{r-1} \frac{t^j}{j!}\right) \qquad \text{if } t > 0.$$

In this paper we shall state some estimates of the rate of uniform convergence of the operators L_n, $n \in N$.

Theorems given in [1, 4–7] are concerned with pointwise approximation. We give theorems on the degree of approximation of functions in B_p by L_n with respect to norm estimates.

Let us introduce the notation

$$A_n(f; r, s; x) := \frac{1}{g(n^s x; r)} \sum_{k=0}^{\infty} \frac{(n^s x)^k}{(k+r)!} f\left(\frac{k+r}{n^s}\right) \tag{6}$$

for $f \in B_p$, $p \in N_0$, $x \in R_0$, $n, r \in N$ and $s > 0$.

We shall apply the method used in [1, 3–11].

2. Auxiliary results

In this section we shall give some properties of the above operators, which we shall apply to the proofs of the main theorems.

From (5) and (6) we easily derive the following formulas

$$
\begin{aligned}
(7)\qquad & A_n(1;r,s;x) = 1,\\
& A_n(t;r,s;x) = x + \frac{1}{n^s(r-1)!g(n^s x;r)},\\
& A_n(t^2;r,s;x) = x^2 + \frac{x}{n^s}\left(1 + \frac{1}{(r-1)!g(n^s x;r)}\right) + \frac{r}{n^{2s}(r-1)!g(n^s x;r)}
\end{aligned}
$$

for every fixed $r \in N$ and for all $n \in N$ and $x \in R_0$.

Using (5), (6) and mathematical induction on $q \in N$ we can prove the following lemma (see [10]).

Lemma 1. *Fix $q \in N_0$, $r \in N$ and $s > 0$. Then there exist positive numbers $\alpha_{q,j}$ depending only on j, q, and $\beta_{q,j}(r)$ depending only on r, j and q, $0 \le j \le q$ such that*

$$
A_n(t^q;r,s;x) = \sum_{j=0}^{q} \frac{x^j}{n^{s(q-j)}}\left(\alpha_{q,j} + \frac{\beta_{q,j}(r)}{g(n^s x;r)}\right) \tag{8}
$$

for all $n \in N$ and $x \in R_0$. Moreover $\alpha_{0,0} = 1$, $\beta_{0,0}(r) = 0$ and $\alpha_{q,0} = \beta_{q,q}(r) = 0$, $\alpha_{q,q} = 1$, $\beta_{q,0}(r) = \dfrac{r^{q-1}}{(r-1)!}$ for $q \in N$.

Next we shall prove

Lemma 2. *Let $p \in N_0$, $r \in N$ and $s > 0$ be fixed numbers. Then there exists a positive constant $M_2 \equiv M_2(p,r)$ such that*

$$
\|A_n(1/w_p(t);r,s;\cdot)\|_p \le M_2, \qquad n \in N. \tag{9}
$$

Proof. The inequality (9) is obvious for $p = 0$ by (2), (3) and (7).

Let $p \in N$. From (5) we get

$$
\frac{1}{g(t;r)} \le r! \qquad \text{for } t \in R_0. \tag{10}
$$

From (10) and by (2) and (6)–(8) we have

$$
\begin{aligned}
w_p(x)A_n(1/w_p(t);r,s;x) &= w_p(x)\{1 + A_n(t^p;r,s;x)\}\\
&= \frac{1}{1+x^p} + \sum_{j=0}^{p} \frac{x^j}{n^{s(p-j)}(1+x^p)}\left(\alpha_{p,j} + \frac{\beta_{p,j}(r)}{g(n^s x;r)}\right)\\
&\le 1 + \sum_{j=0}^{p} \frac{x^j}{1+x^p}(\alpha_{p,j} + r!\beta_{p,j}(r)) \le M_2(p,r)
\end{aligned}
$$

for $x \in R_0$, $n \in N$, $s > 0$ and $r \in N$, where $M_2(p,r)$ is a positive constant depending only p and r. This completes the proof of Lemma 2. ■

Similarly we can prove the following lemma.

