ON ASYMPTOTIC EQUIVALENCE OF THE BOUNDED SOLUTIONS OF TWO SYSTEMS OF DIFFERENTIAL EQUATIONS

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1. INTRODUCTION

Many papers in differential equations have studied the asymptotic equivalence of the solutions of two differential equations (see the references). This paper is another contribution in this area. We shall consider the linear equation

$$\text{(1)} \qquad dy/dt = A(t)y$$

and a perturbation of this equation

$$\text{(2)} \qquad dx/dt = A(t)x + f(t, x).$$

In a recent paper [3, Theorem 1], F. Brauer and J. S. W. Wong considered the problem of establishing a correspondence between the bounded solutions of equations (1) and (2). Their result covers the case where the linear system (1) is, in general, uniformly conditionally stable. We use Schauder's fixed-point theorem and impose a certain restriction on the nonlinear term f in (2), as was done in [3].

In equations (1) and (2), the symbols y, x, and f denote n-vectors, A is a continuous $n \times n$ matrix defined for values of t in $I = [0, \infty)$, and f is continuous on $I \times R^n$. We denote by $Y(t)$ the fundamental matrix of (1) that satisfies $Y(0) = I_n$, where I_n is the $n \times n$ identity matrix. Let P_i $(i = 1, 2)$ be supplementary projections, and define

$$\Phi_i(t; s) = Y(t)P_iY^{-1}(s) \qquad (i = 1, 2).$$

The symbol $\|\cdot\|$ denotes some convenient norm of a vector or matrix.

To determine bounds on the solution vectors of (2), we introduce two positive scalar functions $\psi = \psi(t)$ and $\phi = \phi(t)$ that are continuous on I. In Theorem 1 we require that ψ and ϕ satisfy the condition

$$\int^{\infty} \psi^{-1}(t)\phi(t)\,dt = \infty.$$

A vector function $z = z(t)$ is called *ψ-bounded* if there exists a positive constant M such that

$$\|\psi^{-1}(t)z(t)\| \leq M \qquad (t \in I).$$

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The results of this article extend known results of W. A. Coppel [5, Chapter V], R. Conti [4], and V. A. Staikos [10] in several distinct directions. The references [4], [5], and [10] establish the existence of a bounded solution of a differential equation; our results indicate, in some sense, the number of such solutions. Our results allow more general nonlinearities than were previously allowed. Also, the introduction of the function ψ permits the discussion of ψ-bounded solutions of (1) and (2), as opposed to just bounded solutions. In the last section of this article, we give examples that illustrate the generalizations obtained by using the functions ψ and ϕ.

The introduction of the pair ψ, ϕ also has applications in the area of stability; see [6], [7].

2. A REMARK ON THE HYPOTHESES

To motivate the choice of our hypotheses, we investigate the relationship between two sets of hypotheses that have been used. N. Onuchic [8], [9] used hypotheses (H).

(H) For each positive constant M, there exists a continuous, nonnegative, real-valued function $h_M(t)$ satisfying the inequality $\int_0^\infty h_M(t)\,dt < \infty$. Moreover,

$$\|f(t, x)\| \leq h_M(t)$$

for all (t, x) $(t \in I,\ \|x\| \leq M)$.

Brauer and Wong used hypotheses (C).

(C) There exists a real-valued function $\omega(t, r)$, continuous and nonnegative for $(t, r) \in I \times I$, nondecreasing in r for each fixed $t \in I$, and satisfying the condition $\int_0^\infty \omega(t, \lambda)\,dt < \infty$ for all λ $(0 < \lambda < \infty)$. Moreover,

$$\|f(t, x)\| \leq \omega(t, \|x\|).$$

We point out that Onuchic does not require the functions $h_M(t)$ to be continuous.

