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Solvability of multi-point boundary value problems for 2n-th order ordinary differential equations at resonance(II)

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Abstract. In this paper, we prove existence results for solutions of multi-point boundary value problems at resonance (Theorems 2.1–2.4) and for positive solutions at non-resonance (Theorems 2.5 and 2.6) for a 2n-th order differential equation. Our method is based upon the coincidence degree theory of Mawhin. The interesting is that the degree of some variables among $x_0, x_1, \dots, x_{2n-1}$ in the function $f(t, x_0, x_1, \dots, x_{2n-1})$ are allowable to be greater than 1. The results obtained are new.

1. Introduction

In this paper, we investigate the existence of solutions and positive solutions of the multi-point boundary value problems for 2n-th order differential equations

$$(-1)^{n-1}x^{(2n)} = f(t, x(t), x'(t), \dots, x^{(2n-1)}(t)), \qquad t \in (0,1), \tag{1}$$

subject to one of following boundary value conditions

$$\begin{cases} x^{(2i-1)}(0) = 0, \quad i = 1, \dots, n, \\ x^{(2i-1)}(1) = 0, \quad i = 1, \dots, n-1, \\ x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i), \end{cases} \tag{2}$$

and

$$\begin{cases} x^{(2i-1)}(0) = 0, \quad i = 1, \dots, n, \\ x^{(2i)}(1) = 0, \quad i = 1, \dots, n-1, \\ x(0) = \sum_{i=1}^{m} \beta_i x(\xi_i), \end{cases} \tag{3}$$

where $f : [0,1] \times R^{2n} \to R$ is a continuous function, $n \geq 1$ an integer,

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$0 < \xi_1 < \cdots < \xi_m < 1$ and $\beta_i \in R$ for $i = 1, \ldots, m$. Our purpose here is to provide sufficient conditions for the existence of solutions of boundary value problem (1) and (2) and boundary value problem (1) and (3) at resonance and at non-resonance. These will be done by applying the well known coincidence degree theory and Schauder fixed point theorem.

The motivation for this paper is as follows: There were many papers concerned with the solvability of the second-order differential equations

$$x''(t) + f(t, x(t), x'(t)) = 0, \qquad t \in (0, 1), \tag{4}$$

subject to two-point boundary conditions

$$\alpha x(0) - \beta x'(0) = \delta x(1) + \gamma x'(1) = 0,$$

or the different multi-point boundary conditions at resonance or at non-resonance, we refer the readers to [1–8] and the references therein. For example, in [6], Liu and Yu studied the solvability of equation (4) subject to boundary conditions

$$x'(0) = 0, \qquad x(1) = \sum_{i=1}^{m} \alpha_i x(\xi_i),$$

where $\sum_{i=1}^{m} \alpha_i = 1$, which shows that such a problem is a resonance problem. They proved that under some assumptions it has at least one solution, one of the main assumptions is as follows:

$$|f(t, x, y)| \le a(t)|x| + b(t)|y| + p(t)|x|^{\delta} + q(t)|y|^{\theta} + r(t), \tag{$*$}$$

where a, b, p, q are non-negative continuous functions and r is a continuous function. To the best of our knowledge, the existence of solutions of **multi-point boundary value problems at resonance for higher order differential equations** were not investigated till now. The question is that under what conditions above problems have solutions if $(*)$ is not valid and under what conditions BVP(1) and (2), BVP(1) and (3) have positive solutions?

On the other hand, the solvability of fourth-order differential equations

$$x^{(4)}(t) = f(t, x(t), -x''(t)), \qquad t \in (0, 1), \tag{5}$$

or

$$x^{(4)}(t) = f(t, x(t)), \qquad t \in (0, 1), \tag{6}$$

subject to different boundary conditions have been studied by many authors, please see [16–21]. However, the solvability problems of equations (5) or (6) subject to one of following boundary value conditions

$$x(0) = x'(0) = x''(1) = x'''(0) = 0,$$

and

$$x(1) = x'(0) = x'(1) = x'''(0) = 0$$

have not been studied.

Very recently, Chyan and Henderson, in [14], studied the following $2m^{th}$-order differential equation

$$x^{(2m)}(t) = f(t, x(t), x''(t), \dots, x^{2(m-1)}(t)), \qquad 0 < t < 1, \tag{7}$$

with either the Lidstone boundary value condition

$$x^{(2i)}(0) = x^{(2i)}(1) = 0 \qquad \text{for } i = 0, 1, \dots, m-1, \tag{8}$$

or the focal boundary value condition

$$x^{(2i+1)}(0) = x^{(2i)}(1) = 0 \qquad \text{for } i = 0, 1, \dots, m-1. \tag{9}$$

They proved the existence of at least one positive solution in the case either f is super-linear or f is sub-linear.

The similar problems were also investigated in [15] by Palamides by using an analysis of the corresponding field on the face-plane and the well known Sperner's Lemma. The method there is different from that in [10–14]. In the papers mentioned above, the nonlinearity f depends on $x, x'', \dots, x^{(2(m-1))}$.

For BVP(1) and (2) or BVP(1) and (3), the corresponding linear differential equation is

$$(-1)^{n-1} x^{(2n)} = 0, \qquad t \in (0, 1). \tag{10}$$

It is easy to know that equation (10) subject to boundary conditions (2) or (3) has nontrivial solutions $x(t) = c$ if $\sum_{i=1}^{m} \beta_i = 1$, where $c \in R$. As usual, we say that BVP(1) and (2) and BVP(1) and (3) are resonance problems. The problem appears naturally considering these boundary value problems:

(P). Under what conditions problem (1) and (2) and problem (1) and (3) has at least one positive solution?

In this paper, we will solve above problems, please see theorem 2.1–2.6. The results obtained are new.

By the way, in a recent paper [22], the authors studied the following BVPs which consist of the equation

$$(-1)^{n-1} x^{(2n)} = f(t, x(t), x'(t), \dots, x^{(2n-1)}(t)), \qquad t \in (0, 1),$$

and one of the following boundary value conditions

$$\begin{cases} x^{(2i)}(0) = 0, & i = 0, 1, \dots, n-1, \\ x^{(2i)}(1) = 0, & i = 0, 1, \dots, n-1, \end{cases}$$

and

$$\begin{cases} x^{(2i)}(0) = 0, & i = 0, 1, \ldots, n-1, \\ x^{(2i+1)}(1) = 0, & i = 0, 1, \ldots, n-1. \end{cases}$$

They established some new existence results for the solutions of above BVPs.

2. Main results

In this section, we establish sufficient conditions for the existence of at least one solution of BVP(1)–(2) and BVP(1) and (3). For convenience, we first introduce some notations and an abstract existence theorem by Gaines and Mawhin [9], which can be see in [5–8].

Let X and Y be Banach spaces, $L : \mathrm{dom}\, L \subset X \to Y$ be a Fredholm operator of index zero, $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\mathrm{Im}\, P = \mathrm{Ker}\, L, \quad \mathrm{Ker}\, Q = \mathrm{Im}\, L, \quad X = \mathrm{Ker}\, L \oplus \mathrm{Ker}\, P, \quad Y = \mathrm{Im}\, L \oplus \mathrm{Im}\, Q.$$

It follows that

$$L|_{\mathrm{dom}\, L \cap \mathrm{Ker}\, P} : \mathrm{dom}\, L \cap \mathrm{Ker}\, P \to \mathrm{Im}\, L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $\mathrm{dom}\, L \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact.

Theorem GM[9]. *Let L be a Fredholm operator of index zero and let N be L-compact on Ω. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\mathrm{dom}\, L/\mathrm{Ker}\, L) \cap \partial\Omega] \times (0, 1)$;*

(ii) *$Nx \notin \mathrm{Im}\, L$ for every $x \in \mathrm{Ker}\, L \cap \partial\Omega$;*

(iii) *$\deg(\Lambda QN|_{\mathrm{Ker}\, L}, \Omega \cap \mathrm{Ker}\, L, 0) \neq 0$, where $\Lambda : Y/\mathrm{Im}\, L \to \mathrm{Ker}\, L$ is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $\mathrm{dom}\, L \cap \overline{\Omega}$.

