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A Strong Model of Paraconsistent Logic

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Abstract The purpose of this paper is mainly to give a model of paraconsistent logic satisfying the "Frege comprehension scheme" in which we can develop standard set theory (and even much more as we shall see). This is the continuation of the work of Hinnion and Libert.

1 Introduction

In paraconsistent logic, we have three truth values instead of two as in classical logic: true, false, and inconsistent. In classical logic, any theory containing a paradox proves everything; this is not true in paraconsistent logic where a paradox does not necessarily entail everything.

The idea behind this work is to recover the whole theory of Frege: any formula defines a set, the Russell's paradox being tolerated. This is the continuation of the ideas of Hinnion in [11] where he has constructed a model but his model was too weak to develop standard set theory (due to the lack of the axiom of infinity). As in [11], the paraconsistent logic can be described in a classical setting: we consider the first-order language $\mathcal{L}^{\pm}$: $(\in^+, \in^-, =)$ and the following axiom:

Pd-case $\quad \forall x \forall y (y \in^+ x \vee y \in^- x)$.

Intuitively $\in^+$ is the membership relation and $\in^-$ is the *weak negation* of $\in^+$. In some context there is also an equality $=^+$ and a weak negation of the equality $=^-$, but we will not need these here; our equality will be classical. This corresponds to the logic CLuNs (Batens and De Clercq [2]) or Pac (Avron [1]). For more details, we refer to [11].

In Section 2, we will present the theory HF_∞ where we have, among others, the comprehension scheme for positive formulas on the language $\mathcal{L}^{\pm}$: we can use the

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membership relation and the *weak negation* of $\in^+$: $\in^-$ but not the real negation of $\in^+$! We will see that the hereditarily classical well-founded sets of HF_∞ interpret ZF and much more. The theory HF_∞ appears, as stated here, for the first time in Libert [12] without the axiom of infinity; there it was suggested, however, that such an axiom could be added and there it was also suggested that it would be possible to recover ZF. In Section 3, we construct a model of HF_∞ in an extension of ZF with a large cardinal assumption. Finally, in Subsection 3.3, we give the precise consistency strength of HF_∞.

Before going into the details, it is interesting to recall the history of such ideas. In [9], Gilmore had studied a comprehension scheme analogue to ours but in *partial set theory* (in which the Pt-Case: $\forall x \forall y \neg(y \in^+ x \wedge y \in^- x)$ replaces the Pd-Case). He does not have extensionality and his theory is very weak. Forti and Hinnion have studied in [7] the consistency of some positive comprehension but on the language of standard set theory $(\in, =)$. This leads to *hyperuniverses*, deeply studied by Forti and Honsell (see mainly [7] and [8]) and to the positive set theory GPK^+_∞, which have been studied by the author (see mainly Esser [5]). Using techniques similar to [7], Hinnion has built in [11] a model of paraconsistent logic on the language $\mathcal{L}^\pm$ but his model does not satisfy the axiom of infinity. Libert presents the theory HF in [12]. Here we study the theory HF_∞ in more detail. We will extensively use the results on hyperuniverses and positive set theory.

2 The Theory

2.1 The axioms As mentioned in the introduction, the logic will be classical. Consider the language $\mathcal{L}^\pm$: $(\in^+, \in^-, =)$ and consider the axiom of paraconsistent logic:

Pd-Case $\quad \forall x \forall y (y \in^+ x \vee y \in^- x)$.

Intuitively $\in^+$ is the membership relation and $\in^-$ is the *weak negation* of $\in^-$. Following [12], we say that a set a is *less paraconsistent* than a set b, which we denote $a \leqslant_p b$, if and only if $(\forall x \in^+ a)(x \in^+ b) \wedge (\forall x \in^- a)(x \in^- b)$. Obviously $\leqslant_p$ is a partial order. The class of the Bounded Positive Formulas on the language $\mathcal{L}^\pm$ ($\mathrm{BPF}^\pm$) is the smallest class containing the atomic formulas ($x \in^+ y$, $x \in^- y$, $x = y$) and such that if φ and ψ are $\mathrm{BPF}^\pm$, then so are $\forall x\, \varphi$, $\forall x \in^+ y\, \varphi$, $\forall x \in^- y\, \varphi$, $\exists x\, \varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$.