Hypotheses (C) and (H) are equivalent. It is easy to see that condition (C) implies (H); take $h_M(t) = \omega(t, M)$. To establish the converse, we construct the function $\omega = \omega(t, r)$ in the following manner. For each positive integer n, consider the function $h_n(t)$. For $t \in I$, define H_n $(n = 1, 2, \cdots)$ inductively as follows:

$$H_1(t) = h_1(t), \qquad H_n(t) = \max\{h_n(t), H_{n-1}(t)\} \quad (n = 2, 3, \cdots).$$

We now define $\omega = \omega(t, r)$ on appropriate strips of the first quadrant. Let ε be a positive number $(0 < \varepsilon < 1)$; for each r $(0 < r \leq 1 - \varepsilon)$ and each t $(t \geq 0)$, let

$$\omega(t, r) = H_1(t).$$

For a fixed positive integer n and for $t \in I$ and $r \in (n - \varepsilon, n - \varepsilon/2]$, define

$$\omega(t, r) = H_n(t) - 2\varepsilon^{-1}[H_n(t) - H_{n+1}(t)](r - n + \varepsilon).$$

If $t \in I$ and $r \in (n - \varepsilon/2, n + 1 - \varepsilon)$, let

$$\omega(t, r) = H_{n+1}(t).$$

It follows from the definition of ω that condition (C) is satisfied. Therefore, condition (H) as used by Onuchic and condition (C) as used by Brauer and Wong are equivalent.

Motivated by the discussion above, we use an analogue of the comparison condition (C). In particular, we require that the perturbation term f in (2) satisfies the inequality

$$\|\phi^{-1}(t)f(t, x)\| \leq \omega(t, \|\psi^{-1}(t)x\|), \tag{3}$$

where $\omega(t, r)$ is a continuous function on $I \times I$ and is nondecreasing in r, for each fixed t. We shall impose further restrictions on the function ω when they are required.

3. MAIN RESULTS ON ψ-BOUNDEDNESS

The following result was motivated by a theorem of Coppel [5, p. 74, Theorem 10]. By using Schauder's fixed-point theorem [5, p. 9], we can allow a more general perturbation term f than is considered in [5].

THEOREM 1. *Suppose there exist supplementary projections* P_1, P_2 *and a constant* $K > 0$ *such that for each* t $(t \geq t_0)$, *the inequality*

$$\int_{t_0}^{t} \|\Phi_1(t; s)\phi(s)\| \, ds + \int_t^{\infty} \|\Phi_2(t; s)\phi(s)\| \, ds \leq K\psi(t) \tag{4}$$

holds. Suppose that inequality (3) *holds. Furthermore, let* $\gamma_\lambda(t) = \sup_{s \geq t} \omega(s, \lambda)$, *and assume that* $\lim_{t\to\infty} \gamma_\lambda(t) = 0$ *for each* λ $(0 < \lambda < \infty)$. *Then, corresponding to each* ψ*-bounded solution* $y = y(t)$ *of* (1), *there exists a* ψ*-bounded solution* $x = x(t)$ *of* (2) *with the property that*

$$\|x(t) - y(t)\| = o(\psi(t)) \qquad (t \to \infty). \tag{5}$$

Conversely, to each ψ*-bounded solution* $x = x(t)$ *of* (2) *there corresponds a* ψ*-bounded solution* $y = y(t)$ *of* (1) *such that* (5) *holds.*

Proof. For each $\rho > 0$, define the ψ-ball

$$B_{\psi,\rho} = \{z \mid z \text{ is continuous on } [t_0, \infty) \text{ and } \sup_{t \geq t_0} \|\psi^{-1}(t)z(t)\| \leq \rho\}.$$

Let y be a ψ-bounded solution of (1) such that $y \in B_{\psi,\rho}$. For x in $B_{\psi,2\rho}$, we define

$$Tx(t) = y(t) + \int_{t_0}^{t} \Phi_1(t; s)f(s, x(s))\,ds - \int_t^{\infty} \Phi_2(t; s)f(s, x(s))\,ds. \tag{6}$$