We use the classical Banach space $C^k[0, 1]$, let $X = C^{2n-1}[0, 1]$ and $Y = C^0[0, 1]$. Y is endowed with the norm $\|y\|_\infty = \max_{t \in [0,1]} |y(t)|$, X is endowed with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \ldots, \|x^{(2n-1)}\|_\infty\}$. Define the linear operator L and the nonlinear operator N by

$$L : X \cap \mathrm{dom}\, L \to Y, \qquad Lx(t) = (-1)^{n-1} x^{(2n)}(t) \qquad \text{for } x \in X \cap \mathrm{dom}\, L,$$

$$N : X \to Y \qquad Nx(t) = f(t, x(t), x'(t), \ldots, x^{(2n-1)}(t)), \qquad \text{for } x \in X,$$

respectively, where

$$\operatorname{dom} L = \left\{ x \in C^{n-1}[0,1], x^{(2i-1)}(0) = 0 = x^{(2i-1)}(1) \text{ for } i = 1, \dots, n-1 \right.$$
$$\left. x^{(2n-1)}(0) = 0, x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i) \right\}.$$

Let $G_0(t,s)$ be the Green's function of problem

$$-u''(t) = \alpha(t), \qquad u(0) = u(1) = 0$$

for some α, let

$$G_k(t,s) = \int_0^1 G_0(t,\tau) G_{k-1}(\tau,s) d\tau, \qquad k = 1, \dots, n-1.$$

Lemma 2.1. *For problem* (1) *and* (2), *let* $\sum_{i=1}^{m} \beta_i = 1$. *Suppose there is nonnegative integer* k *such that*

$$\Delta = \int_0^1 \int_0^1 G_{n-2}(s,\tau) \int_0^\tau u^k \, du d\tau ds - \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \int_0^1 G_{n-2}(s,\tau) \int_0^\tau u^k \, du d\tau ds \neq 0.$$

Then the following results hold.

(i) $\operatorname{Ker} L = \{x(t) \equiv c, t \in [0,1], c \in R\}$;

(ii) $\operatorname{Im} L = \left\{ y \in Y, \begin{array}{l} \int_0^1 \int_0^1 G_{n-2}(s,\tau) \int_0^\tau y(u) du d\tau ds \\ \qquad = \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \int_0^1 G_{n-2}(s,\tau) \int_0^\tau y(u) du d\tau ds \end{array} \right\}$;

(iii) *L is a Fredholm operator of index zero;*

(iv) *There are projectors $P : X \to X$ and $Q : Y \to Y$ such that* $\operatorname{Ker} L = \operatorname{Im} P$ *and* $\operatorname{Ker} Q = \operatorname{Im} L$. *Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap \operatorname{dom} L \neq \Phi$, then N is L-compact on $\overline{\Omega}$;*

(v) *$x(t)$ is a solution of BVP(1) and (2) if and only if x is a solution of the operator equation $Lx = Nx$ in* $\operatorname{dom} L$.

Proof. (i) The proof is easy and is omitted.

(ii) If $y \in \operatorname{Im} L$, then

$$\begin{aligned} &(-1)^{n-1} x^{(2n)} = y(t), \qquad t \in (0,1), \\ &x^{(2i-1)}(0) = x^{(2i-1)}(1) = 0, \qquad i = 1, \dots, n-1, \\ &x^{(2n-1)}(0) = 0, \qquad x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i). \end{aligned} \tag{11}$$

This implies $x^{(2n-1)}(t) = (-1)^{n-1} \int_0^t y(u) du$ since $x^{(2n-1)}(0) = 0$. We get

$$x^{(2n-3)}(t) = (-1)^{n-2} \int_0^1 G_0(t,\tau) \int_0^\tau y(u) du d\tau,$$

Similarly, we get

$$x'(t) = \int_0^1 G_{n-2}(t,\tau)\int_0^\tau y(u)du d\tau.$$

So

$$x(t) = c + \int_0^t \int_0^1 G_{n-2}(s,\tau)\int_0^\tau y(u)du d\tau ds. \tag{12}$$

It follows from $x(1) = \sum_{i=1}^m \beta_i x(\xi_i)$ that

$$\int_0^1 \int_0^1 G_{n-2}(s,\tau)\int_0^\tau y(u)du d\tau ds = \sum_{i=1}^m \beta_i \int_0^{\xi_i} \int_0^1 G_{n-2}(s,\tau)\int_0^\tau y(u)du d\tau ds. \tag{13}$$

On the other hand, assume (13) holds. Let

$$x(t) = c + \int_0^t \int_0^1 G_{n-2}(s,\tau)\int_0^\tau y(u)du d\tau ds.$$

Then $x(t)$ satisfies (11). Hence (ii) is complete.

(iii) From (i), $\dim \operatorname{Ker} L = 1$. On the other hand, for $y \in Y$, let

$$y_0 = y - \frac{t^k}{\Delta}\biggl(\int_0^1 \int_0^1 G_{n-2}(s,\tau)\int_0^\tau y(u)du d\tau ds \\ - \sum_{i=1}^m \beta_i \int_0^{\xi_i} \int_0^1 G_{n-2}(s,\tau)\int_0^\tau y(u)du d\tau ds\biggr).$$

It is easy to check that $y_0 \in \operatorname{Im} L$. Let

$$\bar{R} = \{ct^k : t \in [0,1], c \in R\}.$$

We get $Y = \bar{R} + \operatorname{Im} L$. Again, $\bar{R} \cap \operatorname{Im} L = \{0\}$, so $Y = \bar{R} \oplus \operatorname{Im} L$. Hence $\dim Y/\operatorname{Im} L = 1$. On the other hand, f is continuous and $\operatorname{Im} L$ is closed. So L is a Fredholm operator of index zero.

(iv) Define the projectors $P : X \to X$ and $Q : Y \to Y$ by

$$Px(t) = x(0) \qquad \text{for } x \in X,$$

$$Qy(t) = \frac{t^k}{\Delta}\biggl(\int_0^1 \int_0^1 G_{n-2}(s,\tau)\int_0^\tau y(u)du d\tau ds \\ - \sum_{i=1}^m \beta_i \int_0^{\xi_i} \int_0^1 G_{n-2}(s,\tau)\int_0^\tau y(u)du d\tau ds\biggr) \qquad \text{for } y \in Y.$$

It is easy to check that $\operatorname{Ker} L = \operatorname{Im} P$ and $\operatorname{Im} L = \operatorname{Ker} Q$. The generalized inverse $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ of L can be written by

$$K_P y(t) = \int_0^t \int_0^1 G_{n-2}(s,\tau) \int_0^\tau y(u) du d\tau ds.$$

(v) The proof is easy and is omitted.

THEOREM 2.1. *Suppose* $\sum_{i=1}^m \beta_i = 1$ *and the following conditions hold.*

(A_1) *There are a continuous function* $e(t)$ *and nonnegative functions* $g_i(t,x)$ $(i = 0, 1, \dots, 2n-1)$ *such that* f *satisfies*

$$|f(t, x_0, x_1, \dots, x_{2n-1})| \le e(t) + \sum_{i=0}^{2n-1} g_i(t, x_i)$$

for all $t \in [0,1]$ *and* $(x_0, x_1, \dots, x_{2n-1}) \in R^{2n}$ *and*

$$\lim_{|x|\to\infty} \sup_{t\in[0,1]} \frac{|g_i(t,x)|}{|x|} = r_i, \qquad \text{for } i = 0, 1, \dots, 2n-1$$

with $r_i \ge 0$ *for* $i = 0, 1, \dots, 2n-1$*;*

(A_2) *There exist constants* $L \ge 0$, $\alpha > 0$ *and* $\alpha_i \ge 0$ $(i = 1, \dots, 2n-2)$ *such that*

$$|f(t, x_0, x-1, \dots, x_{2n-1})| \ge \alpha|x_0| - \sum_{i=1}^{2n-2} \alpha_i |x_i| - L$$

for all $t \in [0,1]$ *and* $(x_0, x_1, \dots, x_{2n-1}) \in R^{2n}$.

(A_3) *There is a constant* $M > 0$ *such that*

$$f(t, c, 0, \dots, 0) > 0$$

for $t \in [0,1]$ *and* $c > M$ *or*

$$f(t, c, 0, \dots, 0) < 0$$

for $t \in [0,1]$ *and* $c < -M$*;*

Then $BVP(1)$ *and* (2) *has at least one solution provided*

$$\left(1 + \frac{\sum_{i=1}^{2n-2} \alpha_i}{\alpha}\right) r_0 + \sum_{i=1}^{2n-1} r_i < \frac{1}{2}. \tag{14}$$

PROOF. To apply Theorem GM, we should define an open bounded subset Ω of X so that (i), (ii) and (iii) of Theorem GM hold. It is based upon three steps to obtain Ω. The proof of this theorem is divide into four steps.

