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NOTE ON (m,q)-ISOMETRIES ON AN HYPERSPACE OF A NORMED SPACE

ANTONIO MARTINÓN

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Abstract. Given a normed space X we consider the hyperspace $k(X)$ of all non-empty compact convex subsets of X endowed with the Hausdorff distance. We prove that if $T : X \longrightarrow X$ is an (m,q)-isometry, then it is possible that the map $k(T) : k(X) \longrightarrow k(X)$, $k(T)C := TC$, is not an (m,q)-isometry. Moreover, if $\widehat{k(X)}$ is the Rådström space associated to the hyperspace $k(X)$, then $\mathcal{T} : k(X) \longrightarrow k(X)$ is an (m,q)-isometry if and only if $\widehat{\mathcal{T}} : \widehat{k(X)} \longrightarrow \widehat{k(X)}$ is an (m,q)-isometry.

1. Introduction

Throughout this paper, X is a real normed space and $\|\cdot\|$ its norm, $L(X)$ the class of all bounded linear operators $T : X \longrightarrow X$, m a positive integer and q a positive real number, unless stated otherwise.

The notion of (m,q)-isometry in the setting of metric spaces was introduced in [3]: a map $T : E \longrightarrow E$, on a metric space E with distance d, is called an (m,q)-isometry if

$$\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}d(T^ix,T^iy)^q = 0 \qquad (x,y\in E)\,. \tag{1.1}$$

An (m,q)-isometry is called *strict* whenever is not an $(m-1,q)$-isometry. Of course, the $(1,q)$-isometries are the isometries. This definition generalizes the concept of m-isometry firstly introduced on Hilbert spaces by J. Agler [1]. Some time after the notion of (m,q)-isometry on Banach spaces was defined by Bayart [2] and Sid Ahmed [7].

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In [4] it was introduced a notion of m-isometry on certain hyperspaces of a Banach space. In this paper we study (m, q)-isometries on the hyperspace $k(X)$ of all non-empty convex compact subsets of a normed space X. Given an operator $T \in L(X)$ we consider the map $k(T) : k(X) \longrightarrow k(X)$, defined by $k(T)C := TC$. It is possible that T is an (m, q)-isometry but $k(T)$ is not an (m, q)-isometry. More precisely, we prove that any weighted shift operator $S_w \in L(\ell_2)$ which is a $(2, 2)$-isometry induces a map $k(S_w) : k(\ell_2) \longrightarrow k(\ell_2)$ which is not an $(2, 2)$-isometry.

Using a construction by Rådström we associate to $k(X)$ the normed space $\widehat{k(X)}$, being $k(X)$ a subspace of $\widehat{k(X)}$. We prove that $\mathcal{T} : k(X) \longrightarrow k(X)$ is an (m, q)-isometry if and only if $\widehat{\mathcal{T}} : \widehat{k(X)} \longrightarrow \widehat{k(X)}$ is an (m, q)-isometry.

2. The hyperspace $k(X)$

Given a real normed space X, we consider the hyperspace

$$k(X) := \{C \subset X : \emptyset \neq C \text{ compact convex}\} .$$

For $C, D \in k(X)$ and α scalar, we write $C + D := \{x + y : x \in C, y \in D\}$ and $\alpha C := \{\alpha x : x \in C\}$. Some properties of the class $k(X)$ are given in the following proposition:

Proposition 2.1. *For $C, D, E \in k(X)$; $\lambda, \mu \geq 0$ and α scalars,*

(1) $C + D \in k(X)$
(2) $(C + D) + E = C + (D + E)$ *and* $C + D = D + C$
(3) $C + E = D + E \Longrightarrow C = D$
(4) $\alpha C \in k(X)$
(5) $\alpha(C + D) = \alpha C + \alpha D$ *and* $(\lambda + \mu)C = \lambda C + \mu C$