We require that $\gamma_{2\rho}(t_0)K \leq \rho$. It follows that T maps $B_{\psi,2\rho}$ into itself, because relation (6) implies the inequality

$$\|Tx(t)\| \leq [\rho + \gamma_{2\rho}(t_0)K]\psi(t) \leq 2\rho\,\psi(t).$$

Next, we show that T is continuous. Let x, $\{x_n\}_{n=1}^{\infty}$ belong to $B_{\psi,2\rho}$ and suppose $\{x_n\}$ converges uniformly to x on compact subintervals of I. Relation (6) implies that

$$\begin{aligned} \|Tx_n(t) - Tx(t)\| \leq & \int_{t_0}^{t} \|\Phi_1(t;s)\| \cdot \|f(s, x_n(s)) - f(s, x(s))\|\,ds \\ & + \int_{t}^{\infty} \|\Phi_2(t;s)\| \cdot \|f(s, x_n(s)) - f(s, x(s))\|\,ds. \end{aligned} \tag{7}$$

Fix $\varepsilon > 0$, and choose t_1 $(t_1 \geq t_0)$ sufficiently large so that $\gamma_{2\rho}(t_1) \leq \varepsilon/4K$. On the interval $[t_0, t_1]$, the sequence $\{x_n\}$ converges uniformly to x; this, together with the continuity of f, implies that there exists an $N > 0$ such that

$$\phi^{-1}(t)\,\|f(t, x_n(t)) - f(t, x(t))\| < \varepsilon/2K \qquad (t \in [t_0, t_1],\ n \geq N).$$

The use of these results in (7) leads to the inequality

$$\|Tx_n(t) - Tx(t)\| \leq \varepsilon\,\psi(t) \qquad (n \geq N).$$

Since $\psi(t)$ is continuous on I and ε is arbitrary, the sequence $\{Tx_n\}$ converges uniformly to Tx on compact subintervals of $[t_0, \infty)$.

The functions in the image space $TB_{\psi,2\rho}$ are uniformly bounded for each t, since $TB_{\psi,2\rho} \subset B_{\psi,2\rho}$. Because $z = Tx$ is a solution of the nonhomogeneous linear equation

$$z' = A(t)z + f(t, x(t)),$$

the derivatives of the functions in $TB_{\psi,2\rho}$ are uniformly bounded on every finite interval. Thus, the functions in $TB_{\psi,2\rho}$ are equicontinuous on every finite interval. Schauder's fixed-point theorem now establishes that the mapping T has a fixed point in $B_{\psi,2\rho}$. This fixed point is a ψ-bounded solution of (2).

To verify that (5) holds, we need the relation

$$\|Y(t)P_1\| = o(\psi(t)) \qquad (t \to \infty),$$

which is a consequence of Lemma 1 of [6] and the hypothesis $\int^{\infty} \psi^{-1}(t)\,\phi(t)\,dt = \infty$ (actually, the proof in [6] needs to be modified to include the projection P_1).

Fix $\varepsilon > 0$, and select t_2 $(t_2 \geq t_1)$ such that

$$\psi^{-1}(t)\,\|Y(t)P_1\| \int_{t_0}^{t_1} \|Y^{-1}(s)f(s, x(s))\|\,ds \leq \varepsilon/2 \qquad (t \geq t_2). \tag{8}$$

The choice of t_1 guarantees that

$$\psi^{-1}(t)\left\|\int_{t_1}^{t} \Phi_1(t;\, s) f(s,\, x(s))\, ds - \int_{t}^{\infty} \Phi_2(t;\, s) f(s,\, x(s))\, ds\right\| \tag{9}$$
$$\leq K\gamma_{2\rho}(t_1) \leq \varepsilon/4 \qquad (t \geq t_1).$$

From (6), (8), and (9) we obtain the inequalities

$$\|x(t) - y(t)\| \leq \|Y(t) P_1\| \int_{t_0}^{t_1} \|Y^{-1}(s) f(s,\, x(s))\|\, ds + K\gamma_{2\rho}(t_1) \leq \varepsilon\, \psi(t)$$

for sufficiently large t; hence (5) is satisfied.