Step 1. Let

$$\Omega_1 = \{x \in \text{dom } L/\text{Ker } L, Lx = \lambda Nx \text{ for some } \lambda \in (0,1)\}.$$

We prove Ω_1 is bounded. For $x \in \Omega_1$, it is easy to show that there is $\xi_i \in [0,1]$ such that $x^{(2i)}(\xi_i) = 0$ for $i = 1, 2, \dots, n-1$ and

$$\begin{aligned}
|x^{(2n-2)}(t)| &= \left| x^{(2n-2)}(\xi_{n-1}) + \int_t^{\xi_{n-1}} x^{(2n-1)}(s)ds \right| \le \int_0^1 |x^{(2n-1)}(s)|ds, \\
|x^{(2n-3)}(t)| &= \left| x^{(2n-3)}(0) + \int_0^t x^{(2n-2)}(s)ds \right| \\
&\le \int_0^1 |x^{(2n-2)}(s)|ds \le \int_0^1 |x^{(2n-1)}(s)|ds, \\
&\cdots\cdots \ \cdots\cdots, \\
|x'(t)| &\le \int_0^1 |x^{(2n-1)}(s)|ds.
\end{aligned} \tag{15}$$

Since $x \in \text{dom } L$, it follows that $Nx \in \text{Im } L$, so

$$\begin{aligned}
&\int_0^1 \int_0^1 G_{n-2}(s,\tau) \int_0^\tau f(u, x(u), x'(u), \dots, x^{(2n-1)}(u))dud\tau ds \\
&\quad = \sum\nolimits_{i=1}^m \beta_i \int_0^{\xi_i} \int_0^1 G_{n-2}(s,\tau) \int_0^\tau f(u, x(u), x'(u), \dots, x^{(2n-1)}(u))dud\tau ds.
\end{aligned}$$

i.e.

$$\sum_{i=1}^m \beta_i \int_{\xi_i}^1 \int_0^1 G_{n-2}(s,\tau) \int_0^\tau f(u, x(u), x'(u), \dots, x^{(2n-1)}(u))dud\tau ds = 0.$$

Since $\beta_i \ge 0$ for $i = 1, \dots, m$, this inequality implies that that there is $\xi \in (0,1)$ such that $f(\xi, x(\xi), x'(\xi), \dots, x^{(2n-1)}(\xi)) = 0$. By (A_2) and (15), we see that

$$\begin{aligned}
|x(\xi)| &\le \frac{L}{\alpha} + \frac{1}{\alpha} \sum_{i=1}^{2n-2} \alpha_i |x^{(i)}(\xi)| \\
&\le \frac{L}{\alpha} + \frac{1}{\alpha} \sum_{i=1}^{2n-2} \alpha_i \int_0^1 |x^{(2n-1)}(s)|ds.
\end{aligned}$$

Hence

$$
\begin{aligned}
|x(t)| &\le |x(\xi)| + \left|\int_{\xi}^{t} x'(s)ds\right| \\
&\le \frac{L}{\alpha} + \frac{1}{\alpha}\sum\nolimits_{i=1}^{2n-2}\alpha_i \int_0^1 |x^{(2n-1)}(s)|ds + \int_0^1 |x^{(2n-1)}(s)|ds \\
&= A\left(M + \int_0^1 |x^{(2n-1)}(s)|\right)ds, \qquad (16)
\end{aligned}
$$

where $A = 1 + \frac{\sum_{i=1}^{2n-2}\alpha_i}{\alpha}$ and $M = \frac{L}{\alpha A}$. It suffices to prove there is a constant $B > 0$ such that

$$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \ldots, \|x^{(2n-1)}\|_\infty\} \le B.$$

We divide this step into two sub-steps.

Sub-step 1.1. We prove that there is a constant $\overline{M} > 0$ such that

$$\int_0^T |x^{(2n-1)}(s)|^2 ds \le \overline{M}.$$

For $x \in \Omega_1$, we have

$$(-1)^{n-1}x^{(2n)}(t) = \lambda f(t, x(t), x'(t), \ldots, x^{(2n-1)}(t)). \qquad (17)$$

It is easy to know that there is $\eta \in [0, 1]$ such that

$$\int_0^1 |x^{(2n-1)}(s)|^2 ds = |x^{(2n-1)}(\eta)|^2.$$

Multiplying two side of (17) by $x^{(2n-1)}(t)$ and integrating from 0 to η, using (A_1), we get

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 |x^{(2n-1)}(s)|^2 ds \\
&\quad = \frac{1}{2}|x^{(2n-1)}(\eta)|^2 \\
&\quad = \frac{1}{2}|x^{(2n-1)}(\eta)|^2 - \frac{1}{2}|x^{(2n-1)}(0)|^2 \\
&\quad = \lambda\int_0^{\eta}(-1)^{n-1} f(s, x(s), x'(s), \ldots, x^{(2n-1)}(s))x^{(2n-1)}(s)ds
\end{aligned}
$$

$$\leq \int_0^\eta |f(s, x(s), x'(s), \ldots, x^{(2n-1)}(s))|\,|x^{(2n-1)}(s)|ds$$

$$\leq \sum_{i=0}^{2n-1} \int_0^1 g_i(s, x^{(i)}(s))x^{(2n-1)}(s)ds + \int_0^1 |e(s)|\,|x^{(2n-1)}(s)|ds$$

$$\leq \sum_{i=0}^{2n-1} \int_0^1 |g_i(s, x^{(i)}(s))|\,|x^{(2n-1)}(s)|ds + \|e\|_\infty \int_0^1 |x^{(2n-1)}(s)|ds$$

$$\leq \sum_{i=0}^{2n-1} \int_0^1 |g_i(s, x^{(i)}(s))|\,|x^{(2n-1)}(s)|ds + \|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2}.$$

Let $\varepsilon > 0$ satisfy

$$\frac{1}{2} > \left(1 + \frac{\sum_{i=1}^{2n-2} \alpha_i}{\alpha}\right)(r_0 + \varepsilon) + \sum_{i=1}^{2n-1}(r_i + \varepsilon).$$

For such a $\varepsilon > 0$, we find from (A_1) that there is a constant $\delta > M$ such that for every $i = 0, 1, \ldots, 2n-1$,

$$|g_i(t, x)| < (r_i + \varepsilon)|x| \qquad \text{uniformly for } t \in [0, 1] \text{ and } |x| > \delta.$$

Let, for $i = 0, 1, \ldots, 2n-1$,

$$\Delta_{1,i} = \{t : t \in [0, 1], |x^{(i)}(t)| \leq \delta\},$$

$$\Delta_{2,i} = \{t : t \in [0, 1], |x^{(i)}(t)| > \delta\},$$

$$g_{\delta,i} = \max_{t \in [0,1], |x| \leq \delta} |g_i(t, x)|.$$

Then

$$\begin{aligned}\frac{1}{2}\int_0^1 |x^{(2n-1)}(s)|^2 ds \leq{} & \sum_{i=0}^{2n-1} \int_{\Delta_{1,i}} |g_i(s, x^{(i)}(s))|\,|x^{(2n-1)}(s)|ds \\ & + \sum_{i=0}^{2n-1} \int_{\Delta_{2,i}} |g_i(s, x^{(i)}(s))|\,|x^{(2n-1)}(s)|ds \\ & + \|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2} \\ \leq{} & \sum_{i=0}^{2n-1} g_{\delta,i} \int_0^1 |x^{(i)}(s)|ds + \sum_{i=0}^{2n-1}(r_i + \varepsilon) \int_0^1 |x^{(i)}(s)|\,|x^{(2n-1)}(s)|ds \\ & + \|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2}.\end{aligned}$$

Again using (15) and (16), we get

$$
\begin{aligned}
&\int_0^1 |x(s)|ds \le A\left(M + \int_0^1 |x^{(2n-1)}(s)|ds\right),\\
&\int_0^1 |x^{(i)}(s)|ds \le \int_0^1 |x^{(2n-1)}(s)|ds,\\
&\int_0^1 |x(s)|\,|x^{(2n-1)}(s)|ds \le A\left(M + \int_0^1 |x^{(2n-1)}(s)|ds\right)\int_0^1 |x^{(2n-1)}(s)|ds,\\
&\int_0^1 |x^{(i)}(s)|\,|x^{(2n-1)}(s)|ds \le \left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^2, \qquad i = 1,\dots,2n-2.
\end{aligned}
\tag{18}
$$