Proof. The property (3) is [6, Lemma 2]. The other properties are simple. □

We introduce the *norm* of $C \in k(X)$:

$$\|C\| := \sup_{x \in C} \|x\| .$$

Proposition 2.2. *For $C, D \in k(X)$ and α scalar,*

(1) $\|C\| = 0 \Longleftrightarrow C = \{0\}$
(2) $\|C + D\| \leq \|C\| + \|D\|$
(3) $\|\alpha C\| = |\alpha| \|C\|$

Proof. Routine. □

The class $k(X)$ is endowed with the Hausdorff distance h: given $C, D \in k(X)$, we put

$$h(C, D) := \inf\{\varepsilon > 0 : C \subset D + \varepsilon B_X \text{ and } D \subset C + \varepsilon B_X\} ,$$

where B_X is the unit closed ball of X. In the next result we collect some basic facts about the distance h.

Proposition 2.3. *For $C, D, E \in k(X)$ and α scalar,*

(1) *h is a metric on $k(X)$; moreover, if X is a Banach space, then $k(X)$ is complete.*
(2) $h(C+E, D+E) = h(C, D)$
(3) $h(\alpha C, \alpha D) = |\alpha| h(C, D)$
(4) $h(C, \{0\}) = \|C\|$

Proof. The property (1) is well known and (4) is clear. In order to prove (2), notice that, for every $\varepsilon > 0$, we can write

$$\begin{aligned} h(C+E, D+E) < \varepsilon &\implies C+E \subset D+E+\varepsilon B_X \text{ and } D+E \subset C+E+\varepsilon B_X \\ &\implies C \subset D+\varepsilon B_X \text{ and } D \subset C+\varepsilon B_X \\ &\implies h(C, D) \leq \varepsilon\ , \end{aligned}$$

by Proposition 2.1 (3). Analogously, $h(C, D) < \varepsilon \implies h(C+E, D+E) \leq \varepsilon$. Therefore, (2) is true.

Now we prove (3). We have that the equality is obvious if $\alpha = 0$. Assume $\alpha \neq 0$. Then

$$\begin{aligned} h(\alpha C, \alpha D) < \varepsilon &\implies \alpha C \subset \alpha D + \varepsilon B_X \text{ and } \alpha D \subset \alpha C + \varepsilon B_X \\ &\implies C \subset D + \alpha^{-1}\varepsilon B_X = D + |\alpha|^{-1}\varepsilon B_X \\ &\qquad \text{and } D \subset C + \alpha^{-1}\varepsilon B_X = C + |\alpha|^{-1}\varepsilon B_X \\ &\implies h(C, D) \leq |\alpha|^{-1}\varepsilon \\ &\implies |\alpha| h(C, D) \leq \varepsilon\ . \end{aligned}$$

Analogously, $|\alpha| h(C, D) < \varepsilon \implies h(\alpha C, \alpha D) \leq \varepsilon$. Consequently, (3) holds. □

Observe that the property (2) in the above proposition depends on the fact that E is bounded and that both sets $C + \varepsilon B_X$ and $D + \varepsilon B_X$ are convex closed, since C and D are convex compact (see [6, Lemmas 2 and 3]).

It is obvious that we can identify X with $\{\{x\} : x \in X\} \subset k(X)$. For $x, y \in X$ and α scalar we have that $\{x\} + \{y\} = \{x + y\}$, $\alpha\{x\} = \{\alpha x\}$ and $h(\{x\}, \{y\}) = \|x - y\|$. Notice that, in general,

$$h(C, D) \leq \|C - D\| \quad (C, D \in k(X))$$

and it is possible that $h(C, D) < \|C - D\|$. For example, $h(C, C) = 0 < \|C - C\|$ whenever C is not a singleton.