To verify the last statement of the theorem, consider a ψ-bounded solution $x = x(t)$ of (2). Define

$$y(t) = x(t) - \int_{t_0}^{t} \Phi_1(t;\, s) f(s,\, x(s))\, ds + \int_{t}^{\infty} \Phi_2(t;\, s) f(s,\, x(s))\, ds\,. \tag{10}$$

Definition (10) and previous arguments show that y is a ψ-bounded solution of (1) that satisfies (5). This concludes the proof of Theorem 1.

Example. In general, the condition $\lim_{t\to\infty} \gamma_\lambda(t) = 0$ in Theorem 1 may not be replaced with the condition that γ_λ be bounded. The solution of the differential equation

$$dx/dt = -x + x^2 \qquad (t \geq 0)$$

through the point $(t_0,\, x_0)$ is

$$x(t;\, t_0,\, x_0) = x_0 e^{t_0}[x_0(e^{t_0} - e^t) + e^t]^{-1}.$$

Corresponding to each bounded solution x of (2) with $x_0 < 1$, there exists a bounded solution y of (1) such that

$$|x(t) - y(t)| = o(1) \qquad (t \to \infty) \tag{11}$$

(take $y(t) = x_0 e^{t_0 - t}[1 - x_0]^{-1}$ as the solution corresponding to $x(t;\, t_0,\, x_0)$). However, the solution $x(t;\, t_0,\, 1) = 1$ of (2) has no corresponding y such that (11) is satisfied.

The next result is closely related to known results of Conti [4] and Staikos [12] who considered the existence of a bounded solution of (2). Again, our generalizations are in two directions: an improvement in the allowable nonlinearity in (2) and the introduction of the functions ψ and ϕ. Instead of the condition $\int^{\infty} \phi(t)\psi^{-1}(t)\, dt = \infty$ of Theorem 1, we use a similar hypothesis.

THEOREM 2. *Suppose there exist supplementary projections* P_1, P_2 *and a positive constant* K *such that*

$$\left(\int_{t_0}^{t} \|\Phi_1(t; s)\phi(s)\|^q \, ds \right)^{1/q} + \left(\int_{t}^{\infty} \|\Phi_2(t; s)\phi(s)\|^q \, ds \right)^{1/q} \tag{12}$$
$$\leq K\psi(t) \qquad (t \geq t_0, \ 1 < q < \infty).$$

Suppose that (3) *holds and that the function* ω *satisfies the condition*

$$\int_0^\infty \omega^p(t, \lambda)\, dt < \infty \qquad (0 < \lambda < \infty, \ p^{-1} + q^{-1} = 1). \tag{13}$$

Furthermore, assume $\int^\infty \phi^q(t)\psi^{-q}(t)\, dt = \infty$. *Then, corresponding to each* ψ-*bounded solution* $y = y(t)$ *of* (1), *there exists a* ψ-*bounded solution* $x = x(t)$ *of* (2) *such that condition* (5) *is satisfied. Conversely, to each* ψ-*bounded solution* x *of* (2) *there exists a* ψ-*bounded solution* y *of* (1) *such that* (5) *holds.*

Proof. The proof uses Schauder's fixed-point theorem and resembles the proof of Theorem 1. We indicate the necessary modifications.

As above, suppose y is a ψ-bounded solution of (1) in $B_{\psi,\rho}$, and define

$$Tx(t) = y(t) + \int_{t_0}^{t} \Phi_1(t; s) f(s, x(s))\, ds - \int_{t}^{\infty} \Phi_2(t; s) f(s, x(s))\, ds. \tag{14}$$

By virtue of (13), we may choose t_0 sufficiently large so that

$$\left(\int_{t_0}^{\infty} \omega^p(s, 2\rho)\, ds \right)^{1/p} < \rho/K. \tag{15}$$

Using (14) and (15), one can show that $Tx \in B_{\psi,2\rho}$.