So from

$$
\int_0^1 |x^{(2n-1)}(s)|ds \le \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2},
$$

we get

$$
\begin{aligned}
\frac{1}{2}\int_0^1 |x^{(2n-1)}(s)|^2 ds \le{}& g_{\delta,0}A\left(M + \int_0^1 |x^{(2n-1)}(s)|ds\right) + \sum_{i=1}^{2n-1} g_{\delta,i}\int_0^1 |x^{(2n-1)}(s)|ds\\
&+ (r_0+\varepsilon)A\left(M + \int_0^1 |x^{(2n-1)}(s)|ds\right)\int_0^1 |x^{(2n-1)}(s)|ds\\
&+ \sum_{i=1}^{2n-2}(r_i+\varepsilon)\left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^2\\
&+ \|e\|_\infty\left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2} + (r_{2n-1}+\varepsilon)\int_0^1 |x^{(2n-1)}(s)|^2 ds\\
\le{}& g_{\delta,0}A\left[M + \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2}\right]\\
&+ \sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2}\\
&+ (r_0+\varepsilon)A\left[M\left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2} + \int_0^1 |x^{(2n-1)}(s)|^2 ds\right]\\
&+ \sum_{i=1}^{2n-1}(r_i+\varepsilon)\int_0^1 |x^{(2n-1)}(s)|^2 ds + \|e\|_\infty\left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2}.
\end{aligned}
$$

i.e.

$$\left(\frac{1}{2}-A(r_0+\varepsilon)-\sum_{i=1}^{2n-1}(r_i+\varepsilon)\right)\int_0^1|x^{(2n-1)}(s)|^2ds$$
$$\le g_{\delta,0}A\left[M+\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}\right]+\sum_{i=1}^{2n-1}g_{\delta,i}\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}$$
$$+(r_0+\varepsilon)AM\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}+\|e\|_\infty\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}.$$

From the definition of ε, we find that $\frac{1}{2}-A(r_0+\varepsilon)-\sum_{i=1}^{2n-1}(r_i+\varepsilon)>0$ and that there is a constant $\overline{M}>0$ such that

$$\int_0^1|x^{(2n-1)}(s)|^2ds\le\overline{M}.$$

Sub-step 1.2. Prove there is $B>0$ such that $\|x\|\le B$.

From sub-step 1.1, we have

$$\|x\|_\infty\le A\left(M+\int_0^1|x^{(2n-1)}(s)|ds\right)$$
$$\le A\left(M+\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}\right)$$
$$\le A(M+\overline{M}^{1/2}).$$
$$\|x^{(i)}\|_\infty\le\int_0^1|x^{(2n-1)}(s)|ds$$
$$\le\left(\int_0^1|x^{(2n-1)}(s)|^2ds\right)^{1/2}$$
$$\le\overline{M}^{1/2},\qquad i=1,\dots,n-2.$$

Multiplying two side of (17) by $x^{(2n-1)}(t)$, integrating them from 0 to t, using (A_1), we get

$$\frac{1}{2}|x^{(2n-1)}(t)|^2=\lambda\int_0^t(-1)^{n-1}f(s,x(s),x'(s),\dots,x^{(2n-1)}(s))x^{(2n-1)}(s)ds$$
$$=\int_0^1|f(s,x(s),x'(s),\dots,x^{(2n-1)}(s))x^{(2n-1)}(s)|\,|x^{(2n-1)}(s)|ds$$
$$\le\sum_{i=0}^{2n-1}\int_0^1 g_i(s,x^{(i)}(s))|x^{(2n-1)}(s)|ds+\int_0^1|e(s)|\,|x^{(2n-1)}(s)|ds$$

$$\leq \sum_{i=0}^{2n-1}\int_0^1 |g_i(s,x^{(i)}(s))|\,|x^{(2n-1)}(s)|ds + \|e\|_\infty \int_0^1 |x^{(2n-1)}(s)|ds$$

$$\leq \sum_{i=0}^{2n-1}\int_0^1 |g_i(s,x^{(i)}(s))|\,|x^{(2n-1)}(s)|ds + \|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2}.$$

Similarly to step 1.1, we can get

$$\begin{aligned}\frac{1}{2}|x^{(2n-1)}(t)|^2 \leq{} & \sum_{i=0}^{2n-1}\int_{\Delta_{1,i}} |g_i(s,x^{(i)}(s))|\,|x^{(2n-1)}(s)|ds \\ & + \sum_{i=0}^{2n-1}\int_{\Delta_{2,i}} |g_i(s,x^{(i)}(s)|\,|x^{(2n-1)}(s)|ds + \|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2} \\ \leq{} & \sum_{i=0}^{2n-1} g_{\delta,i}\int_0^1 |x^{(i)}(s)|ds + \sum_{i=0}^{2n-1}(r_i+\varepsilon)\int_0^1 |x^{(i)}(s)|\,|x^{(2n-1)}(s)|ds \\ & + \|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2}.\end{aligned}$$

Using (18), we get

$$\begin{aligned}\frac{1}{2}|x^{(2n-1)}(t)|^2 \leq{} & g_{\delta,0}A\left(M+\int_0^1 |x^{(2n-1)}(s)|ds\right) + \sum_{i=1}^{2n-1} g_{\delta,i}\int_0^1 |x^{(2n-1)}(s)|ds \\ & + (r_0+\varepsilon)A\left(M+\int_0^1 |x^{(2n-1)}(s)|ds\right)\int_0^1 |x^{(2n-1)}(s)|ds \\ & + \sum_{i=1}^{2n-2}(r_i+\varepsilon)\left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^2 + \|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2} \\ & + (r_{2n-1}+\varepsilon)\int_0^1 |x^{(2n-1)}(s)|^2 ds \\ \leq{} & g_{\delta,0}A\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2}\right] + \sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2} \\ & + (r_0+\varepsilon)A\left[M\left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2} + \int_0^1 |x^{(2n-1)}(s)|^2 ds\right] \\ & + \sum_{i=1}^{2n-1}(r_i+\varepsilon)\int_0^1 |x^{(2n-1)}(s)|^2 ds + \|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^2 ds\right)^{1/2}\end{aligned}$$

$$\leq g_{\delta,0}A(M+\overline{M}^{1/2})+\sum_{i=1}^{2n-1}g_{\delta,i}\overline{M}^{1/2}$$
$$+(r_0+\varepsilon)A[M\overline{M}^{1/2}+\overline{M}]+\sum_{i=1}^{2n-1}(r_i+\varepsilon)\overline{M}+\|e\|_\infty\overline{M}^{1/2}.$$

So there is $M_3'>0$ such that $|x^{(2n-1)}(t)|\leq M_3'$. It follows from above discussion that there is $B>0$ such that

$$\|x\|\leq B.$$

Hence Ω_1 is bounded. This completes the step 1.

Step 2. Let

$$\Omega_2=\{x\in \operatorname{Ker} L, Nx\in \operatorname{Im} L\}.$$

We prove Ω_2 is bounded. Suppose $x\in\Omega_2$, then $x(t)=c\in R$, we prove $|c|\leq M$. In fact, if $c>M$, then (A_3) implies $f(t,c,0,\dots,0)>0$, then

$$\begin{aligned}&\int_0^1\int_0^1 G_{n-2}(s,\tau)\int_0^\tau f(u,x(u),x'(u),\dots,x^{(2n-1)}(u))du\,d\tau\,ds\\&\quad-\sum_{i=1}^m\beta_i\int_0^{\xi_i}\int_0^1 G_{n-2}(s,\tau)\int_0^\tau f(u,x(u),x'(u),\dots,x^{(2n-1)}(u))du\,d\tau\,ds\\&=\sum_{i=1}^m\beta_i\int_{\xi_i}^1\int_0^1 G_{n-2}(s,\tau)\int_0^\tau f(u,c,0,\dots,0)du\,d\tau\,ds>0.\end{aligned}$$

Similarly, if $c<-M$, then (A_3) implies $f(t,c,0,\dots,0)<0$, we have

$$\begin{aligned}&\int_0^1\int_0^1 G_{n-2}(s,\tau)\int_0^\tau f(u,x(u),x'(u),\dots,x^{(2n-1)}(u))du\,d\tau\,ds\\&\quad-\sum_{i=1}^m\beta_i\int_0^{\xi_i}\int_0^1 G_{n-2}(s,\tau)\int_0^\tau f(u,x(u),x'(u),\dots,x^{(2n-1)}(u))du\,d\tau\,ds\\&=\sum_{i=1}^m\beta_i\int_{\xi_i}^1\int_0^1 G_{n-2}(s,\tau)\int_0^\tau f(u,c,0,\dots,0)du\,d\tau\,ds<0.\end{aligned}$$

On the other hand, if $x\in \operatorname{Ker} L$ and $Nx\in \operatorname{Im} L$, we have $QNx=0$, i.e.

$$\int_0^1 \int_0^1 G_{n-2}(s,\tau) \int_0^\tau f(u, x(u), x'(u), \dots, x^{(2n-1)}(u)) du d\tau ds$$
$$- \sum_{i=1}^m \beta_i \int_0^{\xi_i} \int_0^1 G_{n-2}(s,\tau) \int_0^\tau f(u, x(u), x'(u), \dots, x^{(2n-1)}(u)) du d\tau ds$$
$$= \sum_{i=1}^m \beta_i \int_{\xi_i}^1 \int_0^1 G_{n-2}(s,\tau) \int_0^\tau f(u, c, 0, \dots, 0) du d\tau ds = 0.$$

This is a contradiction. So $|c| \le M$. This shows that Ω_2 is bounded.