The theory HF is the following theory:

Pd-Case

EXT: $\quad \forall x \forall y \big((\forall z\, z \in^+ x \Leftrightarrow z \in^+ y) \wedge (\forall z\, z \in^- x \Leftrightarrow z \in^- y)\big) \Rightarrow x = y.$

$\mathrm{CL}^\pm$: For every formula φ and ψ, the universal closure of '$\forall x(\varphi \vee \psi) \Rightarrow$ there is an $\leqslant_p$-minimal set such that $\forall x(\varphi \Rightarrow x \in y) \wedge \forall x(\psi \Rightarrow x \in^- y)$'.

Comp($\mathrm{BPF}^\pm$): For every pair φ, ψ of $\mathrm{BPF}^\pm$ formula, the universal closure of $\big(\forall x(\varphi \vee \psi)\big) \Rightarrow \big(\exists y(\forall x\, x \in y \Leftrightarrow \varphi) \wedge (\forall x\, x \in^- y \Leftrightarrow \psi)\big)$.

We will use *classes* in the theory HF. If A denotes the class $\{x \mid \varphi(x)\}$, we may write $x \in A$ as an abbreviation for $\varphi(a)$. As usual, this is done only for the clarity of the report; we can speak only about formulas.

A set a will be called *well-founded* if and only if

$$\forall y\big(a \in^+ y \Rightarrow (\exists y' \in^+ y)(\forall t\ t \notin^+ y \vee t \notin^+ y')\big).$$[1]

Let us denote by WF the class of the well-founded sets. Consider the following axiom of infinity:

INF $\quad \big(\exists a \in WF\big)\big(\varnothing \in^+ a \wedge \forall y \in^+ a\ \exists t(t \in^+ a \wedge \forall z(z \in^+ t \Leftrightarrow (z \in^+ y \vee z = y)))\big).$

The axiom INF expresses the idea that there is an infinite well-founded set; any axiom expressing the same idea would work also. We denote by HF_∞ the theory HF + INF.

Before going further, let us recall some results about the positive set theory.

2.2 The positive set theory The positive set theory is a theory on the language $\mathcal{L}: (\in, =)$ of set theory. The class BPF of *Bounded Positive Formulas* is the smallest class containing the atomic formulas ($x = y$ and $x \in y$) and such that if φ and ψ are BPFs then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\forall x \in y\ \varphi$, $(\exists x \in y\ \varphi)$, $(\forall x\ \varphi)$, $\exists x\ \varphi$ (the formulas between brackets can be deduced BPF from the other). The *Bounded Positive Comprehension* (BPC) is the following scheme:

Comp(BPC) The universal closure of $\exists u \forall t\ t \in u \Leftrightarrow \varphi$, where φ is BPF.

The positive set theory GPK^+_∞ is the following theory:

EXT: $\forall x \forall y \forall z(z \in x \Leftrightarrow z \in y) \Rightarrow x = y.$

Comp(BPC)

The following scheme:

CL: For any formula $\varphi(z, y_1, \ldots, y_n)$ whose free variables are among $z, y_1, \ldots, y_n$: $\forall y_1 \ldots \forall y_n \exists x\big(\forall z(\varphi(z, y_1, \ldots, y_n) \Rightarrow z \in x) \wedge \forall y((\forall z(\varphi(z, y_1, \ldots, y_n) \Rightarrow z \in y)) \Rightarrow x \subset y)\big).$

The scheme CL states that for each class $A = \{z \mid \varphi(z, y_1, \ldots, y_n)\}$, there is a smallest set for the inclusion which contains it; we denote this set $\overline{A}$. The closure operation behaves like a topological closure except that it acts only on *classes*, that is, on *definable* collections of the universe.