To establish that the mapping T is continuous, fix $\varepsilon > 0$, and select $t_1 \geq t_0$ such that

$$\int_{t_1}^{\infty} \omega^p(s, 2\rho)\, ds \leq \varepsilon^p / 4^p K^p.$$

If $\{x_n\}_{n=1}^\infty$, x belong to $B_{\psi,2\rho}$ and $\{x_n\}$ converges uniformly to x on compact intervals of $[t_0, \infty)$, then there exists an $N > 0$ such that

$$\phi^{-1}(t)\, \|f(t, x_n(t)) - f(t, x(t))\| < \varepsilon / 2K[t_1 - t_0]^{1/p} \qquad (t \in [t_0, t_1], \ n \geq N). \tag{16}$$

From (14), (16), and Hölder's inequality, we obtain the inequalities

$$\|Tx(t) - Tx_n(t)\| \leq K \left[\int_{t_0}^{t_1} \phi^{-p}(s) \|f(s, x_n(s)) - f(s, x(s))\|^p \, ds \right]^{1/p}$$
$$+ 2K \left[\int_{t_1}^{\infty} \omega^p(s, 2\rho)\, ds \right]^{1/p} \leq \varepsilon\psi(t) \qquad (n \geq N).$$

This shows T is continuous.

The equicontinuity and uniform boundedness follow as in the proof of Theorem 1. Again, Schauder's fixed-point theorem is applicable; hence, there exists an x in $B_{\psi,2\rho}$ such that $Tx = x$. A direct verification shows that x is a ψ-bounded solution of (2).

To complete the proof, we must verify (5). First, we need a modification of some known lemmas (for example [4, Theorem 2], [5, p. 68], or [6, Lemma 1]) to show that

$$\|Y(t)P_1\| = o(\psi(t)) \qquad (t \to \infty). \tag{17}$$

Define

$$h(t) = \phi^q(t)\,\|Y(t)P_1\|^{-q}.$$

We consider the identity

$$Y(t)P_1\int_{t_0}^{t} h(s)\,ds = \int_{t_0}^{t} \|\phi^{-1}(s)Y(s)P_1\|^{-q}\cdot Y(t)P_1Y^{-1}(s)\phi(s)\phi^{-1}(s)Y(s)P_1\,ds. \tag{18}$$

Using the Hölder inequality in (18), we find the inequality

$$\|Y(t)P_1\| \le \left(\int_{t_0}^{t} h(s)\,ds\right)^{-1/q}\left(\int_{t_0}^{t}\|\Phi_1(t;\,s)\phi(s)\|^q\,ds\right)^{1/q}. \tag{19}$$

To establish (17), it suffices (in view of (12) and (19)) to show that $\zeta(t) = \int_{t_0}^{t} h(s)\,ds$ satisfies the condition $\lim_{t\to\infty}\zeta(t) = \infty$. From (19), it follows that

$$[\zeta(t)]^{1-1/p}\,\|Y(t)P_1\|\,\phi^{-1}(t) \le K\,\phi^{-1}(t)\,\psi(t). \tag{20}$$

Substituting

$$[d\zeta/dt]^{-1/q} = \|Y(t)P_1\,\phi^{-1}(t)\|$$

into (20), we find that

$$d\zeta/dt \ge K^{-q}\,\phi^q(t)\,\psi^{-q}(t)\,\zeta(t).$$

Integrating from t_1 $(t_1 > t_0)$ to t, we obtain the inequality

$$\zeta(t) \ge \zeta(t_1)\exp\left[K^{-q}\int_{t_1}^{t}\phi^q(s)\,\psi^{-q}(s)\,ds\right]. \tag{21}$$

Our assumption $\int^{\infty}\phi^q(t)\,\psi^{-q}(t)\,dt = \infty$ now implies that $\lim_{t\to\infty}\zeta(t) = \infty$, which establishes (17).