Step 3. Let

$$\Omega_3 = \{x \in \operatorname{Ker} L, \operatorname{sgn}(\Delta)\lambda \wedge x + (1-\lambda)QNx = 0, \lambda \in [0,1]\},$$

where $\wedge : \operatorname{Ker} L \to \operatorname{Im} Q$ is the linear isomorphism given by $\wedge(c) = ct^k$ for all $c \in R$. Now we show that Ω_3 is bounded. Suppose $x_n(t) = c_n \in \Omega_3$ and $|c_n| \to +\infty$ as n tends to infinity. Then there exist $\lambda_n \in [0,1]$ such that

$$\operatorname{sgn}(\Delta)\lambda_n c_n + \frac{1-\lambda_n}{\Delta}\left(\int_0^1 \int_0^1 G_{n-2}(s,\tau) \int_0^\tau f(u, x(u), x'(u), \dots, x^{(2n-1)}(u)) du d\tau ds \right.$$
$$\left. - \sum\nolimits_{i=1}^m \beta_i \int_0^{\xi_i} \int_0^1 G_{n-2}(s,\tau) \int_0^\tau f(u, x(u), x'(u), \dots, x^{(2n-1)}(u)) du d\tau ds \right)$$
$$= \operatorname{sgn}(\Delta)\lambda_n c_n + \frac{1-\lambda_n}{\Delta} \sum\nolimits_{i=1}^m \beta_i \int_{\xi_i}^1 \int_0^1 G_{n-2}(s,\tau) \int_0^\tau f(u, c, 0, \dots, 0) du d\tau ds = 0.$$

So

$$\operatorname{sgn}(\Delta)\Delta\lambda_n c_n = -(1-\lambda_n) \sum_{i=1}^m \beta_i \int_{\xi_i}^1 \int_0^1 G_{n-2}(s,\tau) \int_0^\tau f(u, c, 0, \dots, 0) du d\tau ds.$$

It is easy to see that λ_n has a convergent subsequence, without loss of generality, suppose $\lambda_n \to \lambda_0$. Again, since $|c_n| \to +\infty$, there two cases to be considered, i.e. there is subsequence of c_n that tends to $+\infty$ (without loss of generality suppose $c_n \to +\infty$) or there is subsequence of c_n that tends to $-\infty$ (without loss of generality suppose $c_n \to -\infty$). If $c_n \to +\infty$ as n tends to infinity. Then for sufficiently large n, we have $c_n > M$. Hence, using (A_3), we see

$$\operatorname{sgn}(\Delta)\Delta\lambda_n c_n^2 = -(1-\lambda_n) c_n \sum_{i=1}^m \beta_i \int_{\xi_i}^1 \int_0^1 G_{n-2}(s,\tau) \int_0^\tau f_c(u) du d\tau ds$$
$$< 0,$$

a contradiction, where $f_c(u) = f(u, c, 0, \dots, 0)$. If $c_n \to -\infty$, then for sufficiently large n, $c_n < -M$. Hence using (A_3), we see

$$\begin{aligned}\operatorname{sgn}(\Delta)\Delta\lambda_n c_n^2 &= -(1-\lambda_n)c_n \sum_{i=1}^{m} \beta_i \int_{\xi_i}^{1} \int_0^1 G_{n-2}(s,\tau) \int_0^{\tau} f_c(u)\,du\,d\tau\,ds \\ &< 0,\end{aligned}$$

a contradiction. So Ω_3 is bounded.

In the following, we shall show that all conditions of Theorem GM are satisfied. Set Ω be a open bounded subset of X such that $\Omega \supset \bigcup_{i=1}^{3} \overline{\Omega_i}$. By Lemma 2.1, L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the definition of Ω, we have

(a) $Lx \neq \lambda Nx$ for $x \in (\operatorname{dom} L/\operatorname{Ker} L) \cap \partial\Omega$ and $\lambda \in (0,1)$;

(b) $Nx \notin \operatorname{Im} L$ for $x \in \operatorname{Ker} L \cap \partial\Omega$.

Step 4. We prove

(c) $\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.

In fact, let $H(x,\lambda) = \lambda \wedge x + (1-\lambda)QNx$. According the definition of Ω, we know $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \operatorname{Ker} L$, thus by homotopy property of degree,

$$\begin{aligned}\deg(QN \mid \operatorname{Ker} L, \Omega \cap \operatorname{Ker} L, 0) &= \deg(H(\cdot\,,0), \Omega \cap \operatorname{Ker} L, 0) \\ &= \deg(H(\cdot\,,1), \Omega \cap \operatorname{Ker} L, 0) \\ &= \deg(I, \Omega \cap \operatorname{Ker} L, 0) \neq 0.\end{aligned}$$

Thus by Theorem GM, $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$, which is a solution of BVP(1)–(2). The proof is complete.

Theorem 2.2. *Suppose* $\sum_{i=1}^{m} \beta_i = 1$ *and all conditions of Theorem 2.1, i.e.* (A_1)–(A_3), *hold. Then BVP*(1) *and* (3) *has at least one solution provided* (14) *holds.*

Proof. The proof is similar to that of Theorem 2.1 and is omitted.

Theorem 2.3. *Suppose* $\sum_{i=1}^{m} \beta_i = 1$ *and the following conditions hold.*

(A_1') *There are continuous functions* $h(t, x_0, x_1, \dots, x_{2n-1})$, $e(t)$ *and non-negative functions* $g_i(t,x)$ $(i = 0, 1, \dots, 2n-1)$ *and positive numbers* β *and* m *such that* f *satisfies*

$$(-1)^{n-1} f(t, x_0, x_1, \dots, x_{2n-1}) = e(t) + h(t, x_0, x_1, \dots, x_{2n-1}) + \sum_{i=0}^{2n-1} g_i(t, x_i),$$

and also that h *satisfies*

$$x_{2n-1} h(t, x_0, x_1, \dots, x_{2n-1}) \leq -\beta |x_{2n-1}|^{m+1}$$

for all $t \in [0,1]$ *and* $(x_0, x_1, \ldots, x_{2n-1}) \in R^{2n}$ *and*

$$\lim_{|x|\to\infty} \sup_{t\in[0,1]} \frac{|g_i(t,x)|}{|x|^m} = r_i, \qquad \textit{for } i = 0, 1, \ldots, 2n-1$$

with $r_i \geq 0$ *for* $i = 0, 1, \ldots, 2n-1$*;*

Furthermore, (A_2) *and* (A_3) *of Theorem 2.1 hold. Then* $BVP(1)$ *and* (2) *has at least one solution provided*

$$\left(1 + \frac{\sum_{i=1}^{2n-2} \alpha_i}{\alpha}\right)^m r_0 + \sum_{i=1}^{2n-1} r_i < \beta. \tag{19}$$

Proof. To apply Theorem GM, we should define an open bounded subset Ω of X so that (i), (ii) and (iii) of Theorem GM hold. It is based upon three steps to obtain Ω. The proof of this theorem is divide into four steps.

Step 1. Let

$$\Omega_1 = \{x \in \operatorname{dom} L/\operatorname{Ker} L, Lx = \lambda Nx \text{ for some } \lambda \in (0,1)\}.$$

We prove Ω_1 is bounded. Similar to Step 1 of Theorem 2.1, we have (15) and (16). It suffices to prove there is a constant $B > 0$ such that

$$\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \ldots, \|x^{(2n-1)}\|_\infty\} \leq B.$$

We divide this step into two sub-steps.