3. Maps on $k(X)$

We say that a map $\mathcal{T} : k(X) \longrightarrow k(X)$ is *linear* if, for $C, D \in k(X)$ and α scalar,

$$\mathcal{T}(C + D) = \mathcal{T}C + \mathcal{T}D \quad \text{ and } \quad \mathcal{T}(\alpha C) = \alpha \mathcal{T}C\ .$$

Given $\mathcal{T} : k(X) \longrightarrow k(X)$ linear we define the *norm* of $\mathcal{T}$ by

$$\|\mathcal{T}\| = \sup_{\{0\} \neq C \in k(X)} \frac{\|\mathcal{T}C\|}{\|C\|} = \sup_{C \in k(X), \|C\| = 1} \|\mathcal{T}C\|\ .$$

Hence, for every $C \in k(X)$, we have that $\|\mathcal{T}C\| \leq \|\mathcal{T}\|\|C\|$. We say that $\mathcal{T}$ is *bounded* if $\|\mathcal{T}\| < \infty$.

The following results are very similar to analogous facts about linear operators between normed spaces and we omit the proof.

Proposition 3.1. *Let $\mathcal{T} : k(X) \longrightarrow k(X)$ a linear map. The following assertions are equivalent:*

(1) *$\mathcal{T}$ is uniformly continuous*
(2) *$\mathcal{T}$ is continuous*
(3) *$\mathcal{T}$ is continuous at $\{0\}$*
(4) *There exists $M > 0$ such that, for every $C \in k(X)$, $\|\mathcal{T}C\| \leq M\|C\|$*
(5) *$\mathcal{T}$ is bounded*

We denote by $L(k(X))$ the class of all bounded linear maps $\mathcal{T} : k(X) \longrightarrow k(X)$.

Proposition 3.2. *For $\mathcal{T}, \mathcal{S} \in L(k(X))$ and scalar α,*

(1) *$\mathcal{T} + \mathcal{S} \in L(k(X))$ and $\|\mathcal{T} + \mathcal{S}\| \leq \|\mathcal{T}\| + \|\mathcal{S}\|$*
(2) *$\alpha\mathcal{T} \in L(k(X))$ and $\|\alpha\mathcal{T}\| = |\alpha|\|\mathcal{T}\|$*
(3) *$\mathcal{T}\mathcal{S} \in L(k(X))$ and $\|\mathcal{T}\mathcal{S}\| \leq \|\mathcal{T}\|\|\mathcal{S}\|$*

Proof. Routine. □

Given $T \in L(X)$ we define the map

$$k(T) : k(X) \longrightarrow k(X) \quad , \quad k(T)C := TC \; .$$

Obviously, the restriction of $k(T)$ to X is T: $k(T)\{x\} = T\{x\} = \{Tx\}$, for any $x \in X$.

Proposition 3.3. *Let $T \in L(X)$. Then $k(T) \in L(k(X))$ and $\|k(T)\| = \|T\|$.*

Proof. For $C \in k(X)$, we have that $\|TC\| \leq \|T\|\|C\|$, hence

$$\|k(T)\| = \sup_{\{0\} \neq C \in k(X)} \frac{\|k(T)C\|}{\|C\|} = \sup_{\{0\} \neq C \in k(X)} \frac{\|TC\|}{\|C\|} \leq \|T\| \; .$$

Moreover

$$\|T\| = \sup_{0 \neq x \in X} \frac{\|Tx\|}{\|x\|} \leq \sup_{\{0\} \neq C \in k(X)} \frac{\|TC\|}{\|C\|} = \|k(T)\|$$

and the proof is completed. □

Proposition 3.4. *Let $T \in L(X)$. Then T is an isometry if and only if the map $k(T)$ is an isometry.*

Proof. It is enough to observe that the equalities

$$\|k(T)C\| = \|C\| = \|TC\|$$

are equivalent to that both $k(T)$ and T are isometries. □

Our main interest is the study of (m,q)-isometries ($m \geq 1$ integer, $q > 0$ real) on the hyperspace $k(X)$. Recall that the general definition was given in (1.1). For $\mathcal{T} : k(X) \longrightarrow k(X)$ the condition (1.1) is equivalent to

$$\sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} h(\mathcal{T}^i C, \mathcal{T}^i D)^q = 0 \quad (C, D \in k(X)) \; . \tag{3.1}$$

The equivalence given in Proposition 3.4 can not be extended to (m,q)-isometries, although an implication is true.