LEMMA 3. *Let $p \in N_0$, $r \in N$ and $s > 0$ be fixed numbers. Then there exists a positive constant $M_3 \equiv M_3(p,r)$ such that*

$$\sup_{x \in R_0} w_p(x) x^k A_n(1/w_{p-k}(t); r, s; x) \le M_3, \qquad n \in N,\ k = 0, 1, 2, \dots, p.$$

LEMMA 4. *Fix $p \in N$, $r \in N$ and $s > 0$. Then for all $x \in R_0$ and $n \in N$ we have*

$$A_n((x-t)^p; r, s; x) = \sum_{j=1}^{p} \frac{x^{j-1}}{n^{s(p-j+1)}} \left(a_{p,j} + \frac{b_{p,j}}{g(n^s x; r)} \right), \tag{11}$$

where $a_{p,j}$, $b_{p,j}$ are numbers depending only on the paremeters r, j and p.

PROOF. Let $p \ge 1$. Applying Lemma 1, we get

$$\begin{aligned}
A_n((x-t)^p; r, s; x) &= \frac{1}{g(n^s x; r)} \sum_{k=0}^{\infty} \frac{(n^s x)^k}{(k+r)!} \left(x - \frac{k+r}{n^s} \right)^p \\
&= \frac{1}{g(n^s x; r)} \sum_{k=0}^{\infty} \frac{(n^s x)^k}{(k+r)!} \sum_{i=0}^{p} \binom{p}{i} (-1)^i x^{p-i} \left(\frac{k+r}{n^s} \right)^i \\
&= \sum_{i=0}^{p} \binom{p}{i} (-1)^i x^{p-i} A_n(t^i; r, s; x) \\
&= \sum_{i=0}^{p} \binom{p}{i} (-1)^i x^{p-i} \sum_{j=0}^{i} \frac{x^j}{n^{s(i-j)}} \left(\alpha_{i,j} + \frac{\beta_{i,j}(r)}{g(n^s x; r)} \right) \\
&= x^p + \sum_{i=1}^{p} \binom{p}{i} (-1)^i x^{p-i} \left[\sum_{j=0}^{i-1} \frac{x^j}{n^{s(i-j)}} \left(\alpha_{i,j} + \frac{\beta_{i,j}(r)}{g(n^s x; r)} \right) + x^i \right] \\
&= x^p \sum_{i=0}^{p} \binom{p}{i} (-1)^i + \sum_{i=1}^{p} \binom{p}{i} (-1)^i \\
&\quad \times \sum_{j=0}^{i-1} \frac{x^{j+p-i}}{n^{s(i-j)}} \left(\alpha_{i,j} + \frac{\beta_{i,j}(r)}{g(n^s x; r)} \right).
\end{aligned}$$

Using the elementary identity

$$\sum_{i=0}^{p} \binom{p}{i} (-1)^i = (1-1)^p = 0, \qquad p \in N,$$

we receive the representation

$$\begin{aligned} A_n((x-t)^p; r, s; x) &= \sum_{i=1}^{p} \binom{p}{i} (-1)^i \sum_{k=1}^{i} \frac{x^{k+p-i-1}}{n^{s(i-k+1)}} \left(\alpha_{i,k-1} + \frac{\beta_{i,k-1}(r)}{g(n^s x; r)} \right) \\ &= \sum_{j=1}^{p} \frac{x^{j-1}}{n^{s(p-j+1)}} \sum_{i=p-j+1}^{p} \binom{p}{i} (-1)^i \left(\alpha_{i,i+j-p-1} + \frac{\beta_{i,i+j-p-1}(r)}{g(n^s x; r)} \right) \\ &= \sum_{j=1}^{p} \frac{x^{j-1}}{n^{s(p-j+1)}} \left(a_{p,j} + \frac{b_{p,j}}{g(n^s x; r)} \right). \quad \blacksquare \end{aligned}$$

LEMMA 5. *Fix* $p, r \in N$ *and* $s > 0$. *Then there exists a positive constant* $M_4 \equiv M_4(f; p, r)$ *such that*

$$\|L_n(f; p, r, s; \cdot)\|_p \le M_4 \tag{12}$$

for all $f \in B_p$.