Sub-step 1.1. We prove that there is a constant $\overline{M} > 0$ such that

$$\int_0^T |x^{(2n-1)}(s)|^{m+1} ds \leq \overline{M}.$$

For $x \in \Omega_1$, we have

$$(-1)^{n-1} x^{(n)}(t) = \lambda f(t, x(t), x'(t), \ldots, x^{(2n-1)}(t)). \tag{20}$$

Multiplying two side of (20) by $x^{(2n-1)}(t)$ and integrating from 0 to 1, using (A_1'), we get

$$\begin{aligned}
0 &\leq \frac{1}{2}|x^{(2n-1)}(1)|^2 \\
&= \lambda \int_0^1 (-1)^{n-1} f(s, x(s), x'(s), \ldots, x^{(2n-1)}(s)) x^{(2n-1)}(s) ds \\
&= \lambda \bigg(\int_0^1 h(s, x(s), x'(s), \ldots, x^{(2n-1)}(s)) x^{(2n-1)}(s) ds \\
&\qquad + \sum_{i=0}^{2n-1} \int_0^1 g_i(s, x^{(i)}(s)) x^{(2n-1)}(s) ds + \int_0^1 e(s) x^{(2n-1)}(s) ds \bigg).
\end{aligned}$$

Thus, from the second part of (A_1'),

$$\begin{aligned}\lambda\beta\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds &\le -\lambda\int_0^1 h(s,x(s),x'(s),\dots,x^{(2n-1)}(s))x^{(2n-1)}(s)ds\\ &= \lambda\sum_{i=0}^{2n-1}\int_0^1 g_i(s,x^{(i)}(s))x^{(2n-1)}(s)ds+\lambda\int_0^1 e(s)x^{(2n-1)}(s)ds\\ &\le \lambda\sum_{i=0}^{2n-1}\int_0^1 |g_i(s,x^{(i)}(s))|\,|x^{(2n-1)}(s)|ds\\ &\quad+\lambda\int_0^1 |e(s)|\,|x^{(2n-1)}(s)|ds.\end{aligned}$$

Hence

$$\begin{aligned}\beta\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds \le{}& \sum_{i=0}^{2n-1}\int_0^1 |g_i(s,x^{(i)}(s))|\,|x^{(2n-1)}(s)|ds\\ &+\int_0^1 |e(s)|\,|x^{(2n-1)}(s)|ds.\end{aligned}$$

Let $\varepsilon>0$ satisfy

$$\beta>\left(1+\frac{\sum_{i=1}^{2n-2}\alpha_i}{\alpha}\right)^m(r_0+\varepsilon)+\sum_{i=1}^{2n-1}(r_i+\varepsilon).$$

By the conditions of theorem, we see $\varepsilon>0$. For such a $\varepsilon>0$, we find from (A_1') that there is a constant $\delta>M$ such that for every $i=0,1,\dots,2n-1$,

$$|g_i(t,x)|<(r_i+\varepsilon)|x|^m \qquad \text{uniformly for } t\in[0,1] \text{ and } |x|>\delta.$$

Let, for $i=0,1,\dots,2n-1$,

$$\begin{aligned}\Delta_{1,i}&=\{t: t\in[0,1], |x^{(i)}(t)|\le\delta\},\\ \Delta_{2,i}&=\{t: t\in[0,1], |x^{(i)}(t)|>\delta\},\\ g_{\delta,i}&=\max\nolimits_{t\in[0,1],|x|\le\delta}|g_i(t,x)|.\end{aligned}$$

Then

$$\begin{aligned}\beta\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds \le{}& \sum_{i=0}^{2n-1}\int_{\Delta_{1,i}} |g_i(s,x^{(i)}(s))|\,|x^{(2n-1)}(s)|ds\\ &+\sum_{i=0}^{2n-1}\int_{\Delta_{2,i}} |g_i(s,x^{(i)}(s))|\,|x^{(2n-1)}(s)|ds\\ &+\int_0^1 |e(s)|\,|x^{(2n-1)}(s)|ds\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=0}^{2n-1} g_{\delta,i}\int_0^1 |x^{(i)}(s)|ds \\
&\quad + \sum_{i=0}^{2n-1}(r_i+\varepsilon)\int_{\Delta_{2,i}} |x^{(i)}(s)|^m |x^{(2n-1)}(s)|ds \\
&\quad + \int_0^1 |e(s)|\,|x^{(2n-1)}(s)|ds \\
&\le \sum_{i=0}^{2n-1} g_{\delta,i}\int_0^1 |x^{(i)}(s)|ds + \sum_{i=0}^{2n-1}(r_i+\varepsilon)\int_0^1 |x^{(i)}(s)|^m |x^{(2n-1)}(s)|ds \\
&\quad + \int_0^1 |e(s)|\,|x^{(2n-1)}(s)|ds.
\end{aligned}
$$

Again

$$
\int_0^1 |x(s)|^m |x^{(2n-1)}(s)|ds \le A^m\left(M+\int_0^1 |x^{(2n-1)}(s)|ds\right)^m \int_0^1 |x^{(2n-1)}(s)|ds,
$$

and

$$
\begin{aligned}
&\int_0^1 |x^{(i)}(s)|^m |x^{(2n-1)}(s)|ds \\
&\qquad \le \left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^m \int_0^1 |x^{(2n-1)}(s)|ds, \qquad i=1,\dots,2n-2.
\end{aligned}
$$

So

$$
\begin{aligned}
\beta\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds &\le A^m\left(M+\int_0^1 |x^{(2n-1)}(s)|ds\right)^m (r_0+\varepsilon)\int_0^1 |x^{(2n-1)}(s)|ds \\
&\quad + \sum_{i=1}^{2n-1}(r_i+\varepsilon)\left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^m \int_0^T |x^{(2n-1)}(s)|ds \\
&\quad + \int_0^1 |e(s)|\,|x^{(2n-1)}(s)|ds + \sum_{i=0}^{2n-1} g_{\delta,i}\int_0^1 |x^{(i)}(s)|ds \\
&\le A^m\left(M+\int_0^1 |x^{(2n-1)}(s)|ds\right)^m (r_0+\varepsilon)\int_0^1 |x^{(2n-1)}(s)|ds \\
&\quad + \sum_{i=1}^{2n-2}(r_i+\varepsilon)\left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^m \int_0^1 |x^{(2n-1)}(s)|ds \\
&\quad + (r_{2n-1}+\varepsilon)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds + \|e\|_\infty \int_0^1 |x^{(2n-1)}(s)|ds
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{2n-1} g_{\delta,i}\int_0^1 |x^{(i)}(s)|ds + g_{\delta,0}\int_0^1 |x(s)|ds\\
&= A^m\left(M+\int_0^1 |x^{(2n-1)}(s)|ds\right)^m (r_0+\varepsilon)\int_0^1 |x^{(2n-1)}(s)|ds\\
&+\sum_{i=1}^{2n-2}(r_i+\varepsilon)\left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^{m+1}\\
&+(r_{2n-1}+\varepsilon)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds + \|e\|_\infty \int_0^1 |x^{(2n-1)}(s)|ds\\
&+\sum_{i=1}^{2n-1} g_{\delta,i}\int_0^1 |x^{(2n-1)}(s)|ds + g_{\delta,0}A\left(M+\int_0^1 |x^{(2n-1)}(s)|ds\right).
\end{aligned}
$$

We claim that there is a constant $\sigma\in(0,1)$, independent of λ, such that $(1+x)^n \le 1+(n+1)x$ for all $x\in(0,\sigma)$. In fact, let $q(x)=(1+x)^n-(1+(n+1)x)$, we see $q(0)=0$, and $q'(0)=-1<0$, so the claim is valid. To obtain $\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds \le \overline{M}$, we consider two cases.

Case 1. $\int_0^1 |x^{(2n-1)}(s)|ds \le \frac{M}{\sigma}$.

So

$$
\left(M+\int_0^1 |x^{(2n-1)}(s)|ds\right)^{m_0} \le M^m\left(1+\frac{1}{\sigma}\right)^{m_0}.
$$

Since

$$
\int_0^1 |x^{(2n-1)}(s)|ds \le \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)},
$$

we get

$$
\begin{aligned}
\beta\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds &\le A^m M^m\left(1+\frac{1}{\sigma}\right)^m (r_0+\varepsilon)\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\\
&+\sum_{i=1}^{2n-1}(r_i+\varepsilon)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\\
&+g_{\delta,0}A\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right]\\
&+\|e\|_\infty\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\\
&+\sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}.
\end{aligned}
$$

i.e.

$$\left(\beta-\sum_{i=1}^{2n-1}(r_i+\varepsilon)\right)\int_0^1|x^{(2n-1)}(s)|^{m+1}ds$$
$$\le\left[A^mM^m\left(1+\frac{1}{\sigma}\right)^m(r_0+\varepsilon)+\|e\|_\infty\right]\left(\int_0^1|x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}$$
$$+\sum_{i=1}^{2n-1}g_{\delta,i}\left(\int_0^1|x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}$$
$$+g_{\delta,0}A\left[M+\left(\int_0^1|x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right].$$

From the definition of ε, we find that $\beta-\sum_{i=1}^{2n-1}(r_i+\varepsilon)>0$ and that there is a constant $M_1'>0$ such that

$$\int_0^1|x^{(2n-1)}(s)|^{m+1}ds\le M_1'.$$

Case 2. $\int_0^1|x^{(2n-1)}(s)|ds>\frac{M}{\sigma}$.