A set a is called *well-founded* if and only if $\forall y\big(a \in y \Rightarrow (\exists y' \in y)(y \cap y' = \varnothing)\big)$. Let us denote by WF the class of the well-founded sets. Consider the following axiom of infinity:

INF $\quad (\exists x \in WF)(\varnothing \in x \wedge (\forall y \in x)(y \cup \{y\} \in x)).$

The axiom INF expresses the idea that there is an infinite well-founded set. Let us denote by GPK^+_∞ the theory $\mathrm{GPK}^+ + \mathrm{INF}$.

One can show that the class WF interprets ZF. Coding classes of well-founded sets by their closure, one can show that we have an interpretation of the Kelley-Morse class theory (KM). The exact power of GPK^+_∞ is KM + 'On is weakly compact'; this last axiom is the natural translation to the class of ordinals of the corresponding property for cardinals in ZF.[2] For more details about this theory, see Esser [4, 5].

2.3 Set theory in HF_∞ It is obvious that HF_∞ interprets $\mathrm{GPK}_\infty^{+\neq}$, which is GPK_∞^+ without extensionality: just take $\in^+$ (or $\in^-$) as the membership symbol. It is known that $\mathrm{GPK}_\infty^{+\neq}$ has the same consistency strength as GPK_∞^+ (see Esser [6]). In order to have a flavor of how HF_∞ works, it is interesting to see how we can directly interpret ZF in HF_∞. A set a is said to be $\in^+$-transitive if and only if $\forall y \forall z (z \in^+ y \in^+ a \Rightarrow z \in^+ a)$. A set a is said to be *hereditarily classical* if and only if there exists an $\in^+$-transitive classical set y with $a \in^+ y$. One can show that the hereditarily classical well-founded sets interpret ZF. The proof is exactly the same as in GPK_∞^+ (see [5]). Notice that, due to the axiom EXT, two classical sets are equal if and only if they have the same $\in^+$-elements.

Using techniques similar to [4] and [5], we can code classes of classical well-founded sets and interpret KM +'On is weakly compact' as in GPK_∞^+.

3 The Model

3.1 The hyperuniverse M_κ Here we recall some properties about the hyperuniverse M_κ (for κ a weakly compact cardinal). Hyperuniverses are natural models of GPK_∞^+. The hyperuniverse M_κ can be seen as the closure of the well-founded sets of the hyperuniverse N_κ of [7]. We will use the presentation of Esser [3] of M_κ. For more details, the reader is referred to [3]. Here the metatheory will be ZFC + 'there exists an uncountable weakly compact cardinal' + U_ℓ where U_ℓ is the axiom of local universality:

U_ℓ For any extensional relation $\in_A$ on a set A, there is a transitive set t such that $(A, \in_A)$ is isomorphic to $(t, \in)$.

The axiom U_ℓ is only assumed for convenience; we could avoid it by replacing 'unexisting transitive sets' by structures $(A, \in_A)$ and working only with these. In the following, κ will denote an uncountable weakly compact cardinal.

Let us introduce the equivalences $\sim_\alpha$ (for $\alpha \leqslant \kappa$) on the universe (first introduced by Malitz in [13]).

$$\begin{cases} \sim_0 &= V \times V \\ \sim_{\alpha+1} &= \{(a,b) \mid (\forall x \in a)(\exists y \in b) x \sim_\alpha y \wedge (\forall y \in b)(\exists x \in a) x \sim_\alpha y\} \\ \sim_\lambda &= \bigcap_{\beta<\alpha} \sim_\alpha \quad (\lambda \text{ limit}) \end{cases}$$

