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Syntax and Semantics of the Logic $\mathcal{L}^{\lambda}_{\omega\omega}$

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Abstract In this paper we study the logic $\mathcal{L}^{\lambda}_{\omega\omega}$, which is first-order logic extended by quantification over functions (but not over relations). We give the syntax of the logic as well as the semantics in Heyting categories with exponentials. Embedding the generic model of a theory into a Grothendieck topos yields completeness of $\mathcal{L}^{\lambda}_{\omega\omega}$ with respect to models in Grothendieck toposes, which can be sharpened to completeness with respect to Heyting-valued models. The logic $\mathcal{L}^{\lambda}_{\omega\omega}$ is the strongest for which Heyting-valued completeness is known. Finally, we relate the logic to locally connected geometric morphisms between toposes.

1 Introduction In this paper we study aspects of completeness of the logic $\mathcal{L}^{\lambda}_{\omega\omega}$, which is intuitionistic first-order logic extended by quantification over functions. This logic may be seen as well as λ-calculus enriched with first-order logic. The details of the syntax are given in Section 2.

The logic $\mathcal{L}^{\lambda}_{\omega\omega}$ is of interest for many reasons: it is reasonably powerful and (therefore) incomplete with respect to models in **Sets**. But the logic $\mathcal{L}^{\lambda}_{\omega\omega}$ is complete with respect to Heyting-valued models. In fact, the infinitary variants $\mathcal{L}^{\lambda}_{\kappa\omega}$ are the strongest logics we know that are complete with respect to Heyting-valued models. Secondly, the logic $\mathcal{L}^{\lambda}_{\omega\omega}$ characterizes a class of geometric morphisms between Grothendieck toposes which are almost locally connected: we show that if the inverse image f^* of a geometric morphism $f\colon \mathcal{F} \longrightarrow \mathcal{E}$ between Grothendieck toposes preserves the internal $\mathcal{L}^{\lambda}_{\omega\omega}$-logic of the topos $\mathcal{E}$, then it is open and each $(f/_E)^*\colon \mathcal{E}/_E \longrightarrow \mathcal{F}/_{f^*E}$ has an $\mathcal{E}$-indexed left adjoint.

The first two sections discuss the syntax and semantics of the logic $\mathcal{L}^{\lambda}_{\omega\omega}$. Models of $\mathcal{L}^{\lambda}_{\omega\omega}$-theories naturally live in Heyting categories with exponentials (that is, cartesian closed Heyting categories). After relating the logic to locally connected geometric morphisms we present some completeness results in Section 5: $\mathcal{L}^{\lambda}_{\omega\omega}$ is complete with respect to models in Grothendieck toposes, therefore as well complete with respect to models in cartesian closed Heyting categories. A recent covering theorem

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for Grothendieck toposes implies that it is enough to look at Heyting-valued models to get completeness. The last section contains some remarks about the infinitary variants $\mathcal{L}^{\lambda}_{\kappa\omega}$.

We assume familiarity with basic notions of categorical logic, see, for example, Lambek and Scott [14] or Freyd and Scedrov [9]. The results presented here are closely related to those found in Awodey and Butz [2]. In fact, they give a detailed exposition of one of the completeness results presented there. In case of pure typed λ-calculus, a more detailed exposition can be found in Awodey [1].

Our overall presentation is in the line of categorical model theory, as was done for geometric logic in Makkaki and Reyes [15] and for first-order logic in Butz and Johnstone [6]. One of the more prominent theories which can be formulated in the logic $\mathcal{L}^{\lambda}_{\omega\omega}$ is SDG, synthetic differential geometry (see Kock [13]). In contrast to this we do not intend to do proof theory here, as was one of the items in [14].

2 Syntax We begin by describing the syntax of the logic $\mathcal{L}^{\lambda}_{\omega\omega}$. Given a set $\underline{\text{type}}$ of basic sorts $A, B, \ldots,$ the set $\underline{\text{type}}^*$ of derived types is the closure of $\underline{\text{type}}$ under products and exponentials:

$$\underline{\text{type}}^* ::= A \mid Y \times Z \mid Z^Y.$$

Thus, the only difference to full higher-order logic is the absence of the type of propositions Ω.

Definition 2.1 A λ-signature $\mathbb{S}$ consists of a list $\underline{\text{type}}_{\mathbb{S}}$ of basic types and sets $\underline{\text{const}}_{\mathbb{S}}$, $\underline{\text{funct}}_{\mathbb{S}}$, and $\underline{\text{rel}}_{\mathbb{S}}$ of constants, functions, and relation symbols, where each of these symbols is typed over $\underline{\text{type}}^*$.