In this case, $0<\frac{M}{\int_0^1|x^{(2n-1)}(s)|ds}<\sigma$. Using $(1+x)^m\le1+(m+1)x$ for $x\in(0,\sigma)$, we have

$$\left(M+\int_0^1|x^{(2n-1)}(s)|ds\right)^m=\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^m\left(1+\frac{M}{\int_0^1|x^{(2n-1)}(s)|ds}\right)^m$$
$$\le\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^m\left(1+\frac{(m+1)M}{\int_0^1|x^{(2n-1)}(s)|ds}\right)$$
$$=\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^m+(m+1)M\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^{m-1}.$$

Thus

$$\beta\int_0^1|x^{(2n-1)}(s)|^{m+1}ds\le A^m(r_0+\varepsilon)\int_0^1|x^{(2n-1)}(s)|ds\left[\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^m\right.$$
$$\left.+(m+1)M\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^{m-1}\right]$$
$$+\sum_{i=1}^{2n-2}(r_i+\varepsilon)\left(\int_0^1|x^{(2n-1)}(s)|ds\right)^{m+1}$$
$$+(r_{2n-1}+\varepsilon)\int_0^1|x^{(2n-1)}(s)|^{m+1}ds+\|e\|_\infty\int_0^1|x^{(2n-1)}(s)|ds$$

$$
\begin{aligned}
&+\sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\\
&+g_{\delta,0}A\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right]\\
=\,& A^m(r_0+\varepsilon)\left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^{m+1}\\
&+\sum_{i=1}^{2n-2}(r_i+\varepsilon)\left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^{m+1}\\
&+A^m(r_0+\varepsilon)(m+1)M\left(\int_0^1 |x^{(2n-1)}(s)|ds\right)^{m}\\
&+(r_{2n-1}+\varepsilon)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds+\|e\|_\infty\int_0^1 |x^{(2n-1)}(s)|ds\\
&+\sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\\
&+g_{\delta,0}A\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right]\\
\le\,& \left(A^m(r_0+\varepsilon)+\sum_{i=1}^{2n-2}(r_i+\varepsilon)\right)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\\
&+(r_{2n-1}+\varepsilon)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\\
&+A^m(r_0+\varepsilon)(m+1)M\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{m/(m+1)}\\
&+\|e\|_\infty\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\\
&+\sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\\
&+g_{\delta,0}A\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right].
\end{aligned}
$$

Hence

$$\left(\beta - A^m(r_0+\varepsilon) - \sum_{i=1}^{2n-1}(r_i+\varepsilon)\right)\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds$$
$$\le A^m(1+m)(r_0+\varepsilon)M\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{m/(m+1)}$$
$$+\|e\|_\infty\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)} + \sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(i)}(s)|^{m+1}ds\right)^{1/(m+1)}$$
$$+g_{\delta,0}A\left[M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right].$$

From the definition of ε, we find that there is $M_2'>0$ such that

$$\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds \le M_2'.$$

Thus we obtain from Case 1 and 2 that

$$\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds \le \max\{M_1', M_2'\} =: \overline{M}.$$

Sub-step 1.2. Prove there is $B>0$ such that $\|x\|\le B$.
From sub-step 1.1, we have

$$\begin{aligned}\|x\|_\infty &\le A\left(M+\int_0^1 |x^{(2n-1)}(s)|ds\right)\\ &\le A\left(M+\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\right)\\ &\le A(M+\overline{M}^{1/(m+1)}).\\ \|x^{(i)}\|_\infty &\le \int_0^1 |x^{(2n-1)}(s)|ds\\ &\le \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1}ds\right)^{1/(m+1)}\\ &\le \overline{M}^{1/(m+1)}, \qquad i=1,\dots,n-2.\end{aligned}$$

Multiplying two side of (20) by $x^{(2n-1)}(t)$, integrating it from 0 to t, using (A_1'), we get

$$\begin{aligned}
\frac{1}{2}|x^{(2n-1)}(t)|^2 &= \lambda \int_0^t (-1)^{n-1} f(s, x(s), x'(s), \dots, x^{(2n-1)}(s)) x^{(2n-1)}(s) ds \\
&= \lambda \int_0^t h(s, x(s), x'(s), \dots, x^{(2n-1)}(s)) x^{(2n-1)}(s) ds \\
&\quad + \lambda \sum_{i=0}^{2n-1} \int_0^t g_i(s, x^{(i)}(s)) x^{(2n-1)}(s) ds + \lambda \int_0^t e(s) x^{(2n-1)}(s) ds \\
&\le -\lambda\beta \int_0^t |x^{(2n-1)}(s)|^{m+1} ds \\
&\quad + \lambda \sum_{i=0}^{2n-1} \int_0^t g_i(s, x^{(i)}(s)) x^{(2n-1)}(s) ds + \lambda \int_0^t e(s) x^{(2n-1)}(s) ds \\
&\le \lambda \sum_{i=0}^{2n-1} \int_0^t g_i(s, x^{(i)}(s)) x^{(2n-1)}(s) ds + \lambda \int_0^t e(s) x^{(2n-1)}(s) ds \\
&\le \sum_{i=0}^{2n-1} \int_0^1 |g_i(s, x^{(i)}(s))| \, |x^{(2n-1)}(s)| ds + \int_0^1 |e(s)| \, |x^{(2n-1)}(s)| ds \\
&\le \sum_{i=0}^{2n-1} \int_{\Delta_{1,i}} |g_i(s, x^{(i)}(s))| \, |x^{(2n-1)}(s)| ds \\
&\quad + \sum_{i=0}^{2n-1} \int_{\Delta_{2,i}} |g_i(s, x^{(i)}(s))| \, |x^{(2n-1)}(s)| ds + \int_0^1 |e(s)| \, |x^{(2n-1)}(s)| ds \\
&\le \sum_{i=1}^{2n-1} g_{\delta,i} \int_0^1 |x^{(2n-1)}(s)| ds + \sum_{i=0}^{2n-1} (r_i + \varepsilon) \int_{\Delta_{2,i}} |x^{(i)}(s)|^m |x^{(2n-1)}(s)| ds \\
&\quad + \int_0^1 |e(s)| \, |x^{(2n-1)}(s)| ds + g_{\delta,0} \int_0^1 |x(s)| ds.
\end{aligned}$$

Similarly to step 1.1, we can get

$$\begin{aligned}
\frac{1}{2}|x^{(2n-1)}(t)|^2 \le{}& \sum_{i=1}^{2n-1} g_{\delta,i} \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1} ds \right)^{1/(m+1)} \\
&+ \sum_{i=0}^{2n-1} (r_i + \varepsilon) \int_0^1 |x^{(i)}(s)|^m |x^{(2n-1)}(s)| ds \\
&+ \int_0^1 |e(s)| \, |x^{(2n-1)}(s)| ds + g_{\delta,0} A \left(M + \int_0^1 |x^{(2n-1)}(s)| ds \right)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1} ds\right)^{1/(m+1)} \\
&\quad + \sum_{i=1}^{2n-1} (r_i+\varepsilon)\left(\int_0^1 |x^{(2n-1)}(s)| ds\right)^{m+1} \\
&\quad + \|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1} ds\right)^{1/(m+1)} \\
&\quad + g_{\delta,0} A\left[M + \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1} ds\right)^{1/(m+1)}\right] \\
&\quad + (r_0+\varepsilon)A^m \left(M + \int_0^1 |x^{(2n-1)}(s)| ds\right)^m \int_0^1 |x^{(2n-1)}(s)| ds \\
&\le \sum_{i=1}^{2n-1} g_{\delta,i}\left(\int_0^1 |x^{(2n-1)}(s)|^{m+1} ds\right)^{1/(m+1)} \\
&\quad + \sum_{i=1}^{2n-1} (r_i+\varepsilon) \int_0^1 |x^{(2n-1)}(s)|^{m+1} ds \\
&\quad + \|e\|_\infty \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1} ds\right)^{1/(m+1)} \\
&\quad + (r_0+\varepsilon)A^m \left[M + \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1} ds\right)^{1/(m+1)}\right]^m \\
&\quad \times \left(\int_0^1 |x^{(2n-1)}(s)|^{m+1} ds\right)^{1/(m+1)} \\
&\le \sum_{i=1}^{2n-1} g_{\delta,i}\overline{M}^{1/(m+1)} + \sum_{i=1}^{2n-1} (r_i+\varepsilon)\overline{M} + \|e\|_\infty \overline{M}^{1/(m+1)} \\
&\quad + (r_0+\varepsilon)A^m[M + \overline{M}^{1/(m+1)}]^m \overline{M}^{1/(m+1)} + g_{\delta,0}A(M + \overline{M}^{1/(m+1)}).
\end{aligned}
$$

So there is $M_3' > 0$ such that $|x^{(2n-1)}(t)| \le M_3'$. It follows from above discussion that there is $B > 0$ such that

$$\|x\| \le B.$$

Hence Ω_1 is bounded. This completes the step 1.