According to (13), for each $\varepsilon > 0$ we can choose $t_1 \ge t_0$ such that

$$\left[\int_{t_1}^{\infty}\omega^p(s,\,2\rho)\,ds\right]^{1/p} < \varepsilon/2K.$$

By virtue of (17) there exists a $t_2 \geq t_1$ such that

$$\psi^{-1}(t)\,\|Y(t)P_1\| \int_{t_0}^{t_1} \|Y^{-1}(s)f(s, x(s))\|\,ds < \varepsilon/2\,.$$

Using these two inequalities, we obtain from (14) that

$$\begin{aligned}\|x(t) - y(t)\| &\leq \|Y(t)P_1\| \int_{t_0}^{t_1} \|Y^{-1}(s)f(s, x(s))\|\,ds + K\left[\int_{t_1}^{\infty} \omega^p(s, 2\rho)\,ds\right]^{1/p} \\ &\leq \varepsilon\,\psi(t) \qquad (t \geq t_2),\end{aligned}$$

which establishes (5).

To establish the converse, let x be a given ψ-bounded solution of (2). One sees easily that

$$y(t) = x(t) - \int_{t_0}^{t} \Phi_1(t; s)f(s, x(s))\,ds + \int_{t}^{\infty} \Phi_2(t; s)f(s, x(s))\,ds$$

is a ψ-bounded solution of (1) that satisfies (5).

Remark. The analogue of Theorem 2 where ψ and ϕ are constants and $p = 1$, $q = \infty$ has already been obtained by Brauer and Wong [3, Theorem 1].

If additional requirements are placed upon the expression on the left-hand side of (12), then further information may be obtained about the solutions. The next theorem considers the case where the solutions of (1) and (2) belong to the Banach space $L^r[t_0, \infty)$ $(1 \leq r < \infty)$.

THEOREM 3. *Assume that the hypotheses of Theorem 2 are satisfied. Furthermore, suppose that the function* $\psi^{-1}\sigma$, *where* $\sigma = \sigma(t)$ *is given by*

$$\sigma(t) = \left(\int_{t_0}^{t} \|\Phi_1(t; s)\phi(s)\|^q\,ds\right)^{1/q} + \left(\int_{t}^{\infty} \|\Phi_2(t; s)\phi(s)\|^q\,ds\right)^{1/q},$$

belongs to $L^r[t_0, \infty)$. *Then, corresponding to each solution* $y = y(t)$ *of* (1) *for which* $\psi^{-1}y$ *belongs to* $L^r[t_0, \infty) \cap L^\infty[t_0, \infty)$, *there exists a solution* $x = x(t)$ *of* (2) *such that* $\psi^{-1}x$ *belongs to* $L^r[t_0, \infty) \cap L^\infty[t_0, \infty)$ *and* (5) *is satisfied. Conversely, if* $x = x(t)$ *is a solution of* (2) *with* $\psi^{-1}x \in L^r[t_0, \infty) \cap L^\infty[t_0, \infty)$, *then there exists a solution* $y = y(t)$ *of* (1) *with* $\psi^{-1}y \in L^r[t_0, \infty) \cap L^\infty[t_0, \infty)$, *and* (5) *is satisfied.*

Proof. The conclusions of the theorem that involve the space $L^\infty[t_0, \infty)$ follow from Theorem 2. The remainder of the proof demonstrates that $\psi^{-1}x$ belongs to $L^r[t_0, \infty)$ provided $\psi^{-1}y$ belongs to $L^r[t_0, \infty)$, and conversely. Equation (14), with $Tx = x$, gives the correspondence between the solutions x and y:

$$x(t) = y(t) + \int_{t_0}^{t} \Phi_1(t; s)f(s, x(s))\,ds - \int_{t}^{\infty} \Phi_2(t; s)f(s, x(s))\,ds\,.$$

By Hölder's inequality, we obtain

$$\|x(t)\| \leq \|y(t)\| + \sigma(t)\left(\int_{t_0}^{\infty} \omega^p(s, 2\rho)\,ds\right)^{1/p}; \tag{22}$$

here we have assumed that x belongs to $B_{\psi,2\rho}$. Since the product of ψ^{-1} and the right-hand side of (22) belongs to $L^r[t_0, \infty)$, the function $\psi^{-1}x$ belongs to $L^r[t_0, \infty)$. The converse follows similarly.