Since $\underline{\text{type}}^*$ has built-in product types, we can assume that all functions and relations are unary. As usual, we write expressions like $c\colon A$, $f\colon Z \longrightarrow Y$, or $R \subset Y$ to indicate the typing.

Next we define the sets $\underline{\text{term}}(Y)$ of terms of type Y, which depend on a given λ-signature $\mathbb{S}$.

1. Each set $\underline{\text{term}}(Y)$ contains countably many variables of type Y, and expressions like $y\colon Y$ have their obvious meanings.
2. If c is a constant of type Y, it is a term of type Y. If t is a term of type Y, and $f\colon Y \longrightarrow Z$ is a function symbol, then $f(t)$ is in $\underline{\text{term}}(Z)$.
3. If $t_1 \in \underline{\text{term}}(Y_1)$ and $t_2 \in \underline{\text{term}}(Y_2)$, then $\langle t_1, t_2\rangle$ is a term of type $Y_1 \times Y_2$. Conversely, if t is in $\underline{\text{term}}(Y_1 \times Y_2)$, then $\pi_1 t$ is a term of type Y_1, and $\pi_2 t$ is a term of type Y_2.
4. If t is a term of type Y and $\alpha \in \underline{\text{term}}(Z^Y)$, then $\alpha(t)$ is a term of type Z. If t is a term of type Z (possibly containing the free variable $y\colon Y$), then $\lambda y.t(y)$ is a term of type Z^Y.

The formulas are generated by the following rules.

1. If t_1 and t_2 are terms of the same type, then $t_1 = t_2$ is a formula.
2. If $R \subset Y$ is a relation symbol and t is a term of type Y, then $R(t)$ is a formula.
3. The logical constants $\perp$ and $\top$ are formulas. If φ and ψ are formulas, so are $\neg\varphi$, $\varphi \wedge \psi$, $\varphi \vee \psi$, and $\varphi \longrightarrow \psi$.

4. If $\varphi(y)$ is a formula (possibly containing the free variable $y{:}\,Y$), then $\forall y{:}\,Y\varphi(y)$ and $\exists y{:}\,Y\varphi(y)$ are formulas.

If we type the formulas by the (imaginary) type Ω, these term and formula forming operations can be summarized in the familiar way:

Y	$Y_1 \times Y_2$	Y_1	Z	Z^Y	Ω
c	$\langle t_1, t_2\rangle$	$\pi_1\bar{t}$	$\alpha(t)$	$\lambda y.t(y)$	$t = t'$
$f(t)$					$R(t)$
					$\bot, \top$
					$\varphi \wedge \psi, \varphi \vee \psi$
					$\neg\varphi, \varphi \longrightarrow \psi$
					$\forall y\varphi(y), \exists y\varphi(y)$

where $c{:}\,Y,\ f{:}\,Z \longrightarrow Y,\ R \subset Z$, and the subterms are of type

Z	Y_1	Y_2	$Y_1 \times Y_2$	Z^Y	Ω
t, t'	t_1	t_2	$\bar{t}$	α	φ, ψ
$t(y)$					$\varphi(y)$

For each finite set X of variables we define a deduction relation $\vdash_X$ between formulas. If we write an expression $p \vdash_X q$ it is always assumed that the free variables occurring on both sides are contained in the set X. Below, $\vdash_X p$ abbreviates $\top \vdash_X p$, and $p \vdash q$ stands for $p \vdash_\varnothing q$. As in [14] we group the rules into different classes.

Structural rules

1.1 $p \vdash_X p$.

1.2 $p \vdash_X q$ and $q \vdash_X r$ implies $p \vdash_X r$.

1.3 $p \vdash_X q$ implies $p \vdash_{X\cup\{y\}} q$.

1.4 $\varphi(y) \vdash_X \psi(y)$ implies $\varphi(b) \vdash_{X\setminus\{y\}} \psi(b)$,

provided that y is a variable of type Y and b is a term of type Y with no free occurrence of variables other than those in $X \setminus \{y\}$. It is being assumed that b is substitutable for y in both sides, that is, no free variable in b becomes bound after substitution.

Logical rules

2.1 $p \vdash_X \top$; $\quad \bot \vdash_X p$.

2.2 $r \vdash_X p \wedge q$ iff $r \vdash_X p$ and $r \vdash_X q$; $\quad p \vee q \vdash_X r$ iff $p \vdash_X r$ and $q \vdash_X r$.