Consider the set V_κ of the well-founded sets of rank less than κ. Call a κ-sequence a function $s : \kappa \to V_\kappa$ and write s_α for $s(\alpha)$. A κ-sequence is said to be *strongly Cauchy* if and only if $(\forall \alpha < \kappa)(\forall \beta \geq \alpha)(s_\alpha \sim_\alpha s_\beta)$. The basis set M_κ is the set of all strongly Cauchy κ-sequences. We define $\approx$ on M_κ by $a \approx b$ if and only if $(\forall \beta < \kappa)(a_\alpha \sim_\alpha b_\alpha)$ and $\overline{\in}$ by $a \mathrel{\overline{\in}} b$ if and only if $(\forall \alpha < \kappa)(\exists a' \sim_\alpha a_\alpha)(\exists b' \sim_\alpha b_\alpha)(a' \in b')$. We extend the definition of the $\sim_\alpha$s on M_κ by saying that, for $a, b \in M_\kappa$, $a \sim_\alpha b$ if and only if $a_\alpha \sim_\alpha b_\alpha$. One can show that $\{\sim_\alpha \mid \alpha < \kappa\}$ forms a basis for a Hausdorff κ-uniformity[3] on M which is κ-compact.[4] The structure $(M_\kappa, \overline{\in}, \approx)$ forms a κ-hyperuniverse. One of its important properties is that it is a topological model: a subset $A \subset M_\kappa$ is *coded*, that is, $\exists a \in M_\kappa A = \{x \in M_\kappa \mid x \mathrel{\overline{\in}} a\}$ if and only if it is closed for the topology of M_κ.

The structure $(M_\kappa, \overline{\in}, \approx)$ being extensional, we can find, following the axiom U_ℓ, a transitive set M such that $(M, \in, =)$ is isomorphic to $(M_\kappa, \overline{\in}, \approx)$. We will denote this last set by M and work only with it. An important property of M is that M

equals the set of its closed subsets. One can show that the equivalences $\sim_\alpha$ of M_κ transported to M coincide with the equivalences $\sim_\alpha$ on M.

3.2 The model We will now construct a model of the theory HF_∞. We will first construct a model of the theory $\mathrm{HF}_\infty^{\neq}$, the theory HF_∞ without extensionality. We will recover the extensionality later by techniques of Hinnion [10] and Esser [6] adapted to handle both membership symbols, $\in^+$ and $\in^-$.

Consider M^2: the ordered pairs of elements of M (we use the Kuratowski pair: $(a, b) := \{\{a\}, \{a, b\}\}$). Recall that since M is a hyperuniverse, we have that $M^2 \subseteq M$ is closed for the topology of M and so $M^2 \in M$. For $(a, b), (a', b') \in M^2$, we have that $(\forall \alpha < \kappa)(a \sim_\alpha a' \wedge b \sim_\alpha b' \Rightarrow (a, b) \sim_{\alpha^{++}} (a', b'))$.[5] A set $a \in M$ is called a *hereditarily ordered pair* if and only if there is some set $t \in M$ with $t \subseteq M^2$ and with $a \in t$ which satisfies $(\forall (x, y) \in t)(x \subseteq t \wedge y \subseteq t)$. Since it is defined by a BPF, we have that the hereditarily ordered pairs form a set, which we denote by h.

For $a \in h$ and $\alpha \leqslant \kappa$, we define $[a]_\alpha := \{x \in h \mid (\forall \beta < \alpha)(\exists y \in a)\ x \sim_\beta y\}$. We have that $[a]_\alpha \subseteq h$ is closed for the topology of M. For each $(a, b) \in h$, we define $\gamma := \sup\{\alpha \leqslant \kappa \mid [a]_\alpha \cup [b]_\alpha = h\}$. Now we define the following function $F : h \to h$ $F(a, b) := ([a]_\gamma, [b]_\gamma)$. It is easy to see that $p_1(F(a, b)) \cup p_2(F(a, b)) = h$.[6] Moreover, F is continuous for the topology of M:

$$\big(\forall (a, b) \in h\big)\big(\forall \alpha < \kappa\big)\big(\exists \beta < \kappa\big)$$
$$\big(\forall (a', b') \in h((a, b) \sim_\alpha (a', b') \Rightarrow F(a, b) \sim_\beta F(a', b'))\big).$$