The formulas (4), (5) *and* (12) *show that* $L_n(f; p, r, s)$ *is well-defined on the space* B_p, $p \in N$.

PROOF. First we suppose that $f \in B_p$, $p \in N$. From this, using the elementary inequality $(a+b)^k \le 2^{k-1}(a^k + b^k)$, $a, b > 0$, $k \in N_0$, we get

$$\begin{aligned} |x-t|^k |f^{(k)}(t)| &\le 2^{k-1} |f^{(k)}(t)| \{x^k + t^k\} \\ &\le M_5(f; p, k) \left\{ \frac{1}{w_p(t)} + \frac{x^k}{w_{p-k}(t)} \right\}, \qquad k = 0, 1, 2, \dots, p, \ x, t \in R_0. \end{aligned}$$

This implies that

$$\begin{aligned} &w_p(x) |L_n(f; p, r, s; x)| \\ &\quad \le M_6(f; p) \frac{w_p(x)}{g(n^s x; r)} \sum_{k=0}^{\infty} \frac{(n^s x)^k}{(k+r)!} \left\{ \frac{1}{w_p((k+r)/n^s)} + \sum_{j=0}^{p} \frac{x^j}{w_{p-j}((k+r)/n^s)} \right\} \\ &\quad = M_6(f; p) w_p(x) \left\{ A_n(1/w_p(t); r, s; x) + \sum_{j=0}^{p} x^j A_n(1/w_{p-j}(t); r, s; x) \right\}. \end{aligned}$$

From this and in view of Lemmas 2 and 3 we get

$$w_p(x) |L_n(f; p, r, s; x)| \le M_4(f; p, r).$$

This ends the proof of (12). $\blacksquare$

3. Rate of convergence

In this section we shall study properties of $L_n(f;p,r,s)$. We shall give theorems on the degree of approximation of $f \in B_p$, $p \in N$, by these operators.

We can state now the main results of this paper.

Theorem 1. *Fix $p \in N_0$, $r \in N$ and $s > 0$. Then there exists a positive constant $M_7 \equiv M_7(p,r,s)$ such that for every $f \in B_{2p+1}$ we have*

$$\|L_n(f;2p+1,r,s;\cdot) - f(\cdot)\|_{2p+1} \le M_7 \omega_1\left(f^{(2p+1)};C_0;\frac{1}{n^s}\right), \qquad n \in N. \tag{13}$$

Proof. First we suppose that $f \in B_{2p+1}$. This implies that $f^{(2p+1)} \in C_0$.

Let $p \in N_0$. Using the modified Taylor formula

$$f(x) = \sum_{j=0}^{2p+1} \frac{f^{(j)}\left(\frac{k+r}{n^s}\right)\left(x - \frac{k+r}{n^s}\right)^j}{j!} + \frac{\left(x - \frac{k+r}{n^s}\right)^{2p+1}}{(2p)!}$$

$$\times \int_0^1 (1-t)^{2p}\left[f^{(2p+1)}\left(\frac{k+r}{n^s} + t\left(x - \frac{k+r}{n^s}\right)\right) - f^{(2p+1)}\left(\frac{k+r}{n^s}\right)\right]dt$$

and the definition of modulus of continuity and (4), (5), we obtain

$$w_{2p+1}(x)|L_n(f;2p+1,r,s;x) - f(x)|$$

$$\le \frac{w_{2p+1}(x)}{g(n^s x;r)} \sum_{k=0}^{\infty} \frac{(n^s x)^k}{(k+r)!}\left|\sum_{j=0}^{2p+1} \frac{f^{(j)}\left(\frac{k+r}{n^s}\right)\left(x - \frac{k+r}{n^s}\right)^j}{j!} - f(x)\right|$$

$$\le \frac{w_{2p+1}(x)}{g(n^s x;r)} \sum_{k=0}^{\infty} \frac{(n^s x)^k}{(k+r)!} \frac{\left|x - \frac{k+r}{n^s}\right|^{2p+1}}{(2p)!}$$