2.3 $p \vdash_X q \longrightarrow r$ iff $p \wedge q \vdash_X r$.

2.4 $p \vdash_X \forall y\psi(y)$ iff $p \vdash_{X\cup\{y\}} \psi(y)$; $\quad \exists y\psi(y) \vdash_X p$ iff $\psi(y) \vdash_{X\cup\{y\}} p$.

Extralogical axioms

3.1 $\vdash \forall z{:}\,Y_1 \times Y_2\ (z = \langle \pi_1 z, \pi_2 z\rangle)$.

3.2 $\vdash \forall z{:}\,Y_1 \times Y_2 \forall z'{:}\,Y_1 \times Y_2\ (z = z' \longrightarrow (\pi_1 z = \pi_1 z' \wedge \pi_2 z = \pi_2 z'))$.

3.3 $\vdash \forall y_1{:}\,Y_1 \forall y_2{:}\,Y_2\ (\pi_1\langle y_1, y_2\rangle = y_1 \wedge \pi_2\langle y_1, y_2\rangle = y_2)$.

3.4 (Comprehension) $\quad \vdash \forall y{:}\,Y[\lambda y'.t(y')](y) = t(y)$.

3.5 (Extensionality) $\quad \forall f{:}\,Z^Y \forall g{:}\,Z^Y((\forall y{:}\,Y f(y) = g(y)) \longrightarrow f = g)$.

Axioms for equality

4.1 $\vdash_{\{y\}} y = y;\quad y = y' \vdash_{\{y,y'\}} y' = y;$
$y_1 = y_2 \wedge y_2 = y_3 \vdash_{\{y_1,y_2,y_3\}} y_1 = y_3.$

4.2 $y = y' \vdash_{\{y,y'\}} f(y) = f(y'),$ for each functions symbol $f\colon Y \longrightarrow Z;$
$y = y' \vdash_{\{y,y'\}} R(y) \longleftrightarrow R(y'),$ for each relation symbol $R \subset Y.$

The calculus defined so far is intuitionistic. The deduction relations $\vdash^c_X$ are defined by adding the logical rule

$$\top \vdash_X p \vee \neg p.$$

In general, we write $T \vdash p$ (or $T \vdash_X p$) for derivability in the calculus with added axioms $\vdash \tau$ for τ in T. In case T consists of just one formula, the two notions $\{\tau\} \vdash p$ and $\tau \vdash_p$ coincide, so that $T \vdash p$ just extends our definition of $\vdash$. Similar calculi as above for full second-order logic can be found in Boileau and Joyal [4] and in [14].

3 Semantics It should be clear from the syntax that the right categories hosting models of $\mathcal{L}^\lambda_{\omega\omega}$-theories are ($\omega$-) Heyting categories with exponentials (i.e., cartesian closed Heyting categories or logoi with exponentials in the language of [9]). Recall that a Heyting category is a regular category C that has, in addition to finite intersections of subobjects, unions of finite families of subobjects. Moreover, pulling back subobjects along a fixed morphism has a right adjoint. (It follows that the lattice of subobjects of each object in C is a Heyting algebra, and this Heyting algebra structure is preserved under pullbacks.) The most prominent examples of Heyting categories with exponentials are elementary toposes, in particular, Grothendieck toposes.

Let C be a Heyting category with exponentials. An *interpretation* M of a λ-signature $\mathbb{S}$ in C assigns first of all to each basic sort $A \in \underline{\text{type}}_{\mathbb{S}}$ an object $A^{(M)}$. This assignment extends naturally to all types, by $(Y \times Z)^{(M)} = Y^{(M)} \times Z^{(M)}$ and $(Z^Y)^{(M)} = {Z^{(M)}}^{Y^{(M)}}$. Furthermore, the interpretation M assigns a global element $c^{(M)}\colon 1 \longrightarrow Y^{(M)}$ for each constant $c\colon Y$ in $\underline{\text{const}}_{\mathbb{S}}$, a function $f^{(M)}\colon Y^{(M)} \longrightarrow Z^{(M)}$ for each function symbol $f\colon Y \longrightarrow Z$ in $\underline{\text{funct}}_{\mathbb{S}}$, and a subobject $R^{(M)} \rightarrowtail Y^{(M)}$ for each relation symbol $R \subset Y$ in $\underline{\text{rel}}_{\mathbb{S}}$. Using the structure of the category C, we extend this interpretation to arbitrary terms and formulas. In particular, for a formula $\psi(\overline{y}\colon\overline{Y})$ ($\overline{y} = (y_1, \ldots, y_n)$ of type $\overline{Y} = Y_1, \ldots, Y_n$) we get a subobject

$$\{\overline{y} \mid \psi(\overline{y})\}^{(M)} \rightarrowtail \overline{Y}^{(M)} = Y_1^{(M)} \times \cdots \times Y_n^{(M)}.$$

As usual, we say that M is a model of a closed formula τ ($M \models \tau$) if $\{\cdot \mid \tau\}^{(M)} \rightarrowtail \varnothing^{(M)} = 1_C$ is the top element in the Heyting algebra of subobjects of 1_C. This way we get a sound notion of models.