In order to prove this, let us take $(a, b) \in h$ and consider two cases. If $a \cup b \neq h$, taking γ as before, we have $\gamma < \kappa$. We see that if $(a', b') \sim_{\gamma^{++}} (a, b)$ then $F(a', b') \sim_{\gamma^{++}} F(a, b)$. Now suppose that $a \cup b = h$, so $F(a, b) = (a, b)$. Let us take $\alpha < \kappa$ and suppose without loss of generality that $\alpha = \alpha'^{++}$ for some $\alpha' < \kappa$. We have

$$(a', b') \sim_{\alpha'^{++}} (a, b) \Rightarrow (a', b') \subseteq F(a', b') \subseteq ([a']_{\alpha'}, [b']_{\alpha'}).$$

Since $(a', b') \sim_{\alpha'^{++}} (a, b)$ and $([a']_{\alpha'}, [b']_{\alpha'}) \sim_{\alpha'^{++}} (a, b)$, this shows that $F(a', b') \sim_{\alpha'^{++}} F(a, b)$ and F is continuous. So we have $F \in M$.[7]

For $a, b \in h$, define $\in^+$ by $a \in^+ b$ if and only if $a \in p_1(F(b))$ and $a \in^- b$ if and only if $a \in p_2(F(b))$. Using the fact that the hyperuniverse M satisfies GPK_∞^+, one can show that $(h, \in^+, \in^-, =)$ as defined forms a model of $\mathrm{HF}_\infty^{\neq}$. The Pd-Case is obvious. For $\mathrm{CL}^\pm$, if φ and ψ are formulas involving $\in^+$ and $\in^-$ and where all quantifiers are restricted to h, just take $\big(\overline{\{x \in h \mid \varphi(x)\}}, \overline{\{x \in h \mid \psi(x)\}}\big)$ for the $\leqslant_p$-minimal set of $\mathrm{CL}^\pm$. For Comp($\mathrm{BPF}^\pm$), first notice that any $\mathrm{BPF}^\pm$ formula where all quantifiers are restricted to h can be translated to a BPF formula using the definition of $\in^+$ and $\in^-$. If φ and ψ are such formulas satisfying $\forall x(\varphi \vee \psi)$, just take $(\{x \in h \mid \varphi(x)\}, \{x \in h \mid \psi(x)\})$ for the set y of Comp($\mathrm{BPF}^\pm$). Notice that we do not have extensionality since, for example, $(\varnothing, \varnothing)$ and (h, h) have the same $\in^+$- and $\in^-$-elements.

Now we will recover EXT using techniques of [6] and [10]. We will prove that $\mathrm{HF}_\infty^{\neq}$ and HF_∞ are mutually interpretable. So let us place ourselves in $\mathrm{HF}_\infty^{\neq}$. Since the proof follows [6], we will only give the pattern and we refer the reader to this paper for more details. First note that $(\mathrm{HF}_\infty^{\neq}, \in^+, =)$ is an interpretation of $\mathrm{GPK}_\infty^{+\,\neq}$, similarly for $(\mathrm{HF}_\infty^{\neq}, \in^-, =)$. In the following, all set constructions are supposed to be relative to $\in^+$. In $\mathrm{HF}_\infty^{\neq}$, unless mentioned explicitly, we will still use terms $\{x \mid \varphi\}$ although they are ambiguous. If a formula φ contains a term, just rewrite it, as usual, using $\in^+$ as the default membership symbol (so without using terms). For instance, a formula such as $a \in^+ \{x \mid \varphi(x)\}$ must be interpreted as $\varphi(a)$. If we write $a = \{x \mid \varphi(x)\}$ we mean $\forall x(x \in^+ a \Leftrightarrow \varphi(x))$; so there can be several as satisfying $a = \{x \mid \varphi(x)\}$. Many sets will be in fact defined "up to $\doteq$" in $\mathrm{HF}_\infty^{\neq}$ (where $a \doteq b$ if and only if a and b have the same $\in^+$-elements). We first need a few definitions.