Proposition 3.5. *Let $T \in L(X)$. If the map $k(T)$ is an (m,q)-isometry, then T is an (m,q)-isometry.*

Proof. It is enough to observe that any restriction of an (m,q)-isometry to an invariant subset is also an (m,q)-isometry and that T is the restriction of $k(T)$ to X as explained before. □

The converse of above proposition is false, as we show in the next example.

Example 3.6. Let $S_w : \ell_2 \longrightarrow \ell_2$ the weighted shift operator on ℓ_2 with weight sequence $w = (w_n)_{n\geq 1} \in \ell_\infty$. That is, for $x = (x_n)_{n\geq 1} \in \ell_2$,

$$S_w x = S_w(x_1, x_2, x_3 ...) = (0, w_1 x_1, w_2 x_2, w_3 x_3 ...) \ .$$

If S_w is a strict $(2,2)$-isometry, then $k(S_w)$ is not a $(2,2)$-isometry.

Proof. We put $\alpha := |w_1|^2$. Then, for $n \geq 1$ [4, Remark 3.9(1)(b)]

$$|w_n|^2 = \frac{\alpha n - (n-1)}{\alpha(n-1) - (n-2)} \ ,$$

hence

$$|w_2|^2 = \frac{2\alpha - 1}{\alpha} \quad \text{and} \quad |w_3|^2 = \frac{3\alpha - 2}{2\alpha - 1} \ .$$

We have that $\alpha \neq 1$ since S_w is not an isometry, and $\alpha > 1$ since S_w is a $(2,2)$-isometry ([4, Remark 3.9(1)(b)], [5, Corollary 2.3]).

Let $(e_n)_{n\geq 1}$ be the canonical basis of ℓ_2. Take $x = e_1$ and $y = \lambda e_2$, such that λ is a scalar with

$$1 < |\lambda|^2 < \frac{\alpha^2}{2\alpha - 1} \ .$$

We obtain

$$\|x\|^2 = 1 \ , \ \|S_w x\|^2 = \alpha \ , \ \|S_w^2 x\|^2 = 2\alpha - 1 \ ,$$

$$\|y\|^2 = |\lambda|^2 \ , \ \|S_w y\|^2 = |\lambda|^2 \frac{2\alpha - 1}{\alpha} \ , \ \|S_w^2 y\|^2 = |\lambda|^2 \frac{3\alpha - 2}{\alpha} \ .$$

Consider the segment

$$C = [x, y] := \{tx + (1-t)y : 0 \leq t \leq 1\} \in k(\ell_2) \ .$$

Then

$$\begin{aligned} \|C\|^2 &= \sup_{0\leq t\leq 1} \|tx + (1-t)y\|^2 \\ &= \sup_{0\leq t\leq 1} \|(t, (1-t)\lambda, 0, 0, 0 ...)\|^2 \\ &= \sup_{0\leq t\leq 1} (t^2 + (1-t)^2 |\lambda|^2) \\ &= |\lambda|^2 \ , \end{aligned}$$

since $1 < |\lambda|^2$. Moroever,

$$\begin{aligned}\|S_w C\|^2 &= \sup_{0\le t\le 1} \|(0, w_1 t, w_2(1-t)\lambda, 0, 0, 0...)\|^2 \\ &= \sup_{0\le t\le 1} (|w_1|^2 t^2 + |w_2|^2 (1-t)^2 |\lambda|^2) \\ &= \sup_{0\le t\le 1} (\alpha t^2 + \frac{2\alpha - 1}{\alpha}(1-t)^2|\lambda|^2) \\ &= \alpha\end{aligned}$$