4. EXAMPLES

In this section, we give two examples to indicate some advantages that are obtained by introducing the scalar functions ψ and ϕ. We compare the results obtainable for Theorem 1 in the case where $\psi \equiv \phi \equiv 1$ and in an instance where the functions ψ, ϕ are chosen more judiciously. Example 1 shows that by specifying the function ψ properly, we cannot only strengthen the asymptotic growth condition (5) but also improve the hypothesis (3), which relates to the perturbation term f, over the case $\psi \equiv 1$. Example 2 similarly points out that a suitable choice of the function ϕ allows a more general perturbation term f than the case $\phi \equiv 1$. It is clear that the introduction of the function ϕ does not alter the conclusion (5) of the theorem as the function ψ does; the function ϕ can only serve to weaken the required hypotheses on f.

Example 1. Consider the linear equation

$$dy/dt = -(2t + t^{-1})y \qquad (t \geq 1) \tag{23}$$

and the nonlinear perturbation of (23)

$$dx/dt = -(2t + t^{-1})x + a(t)x^r \qquad (t \geq 1), \tag{24}$$

where $a(t)$ is continuous for $t \geq 1$ and $r > 0$.
The solutions of (23) are of the form $y(t) = ct^{-1}e^{-t^2}$, for some constant c.

First, we indicate the nature of the hypotheses of Theorem 1 for the choice $\psi \equiv \phi \equiv 1$. Since the relation

$$t^{-1}e^{-t^2}\int_1^t e^{s^2}s\,ds = o(1) \qquad (t \to \infty)$$

holds, (4) is satisfied. The condition $\lim_{t\to\infty} \gamma_\lambda(t) = 0$ $(0 < \lambda < \infty)$ for the particular choice $f(t, x) = a(t)x^r$ reduces to the condition

$$\lim_{t\to\infty} a(t) = 0. \tag{25}$$

If condition (25) is satisfied, Theorem 1 implies that corresponding to each bounded solution y of (23), there exists a bounded solution x of (24) such that

$$|x(t) - y(t)| = o(1) \qquad (t \to \infty), \tag{26}$$

and conversely.

The choice $\psi(t) = t^{-1}$, $\phi(t) \equiv 1$ still implies that (4) is valid. In this case, the specification of the function ω in (3) and the condition $\lim_{t \to \infty} \gamma_\lambda(t) = 0$ lead to the hypothesis $\lim_{t \to \infty} a(t) t^{-r} = 0$. This is a weaker condition than (25), since $r > 0$. Also, the corresponding solutions x and y of (23) and (24), respectively, satisfy the condition

$$|x(t) - y(t)| = o(t^{-1}) \qquad (t \to \infty),$$

which is a stronger result than (26).

Example 2. We consider the equations

$$dy/dt = -2ty \qquad (t \geq 0) \tag{27}$$

and

$$dx/dt = -2tx + a(t)x^r \qquad (t \geq 0), \tag{28}$$

where a(t) is continuous and r is a positive number. The solutions of (27) are $y = c\,e^{-t^2}$, where c is a constant. As in Example 1, inequality (4) is satisfied for $\psi \equiv \phi \equiv 1$, since

$$0 \leq e^{-t^2} \int_1^t e^{s^2} ds \leq 1;$$

and if we require condition (25), then the hypotheses of Theorem 1 are satisfied. However, if one takes $\psi \equiv 1$ and $\phi(t) = 2t$, then (4) still holds, but we may replace (25) by the weaker condition $\lim_{t \to \infty} t^{-1} a(t) = 0$.

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