Definition 3.1

1. A *set* r is a *relation* if and only if $\forall z \in^+ r\ \exists x \exists y\ z = (x, y)$. We will adopt, as usual, the following notation: $x\ r\ y$ for $(x, y) \in^+ r$. It is easy to see that "r is a relation" is a $\mathrm{BPF}^{\pm}$.
2. The relation $\doteq$ is defined as follows: $a \doteq b$ iff $\forall x(x \in^+ a \Leftrightarrow x \in^+ b)$.
3. If r is a relation, r^+ is the following relation:
$$r^+ = \{(a, b) \mid (\forall x \in^+ a)(\exists y \in^+ b) x\ r\ y \wedge\ (\forall y \in^+ b)(\exists x \in^+ a) x\ r\ y$$
$$\wedge\ (\forall x \in^- a)(\exists y \in^- b) x\ r\ y \wedge (\forall y \in^- b)(\exists x \in^- a) x\ r\ y\}.$$
We easily see that r^+ is defined by a BPF and so it is a set.
4. For a relation r, $r \circ r$ is defined as follows:
$$r \circ r = \{(x, y) \mid \exists z (x, z) \in^+ r \wedge (z, y) \in^+ r\}.$$
We see that $r \circ r$ is defined by a $\mathrm{BPF}^{\pm}$.
5. A relation r is an *equivalence* if and only if
$$\forall x (x, x) \in^+ r \wedge \forall x, y\big((x, y) \in^+ r \Rightarrow (y, x) \in^+ r\big) \wedge r \circ r \subseteq r.$$
Notice that "to be an equivalence" is not described by a $\mathrm{BPF}^{\pm}$.
6. r is a *final* relation if and only if $r \subseteq r^+$. We see that "to be a final relation" is defined by a $\mathrm{BPF}^{\pm}$; notice that contrary to [10], we do not ask r to be an equivalence . The reason is that "to be an equivalence" is not described by a $\mathrm{BPF}^{\pm}$.[8]
7. r is a contraction if and only if r is an equivalence and $r \doteq r^+$.

Now we define in $\mathrm{HF}_\infty^{\neq}$: $a \equiv b \Leftrightarrow \exists r($"$r$ is a final relation" $\wedge\ a\ r\ b)$ and
$$a \in^+_{\equiv} b \Leftrightarrow \exists a' \equiv a \exists b' \equiv b\ a' \in^+ b';$$
$$a \in^-_{\equiv} b \Leftrightarrow \exists a' \equiv a \exists b' \equiv b\ a' \in^- b'.$$

As in [5], we check that we have an interpretation of HF_∞ in $\mathrm{HF}_\infty^{\neq}$ with the equality interpreted as $\equiv$, $\in^+$ interpreted as $\in^+_{\equiv}$, and $\in^-$ interpreted as $\in^-_{\equiv}$.

3.3 The consistency strength of $\mathbf{HF}_\infty$ We have constructed a model of HF_∞ using M_κ, the Cauchy completion of V_κ, the set of well-founded sets of rank less than κ. It is possible to adapt this result and to obtain an interpretation of HF_∞ inside KM + 'On is weakly compact' by taking the Cauchy-completion of the class of well-founded sets. This last theory is known to be equiconsistent with the theory GPK^+_∞ (see [5]). In an adaptation of this last paper, see also that HF is mutually interpretable with PA_2 (second-order arithmetic). Let us summarize these last results in the following theorem.