Step 2. Let

$$\Omega_2 = \{x \in \mathrm{Ker}\, L,\ Nx \in \mathrm{Im}\, L\}.$$

It is similar to that of Step 2 of the proof of Theorem 2.1 to prove that Ω_2 is bounded.

Step 3. Let

$$\Omega_3 = \{x \in \mathrm{Ker}\, L,\ \mathrm{sgn}(\Delta)\lambda \wedge x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\},$$

where $\wedge : \mathrm{Ker}\, L \to \mathrm{Im}\, Q$ is the linear isomorphism given by $\wedge(c) = ct^k$ for all $c \in R$. It is similar to that of proof of Theorem 2.1 to show that Ω_3 is bounded.

In the following, we shall show that all conditions of Theorem GM are satisfied. Set Ω be a open bounded subset of X such that $\Omega \supset \bigcup_{i=1}^{3} \overline{\Omega_i}$. By Lemma 2.1, L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the definition of Ω, we have

(a) $Lx \neq \lambda Nx$ for $x \in (\mathrm{dom}\, L/\mathrm{Ker}\, L) \cap \partial\Omega$ and $\lambda \in (0,1)$;

(b) $Nx \notin \mathrm{Im}\, L$ for $x \in \mathrm{Ker}\, L \cap \partial\Omega$.

Step 4. We prove

(c) $\deg(QN|_{\mathrm{Ker}\, L}, \Omega \cap \mathrm{Ker}\, L, 0) \neq 0$.

In fact, let $H(x,\lambda) = \lambda \wedge x + (1-\lambda)QNx$. According the definition of Ω, we know $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \mathrm{Ker}\, L$, thus by homotopy property of degree,

$$\begin{aligned}\deg(QN \mid \mathrm{Ker}\, L, \Omega \cap \mathrm{Ker}\, L, 0) &= \deg(H(\cdot\,,0), \Omega \cap \mathrm{Ker}\, L, 0)\\ &= \deg(H(\cdot\,,1), \Omega \cap \mathrm{Ker}\, L, 0)\\ &= \deg(I, \Omega \cap \mathrm{Ker}\, L, 0) \neq 0.\end{aligned}$$

Thus by Theorem GM, $Lx = Nx$ has at least one solution in $\mathrm{dom}\, L \cap \overline{\Omega}$, which is a solution of BVP(1)–(2). The proof is complete.

Similarly, we can prove the following theorem and its proof is omitted.

Theorem 2.4. *Suppose* $\sum_{i=1}^{m} \beta_i = 1$ *and the conditions of Theorem 2.2, i.e.* (A_1'), (A_2), (A_3), *hold. Then* $BVP(1)$ *and* (3) *has at least one solution provided* (19) *holds.*

Remark 1. In Theorems 2.1 and 2.2, the degree of the variables $x_0, x_1, \ldots, x_{2n-1}$ in function f may be different from each other.

Now, we suppose

$$\sum_{i=1}^{m} \beta_i \neq 1.$$

In this case, problems (1) and (2), problem (1) and (3) are non-resonance boundary value problems. We have the following results.

THEOREM 2.5. *Suppose* $\beta_i \geq 0$, $\sum_{i=1}^{m} \beta_i \neq 1$ *and* $(-1)^{n-1} f(t, x_0, \ldots, x_{2n-1}) \geq 0$ *for all* $t \in [0,1]$ *and* $(x_0, \ldots, x_{2n-1}) \in R^{2n}$ *and the conditions of Theorem 2.1,* (A_1)–(A_3)*, hold. Then* $BVP(1)$ *and* (2) *and* $BVP(1)$ *and* (3) *have at least one positive solution, respectively, provided* (14) *holds.*

PROOF. For problem (1) and (2), from (12) together with $x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i)$, we get

$$c = \frac{1}{\sum_{i=1}^{m} \beta_i - 1} \left(\int_0^1 \int_0^1 G_{n-2}(s, \tau) \int_0^\tau y(u) du d\tau ds \right. \\ \left. - \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \int_0^1 G_{n-2}(s, \tau) \int_0^\tau y(u) du d\tau ds \right).$$

Define an operator T by

$$Tx(t) = \frac{1}{\sum_{i=1}^{m} \beta_i - 1} \left(\int_0^1 \int_0^1 G_{n-2}(s, \tau) \int_0^\tau f(u, x(u), x'(u), \ldots, x^{(2n-1)}(u)) du d\tau ds \right. \\ \left. - \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \int_0^1 G_{n-2}(s, \tau) \int_0^\tau f(u, x(u), x'(u), \ldots, x^{(2n-1)}(u)) du d\tau ds \right) \\ + \int_0^t \int_0^1 G_{n-2}(s, \tau) \int_0^\tau f(u, x(u), x'(u), \ldots, x^{(2n-1)}(u)) du d\tau ds$$

for every $x \in C^{n-1}[0,1]$.

Consider the set $\Omega = \{x \in C^{n-1}[0,1], x = Tx\}$. For $x \in \Omega$, we have

$$(-1)^{n-1} x^{(2n-1)}(t) = f(t, x(t), x'(t), \ldots, x^{(2n-1)}(t)).$$

Similar to the step 1 of the proof of Theorem 2.1, we can prove that there is a constant $B > 0$ such that $\|x\| \leq B$ for every $x \in \Omega$. Then by Schauder fixed point theorem, T has at least one fixed point, which is a solution of BVP(1) and (2) since

$$
\begin{aligned}
x(t) &= \frac{1}{\sum_{i=1}^{m}\beta_i - 1}\left(\int_0^1\int_0^1 G_{n-2}(s,\tau)\int_0^\tau f(u,x(u),\dots,x^{(2n-1)}(u))dud\tau ds\right.\\
&\quad \left. - \sum_{i=1}^{m}\beta_i\int_0^{\xi_i}\int_0^1 G_{n-2}(s,\tau)\int_0^\tau f(u,x(u),\dots,x^{(2n-1)}(u))dud\tau ds\right)\\
&\quad + \int_0^t\int_0^1 G_{n-2}(s,\tau)\int_0^\tau f(u,x(u),x'(u),\dots,x^{(2n-1)}(u))dud\tau ds\\
&= \sum_{i=1}^{m}\beta_i\int_{\xi_i}^1\int_0^1 G_{n-2}(s,\tau)\int_0^\tau f(u,x(u),x'(u),\dots,x^{(2n-1)}(u))dud\tau ds\\
&\quad + \int_0^t\int_0^1 G_{n-2}(s,\tau)\int_0^\tau f(u,x(u),x'(u),\dots,x^{(2n-1)}(u))dud\tau ds\\
&\geq 0.
\end{aligned}
$$

For BVP(1) and (3), the proof is similar and is omitted.

Theorem 2.6. *Suppose* $\beta_i \geq 0$, $\sum_{i=1}^{m}\beta_i \neq 1$ *and* $(-1)^{n-1}f(t,x_0,\dots,x_{2n-1}) \geq 0$ *for all* $t \in [0,1]$ *and* $(x_0,\dots,x_{2n-1}) \in R^{2n}$ *and the conditions of Theorem* 2.1, (A_1'), (A_2), (A_3), *hold. Then BVP*(1) *and* (2) *and BVP*(1) *and* (3) *have at least one positive solution, respectively, provided* (19) *holds.*

Proof. The proof is similar to that of Theorem 2.5 and is omitted.

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