$$\times \int_0^1 (1-t)^{2p}\left|f^{(2p+1)}\left(\frac{k+r}{n^s} + t\left(x - \frac{k+r}{n^s}\right)\right) - f^{(2p+1)}\left(\frac{k+r}{n^s}\right)\right|dt$$

$$\le \frac{w_{2p+1}(x)}{g(n^s x;r)} \sum_{k=0}^{\infty} \frac{(n^s x)^k}{(k+r)!} \frac{\left|x - \frac{k+r}{n^s}\right|^{2p+1}}{(2p)!}$$

$$\times \int_0^1 (1-t)^{2p} \omega_1(f^{(2p+1)};C_0;t|x-(k+r)/n^s|)dt.$$

Observe that

$$\omega_1(f^{(2p+1)}; C_0; t|x-(k+r)/n^s|) \le (1+t|x-(k+r)/n^s|n^s)\omega_1(f^{(2p+1)}; C_0; 1/n^s).$$

From this, using the elementary inequality $(a+b)^k \le 2^{k-1}(a^k+b^k)$, $a,b>0$, $k \in N_0$, and (6) and (7), we get

$$\begin{aligned} &w_{2p+1}(x)|L_n(f;2p+1,r,s;x)-f(x)| \\ &\quad \le \frac{w_{2p+1}(x)}{g(n^s x;r)}\sum_{k=0}^{\infty}\frac{(n^s x)^k}{(k+r)!}\left[\left(x-\frac{k+r}{n^s}\right)^{2p+2}n^s+\left|x-\frac{k+r}{n^s}\right|^{2p+1}\right] \\ &\qquad \times \omega_1(f^{(2p+1)}; C_0; 1/n^s) \\ &\quad \le w_{2p+1}(x)\{n^s A_n((x-t)^{2p+2}; r,s;x) + A_n((t+x)^{2p+1}; r,s;x)\} \\ &\qquad \times \omega_1(f^{(2p+1)}; C_0; 1/n^s) \\ &\quad \le w_{2p+1}(x)\{n^s A_n((t-x)^{2p+2}; r,s;x) + 2^{2p}A_n(t^{2p+1}; r,s;x) + 2^{2p}x^{2p+1}\} \\ &\qquad \times \omega_1(f^{(2p+1)}; C_0; 1/n^s). \end{aligned}$$

Applying Lemmas 1 and 4, (10) and (2), we immediately obtain

$$\begin{aligned} &w_{2p+1}(x)|L_n(f;2p+1,r,s;x)-f(x)| \\ &\quad \le w_{2p+1}(x)\left\{\sum_{j=1}^{2p+2}\frac{x^{j-1}}{n^{s(2p-j+2)}}\left(a_{2p+2,j}+\frac{b_{2p+2,j}}{g(n^s x;r)}\right)\right. \\ &\qquad \left. + 2^{2p}\sum_{j=0}^{2p+1}\frac{x^j}{n^{s(2p+1-j)}}\left(\alpha_{2p+1,j}+\frac{\beta_{2p+1,j}(r)}{g(n^s x;r)}\right) + 2^{2p}x^{2p+1}\right\}\omega_1(f^{(2p+1)}; C_0; 1/n^s) \\ &\quad \le M_8(p,s)\left\{\sum_{j=1}^{2p+2}(|a_{2p+2,j}|+r!|b_{2p+2,j}|) + 2^{2p}\sum_{j=1}^{2p+1}(\alpha_{2p+1}+r!\beta_{2p+1}(r)) + 2^{2p}\right\} \\ &\qquad \times \omega_1(f^{(2p+1)}; C_0; 1/n^s) \le M_7(p,r,s)\omega_1(f^{(2p+1)}; C_0; 1/n^s) \end{aligned}$$

for $x \in R_0$, $n, r \in N$, $s>0$. This completes the proof of Theorem 1. ■

Theorem 2. *Fix $p \in N_0$, $r \in N$ and $s>0$. Then there exists a positive constant $M_9 \equiv M_9(p,r,s)$ such that for every $f \in B_{2p+2}$ we have*

$$(14) \qquad \|L_n(f;2p+2,r,s;\cdot)-f(\cdot)\|_{2p+2} \le \frac{M_9(p,r,s)}{n^s}\|f^{(2p+2)}\|_0, \qquad n \in N.$$