Theorem 3.2 *The theory* HF_∞ *is mutually interpretable with* GPK^+_∞ *which is also* KM + *'On is weakly compact'. The theory* HF *is mutually interpretable with* PA_2.

Notes

1. $\notin^+$ and $\notin^-$ denote the classical negation of $\in^+$ and $\in^-$.
2. One can show that the presence or the absence of the axiom of choice does not change the consistency strength of this theory.
3. A κ-uniform space is a uniform space where the set of entourages is stable by intersection of cardinality less than κ.
4. This means that any open cover contains a κ-finite subcover; κ-finite means of cardinality less than κ.
5. For an ordinal α, α^+ denotes the successor of α: $\alpha \cup \{\alpha\}$.
6. p_1 and p_2 denote the projection on the first and second component.
7. Recall that, as is usual in set theory, a function is identified with its graph.
8. A final equivalence is also called a *bisimulation* in the literature.

References

[1] Avron, A., "Natural 3-valued logics—characterization and proof theory," *The Journal of Symbolic Logic*, vol. 56 (1991), pp. 276–94. Zbl 0745.03017. MR 93b:03032. 149

[2] Batens, D., and K. De Clercq, "A rich paraconsistent extension of full positive logic," available at http://logica.rug.ac.be/centrum/writings/. 149

[3] Esser, O., "Mildly ineffable cardinals and hyperuniverses," forthcoming in Reports on Mathematical Logic. 152

[4] Esser, O., "An interpretation of the Zermelo-Fraenkel set theory and the Kelley-Morse set theory in a positive theory," *Mathematical Logic Quarterly*, vol. 43 (1997), pp. 369–77. Zbl 0880.03030. MR 98i:03068. 151, 152

[5] Esser, O., "On the consistency of a positive theory," *Mathematical Logic Quarterly*, vol. 45 (1999), pp. 105–16. Zbl 0924.03102. MR 2000e:03145. 150, 151, 152, 154

[6] Esser, O., "On the axiom of extensionality in the positive set theory," *Mathematical Logic Quarterly*, vol. 49 (2003), pp. 97–100. Zbl 1015.03054. MR 2004b:03099. 152, 153

[7] Forti, M., and R. Hinnion, "The consistency problem for positive comprehension principles," *The Journal of Symbolic Logic*, vol. 54 (1989), pp. 1401–18. Zbl 0702.03026. MR 91d:03020. 150, 152

[8] Forti, M., and F. Honsell, "A general construction of hyperuniverses," *Theoretical Computer Science*, vol. 156 (1996), pp. 203–15. Zbl 0871.68131. MR 97d:03072. 150

[9] Gilmore, P. C., "The consistency of partial set theory without extensionality," pp. 147–53 in *Axiomatic Set Theory (Proceedings of Symposia in Pure Mathematics, vol. 13, part 2, University of California, Los Angeles, 1967)*, American Mathematical Society, Providence, 1974. Zbl 0309.02065. MR 50:12721. 150

[10] Hinnion, R., "Extensional quotients of structures and applications to the study of the axiom of extensionality," *Bulletin de la Société Mathématique de Belgique. Série B*, vol. 33 (1981), pp. 173–206. Zbl 0484.03029. MR 84c:03089. 153, 154

[11] Hinnion, R., "Naïve set theory with extensionality in partial logic and in paradoxical logic," *Notre Dame Journal of Formal Logic*, vol. 35 (1994), pp. 15–40. Zbl 0801.03019. MR 95c:03063. 149, 150

[12] Libert, T., "ZF and the axiom of choice in some paraconsistent set theories," forthcoming. 150

[13] Malitz, R. J., *Set Theory in Which the Axiom of Foundation Fails*, Ph.D. thesis, University of California, 1976. Available from University Microfilms International, Ann Arbor, MI 48106. 152

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