and

$$\begin{aligned}\|S_w^2 C\|^2 &= \sup_{0\le t\le 1} \|(0, 0, w_1 w_2 t, w_2 w_3 (1-t)\lambda, 0, 0, 0...)\|^2 \\ &= \sup_{0\le t\le 1} (|w_1 w_2|^2 t^2 + |w_2 w_3|^2 (1-t)^2 |\lambda|^2) \\ &= \sup_{0\le t\le 1} ((2\alpha - 1) t^2 + \frac{3\alpha - 2}{\alpha}(1-t)^2|\lambda|^2) \\ &= 2\alpha - 1\ .\end{aligned}$$

We have that

$$h(k(S_w)^2 C, k(S_w)^2\{0\})^2 - 2h(k(S_w)C, k(S_w)\{0\})^2 + h(C, \{0\})^2 =$$

$$= \|k(S_w)^2 C\|^2 - 2\|k(S_w)C\|^2 + \|C\|^2 = 2\alpha - 1 - 2\alpha + |\lambda|^2 = |\lambda|^2 - 1 \neq 0\ ,$$

because of $1 < |\lambda|^2$. By (3.1) we obtain that S_w is not a $(2, 2)$-isometry. □

4. The Rådström space $\widehat{k(X)}$

Rådström [6] proved that $k(X)$ endowed with the Hausdorff distance can be isometrically embedded in a normed space $\widehat{k(X)}$ in such a way that addition in $\widehat{k(X)}$ induces addition in $k(X)$ and multiplication by scalars in $\widehat{k(X)}$ induces multiplication by scalars in $k(X)$.

Now we give a description of the Rådström space associated to the hyperspace $k(X)$ (see [6]). On $k(X) \times k(X)$ we consider the equivalence relation $(C, D) \sim (E, F) \iff C + F = D + E$, where $C, D, E, F \in k(X)$. The class of (C, D) is denoted by $[C, D]$.

The quotient space

$$\widehat{k(X)} := \frac{k(X) \times k(X)}{\sim}$$

is a normed space with the following: for $C, D, E, F \in k(X)$ and $\lambda \geq 0$ scalar,

$$\|[C, D]\| = h(C, D)\ ,\ [C, D] + [E, F] = [C + E, D + F]\ ,$$

$$\lambda[C, D] = [\lambda C, \lambda D]\ ,\ (-\lambda)[C, D] = [\lambda D, \lambda C]\ ,\ \ .$$

From this, the distance between two classes of $\widehat{k(X)}$ is given by

$$\widehat{h}([C, D], [E, F]) = \|[C, D] - [E, F]\| = \|[C + F, D + E]\| = h(C + F, D + E)\ .$$

Moreover the map $\psi : k(X) \longrightarrow \widehat{k(X)}$ defined by $\psi C := [C, \{0\}]$, is an isometric embedding of $k(X)$ into $\widehat{k(X)}$; in fact, we have that $\psi(C + D) = \psi(C) + \psi(D)$, $\psi(\lambda C) = \lambda\psi(C)$ and $\|\psi(C)\| = \|C\|$.

Given a map $\mathcal{T} : k(X) \longrightarrow k(X)$, we define

$$\widehat{\mathcal{T}} : \widehat{k(X)} \longrightarrow \widehat{k(X)} \quad , \quad \widehat{\mathcal{T}}[C, D] := [\mathcal{T}C, \mathcal{T}D] \ .$$

Notice that the restriction of $\widehat{\mathcal{T}}$ to $k(X)$ is $\mathcal{T}$.