Proposition 3.1 (Soundness) *The deduction relation $\vdash$ is sound for the notion of models just defined, that is, for any set of $\mathcal{L}^\lambda_{\omega\omega}$-formulas T and any λ-formula τ, $T \vdash \tau$ implies $T \models \tau$.*

One of our main goals will be to prove the converse of Proposition 3.1, that is, completeness. Next we turn the class of models of a theory in a fixed Heyting category C with exponentials into a category. A *morphism* h between $\mathbb{S}$-interpretations M and M' is a family of maps $\{h_Y\colon Y^{(M)} \longrightarrow Y^{(M')}\}_{Y \in \underline{\text{type}}^*_{\mathbb{S}}}$, satisfying the following three conditions:

1. $h_{Y_1\times Y_2} = \langle h_{Y_1}, h_{Y_2}\rangle : Y_1^{(M)} \times Y_2^{(M)} \longrightarrow Y_1^{(M')} \times Y_2^{(M')}$
   for all types $Y_1, Y_2 \in \underline{\text{type}}^*_{\mathbb{S}}$.
2. For all Y and Z in $\underline{\text{type}}^*_{\mathbb{S}}$ the following two diagrams commute:

$$\begin{tikzcd} Y^{(M)} \times (Z^Y)^{(M)} \arrow[r, "\langle h_Y, h_{Z^Y}\rangle"] \arrow[d, "\epsilon"'] & Y^{(M')} \times (Z^Y)^{(M')} \arrow[d, "\epsilon"] \\ Z^{(M)} \arrow[r, "h_Z"'] & Z^{(M')} \end{tikzcd} \qquad \begin{tikzcd} Z^{(M)} \arrow[r, "h_Z"] \arrow[d, "\widehat{\text{const}}"'] & Z^{(M')} \arrow[d, "\widehat{\text{const}}"] \\ (Z^Y)^{(M)} \arrow[r, "h_{Z^Y}"'] & (Z^Y)^{(M')} \end{tikzcd}$$

   where $\widehat{\text{const}}$ is the transposed of the projection map $\pi_2 : Z^{(-)} \times Y^{(-)} \longrightarrow Z^{(-)}$.
3. The maps $\{h_Y\}_{Y\in\underline{\text{type}}^*_{\mathbb{S}}}$ preserve the interpretation of constants, function and relation symbols. For example, for a constant $c : Y$ this means that

$$\begin{tikzcd} Y^{(M)} \arrow[r, "h_Y"] & Y^{(M')} \\ 1 \arrow[u, "c^{(M)}"] \arrow[ur, "c^{(M')}"'] & \end{tikzcd}$$

   commutes.

For the following definition we remind the reader of the forcing relation $\Vdash$ in C (usually only defined if C is a topos): For a λ-formula $\psi(\overline{y}:\overline{Y})$, for U in $\mathcal{E}$, and for generalized elements $\alpha_i : U \longrightarrow Y_i^{(M)}$ we write $U \Vdash \psi(\alpha_1, \ldots, \alpha_n)$ if the map $\langle \alpha_1, \ldots, \alpha_n\rangle : U \longrightarrow \overline{Y}^{(M)}$ factors through $\{\overline{y} \mid \psi(\overline{y})\}^{(M)} \hookrightarrow \overline{Y}^{(M)}$.