PROOF. First we suppose that $f \in B_{2p+2}$. This implies that $f^{(2p+2)} \in C_0$. Arguing as in the first part of the proof of Theorem 1 we obtain

$$w_{2p+2}(x)|L_n(f;2p+2,r,s;x)-f(x)|$$
$$\leq \frac{w_{2p+2}(x)}{g(n^s x;r)}\sum_{k=0}^{\infty}\frac{(n^s x)^k}{(k+r)!}\frac{\left(x-\frac{k+r}{n^s}\right)^{2p+2}}{(2p+1)!}$$
$$\times \int_0^1 (1-t)^{2p+1}\left\{\left|f^{(2p+2)}\left(\frac{k+r}{n^s}+t\left(x-\frac{k+r}{n^s}\right)\right)\right|+\left|f^{(2p+2)}\left(\frac{k+r}{n^s}\right)\right|\right\}dt.$$

From this and by our assumption we get

$$w_{2p+2}(x)|L_n(f;2p+2,r,s;x)-f(x)|$$
$$\leq 2\|f^{(2p+2)}\|_0\frac{w_{2p+2}(x)}{g(n^s x;r)}\sum_{k=0}^{\infty}\frac{(n^s x)^k}{(k+r)!}\left(x-\frac{k+r}{n^s}\right)^{2p+2}$$
$$=2\|f^{(2p+2)}\|_0 w_{2p+2}(x)A_n((x-t)^{2p+2};r,s;x).$$

From this, applying Lemma 4, (10) and (2), we immediately obtain

$$w_{2p+2}(x)|L_n(f;2p+2;r,s;x)-f(x)|$$
$$\leq \frac{2}{n^s}\|f^{(2p+2)}\|_0 w_{2p+2}(x)\sum_{j=1}^{2p+2}\frac{x^{j-1}}{n^{s(2p-j+2)}}\left(|a_{p+2,j}|+\frac{|b_{p+2,j}|}{g(n^s x;r)}\right)$$
$$\leq \frac{M_9(p,r,s)}{n^s}\|f^{(2p+2)}\|_0$$

for $x \in R_0$, $n, r \in N$, $s > 0$. This ends the proof of Theorem 2. ■

From above Theorems we obtain

COROLLARY. *For every fixed* $r \in N$, $p \in N$, $s > 0$ *and* $f \in B_p$ *we have*

$$\lim_{n\to\infty}\|L_n(f;p,r,s;\cdot)-f(\cdot)\|_p=0.$$

REMARK. In [1] it was proved that if $f \in C_p$, $p \in N_0$, then

$$w_p(x)|S_n(f;x)-f(x)| \leq M_9\omega_2\left(f;C_p;\sqrt{\frac{x}{n}}\right),\qquad x\in R_0,\ n\in N,$$

where $M_{10} = const. > 0$ and $\omega_2(f;\cdot)$ is the modulus of smoothness defined by

$$\omega_2(f;C_p;t) := \sup_{0\leq h\leq t}\|\Delta_h^2 f(\cdot)\|_p,\qquad t\in R_0,$$

with $\Delta_h^2 f(x) := f(x) - 2f(x+h) + f(x+2h)$. In particular, if $f \in C_p^1$, $p \in N_0$, then

$$w_p(x)|S_n(f;x) - f(x)| \le M_{10}\sqrt{\frac{x}{n}}, \qquad x \in R_0,\ n \in N$$

where $M_{10} = const. > 0$.

Theorem 1, Theorem 2 and Corollary in this paper show that operators $L_n(f;p;r;1;\cdot)$, $n \in N$, give better the degree of approximation of functions $f \in B_p$, $p \in N$, than S_n and the operators examined in [1, 4, 6, 7].

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