Proposition 4.1. *Let $\mathcal{T} : k(X) \longrightarrow k(X)$ a linear map.Then*

(1) *$\widehat{\mathcal{T}}$ is linear*
(2) *$\mathcal{T}$ bounded $\Longrightarrow$ $\widehat{\mathcal{T}}$ bounded and $\|\widehat{\mathcal{T}}\| = \|\mathcal{T}\|$.*

Proof. (1) Straightforward.

(2) As $\mathcal{T}$ is restriction of $\widehat{\mathcal{T}}$, we have that $\|\mathcal{T}\| \leq \|\widehat{\mathcal{T}}\|$. Now we show $\|\mathcal{T}\| \geq \|\widehat{\mathcal{T}}\|$. For this purpose, first we prove

$$h(\mathcal{T}C, \mathcal{T}D) \leq \|\mathcal{T}\| h(C, D) \quad (C, D \in k(X)) \ . \tag{4.1}$$

Fix $C, D \in k(X)$. Let $\varepsilon > h(C, D)$. Then $C \subset D + \varepsilon B_X$ and $D \subset C + \varepsilon B_X$. Hence $\mathcal{T}C \subset \mathcal{T}D + \varepsilon\widetilde{\mathcal{T}}B_X$ and $\mathcal{T}D \subset \mathcal{T}C + \varepsilon\widetilde{\mathcal{T}}B_X$, where

$$\widetilde{\mathcal{T}}B_X := \bigcup_{b \in B_X} \mathcal{T}\{b\} \ .$$

(Observe that $\mathcal{T}B_X$ is not always defined because of $B_X \notin k(X)$ if X is infinitedimensional). Notice that from $\mathcal{T}\{b\} \subset \|\mathcal{T}\|\|b\|B_X \subset \|\mathcal{T}\|B_X$, we obtain $\widetilde{\mathcal{T}}B_X \subset \|\mathcal{T}\|B_X$ and consequently $\mathcal{T}C \subset \mathcal{T}D + \varepsilon\|\mathcal{T}\|B_X$ and $\mathcal{T}D \subset \mathcal{T}C + \varepsilon\|\mathcal{T}\|B_X$. Therefore $h(\mathcal{T}C, \mathcal{T}D) \leq \varepsilon\|\mathcal{T}\|$. Hence (4.1) follows. From this

$$\begin{aligned} \|\widehat{\mathcal{T}}\| &= \sup_{\|[C,D]\| \leq 1} \|\widehat{\mathcal{T}}[C, D]\| \\ &= \sup_{h(C,D) \leq 1} \|[\mathcal{T}C, \mathcal{T}D]\| \\ &\leq \sup_{h(C,D) \leq 1} \|\mathcal{T}\| h(C, D) \\ &= \|\mathcal{T}\| \ . \end{aligned}$$

So the proof is completed. □

Proposition 4.2. *Let $\mathcal{T} \in L(k(X))$. The following assertions are equivalent:*

(1) *$\mathcal{T}$ is a strict (m, q)-isometry*
(2) *$\widehat{\mathcal{T}}$ is a strict (m, q)-isometry*

Proof. For $C, D \in k(X)$ and $1 \leq k \leq m$, we have the following equalities

$$\|\widehat{\mathcal{T}}^k[C, D]\| = \|[\mathcal{T}^k C, \mathcal{T}^k D]\| = h(\mathcal{T}^k C, \mathcal{T}^k D) \ .$$

Consequently, $\mathcal{T}$ is an (m,q)-isometry, that is it verifies (3.1), if and only if $\widehat{\mathcal{T}}$ verifies

$$\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i}\|\widehat{\mathcal{T}}^i[C,D]\|^q = 0 \quad (C,D\in k(X))\ ;$$

that is, $\widehat{\mathcal{T}}$ is an (m,q)-isometry. From this, it is obvious that $\mathcal{T}$ is a strict (m,q)-isometry if and only if $\widehat{\mathcal{T}}$ is also a strict (m,q)-isometry. □

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Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna (Tenerife), Spain
E-mail address: anmarce@ull.edu.es