Definition 3.2 Let M and M' be two S-interpretations in C. A morphism of $\mathbb{S}$-structures $h : M \longrightarrow M'$ is called an *$\mathcal{L}^{\lambda}_{\omega\omega}$-homomorphism* if for each $\mathcal{L}^{\lambda}_{\omega\omega}$-formula $\psi(\overline{y}:\overline{Y})$ and generalized elements $\alpha_i : U \longrightarrow Y_i^{(M)}$

$$U \Vdash \psi(\alpha_1, \ldots, \alpha_n) \qquad \text{implies} \qquad U \Vdash \psi(h_{Y_1}\circ\alpha_1, \ldots, h_{Y_n}\circ\alpha_n).$$

We denote by $\underline{\text{Mod}}^{\lambda}(T, C)$ the category of models of T in C, with morphisms the $\mathcal{L}^{\lambda}_{\omega\omega}$-homomorphisms. Note that the condition of the definition is equivalent to the following: $h : M \longrightarrow M'$ is a $\mathcal{L}^{\lambda}_{\omega\omega}$-homomorphism if and only if for each formula $\psi(\overline{y}:\overline{Y})$ the composite $h_{\overline{Y}} \circ i : \{\overline{y} \mid \psi(\overline{y})\}^{(M)} \hookrightarrow \overline{Y}^{(M)} \longrightarrow \overline{Y}^{(M')}$ factors through $\{\overline{y} \mid \psi(\overline{y})\}^{(M')}$, viz.

$$\begin{tikzcd} \overline{Y}^{(M)} \arrow[r] & \overline{Y}^{(M')} \\ \{\overline{y} \mid \psi(\overline{y})\}^{(M)} \arrow[u, hook] \arrow[r, dashed] & \{\overline{y} \mid \psi(\overline{y})\}^{(M')} . \arrow[u, hook] \end{tikzcd}$$

4 A topos theoretical characterization of $\mathcal{L}^{\lambda}_{\omega\omega}$ Recall that a geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ between Grothendieck toposes is called *locally connected* (or molecular in Barr and Paré [3]) if the inverse image f^* commutes with $\prod$-functors. Equivalently, f is locally connected if and only if for all E in $\mathcal{E}$ the inverse image of the induced geometric morphism $f/_E$ in

$$\begin{tikzcd} \mathcal{F}/_{f^*E} \arrow[r] \arrow[d, "f/_E"'] & \mathcal{F} \arrow[d, "f"] \\ \mathcal{E}/_E \arrow[r] & \mathcal{E} \end{tikzcd}$$

preserves exponentials. Locally connected geometric morphisms are open (see Johnstone [11]) and hence preserve the internal first-order logic. We sum this up in the following lemma.

Lemma 4.1 *The inverse image of a locally connected geometric morphism $f\colon \mathcal{F} \longrightarrow \mathcal{E}$ induces a functor*

$$f^*\colon \underline{\mathrm{Mod}}^{\lambda}(T, \mathcal{E}) \longrightarrow \underline{\mathrm{Mod}}^{\lambda}(T, \mathcal{F}),$$

for any $\mathcal{L}^{\lambda}_{\omega\omega}$-theory T.

The next natural question is whether this property characterizes locally connected geometric morphisms. The following proposition shows that the logic $\mathcal{L}^{\lambda}_{\omega\omega}$ captures a class of morphisms which is slightly larger than that of locally connected geometric morphisms.

Proposition 4.2 *Let $f\colon \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism between Grothendieck toposes.*

1. *If f^* preserves internal products, that is, reindexing of the form $\prod_{E \longrightarrow 1}$ for $E \in \mathcal{E}$, then f^* preserves exponentials.*
2. *If f is open and f^* preserves exponentials, then f^* preserves internal products.*

Proof: The first claim holds since for A and E in $\mathcal{E}$ the exponential A^E equals $\prod_{E\longrightarrow 1}(A \times E \longrightarrow E)$. For the second claim note that internal products can be expressed using exponentials and the internal first-order logic.

$$\prod_{E\longrightarrow 1}(\alpha\colon A \longrightarrow E) = \{\gamma \in A^E \mid \forall e \in E.\alpha(\gamma(e)) = e\}.$$

If f is open it preserves the internal first-order logic ([11], Theorem 3.2), so f^* preserves internal products if f^* preserves in addition exponentials. □

Finally we show that in case that f^* preserves the internal $\mathcal{L}^{\lambda}_{\omega\omega}$-logic of $\mathcal{E}$, each slice map $(f/_E)^*$ has an $\mathcal{E}$-internal left adjoint.

Proposition 4.3 *Suppose that the inverse image of $f\colon \mathcal{F} \longrightarrow \mathcal{E}$ preserves the internal $\mathcal{L}^{\lambda}_{\omega\omega}$-logic of the topos $\mathcal{E}$. Then for each E in $\mathcal{E}$, the inverse image $(f/_E)^*$ of the geometric morphism $f/_E\colon \mathcal{F}/_{f^*E} \longrightarrow \mathcal{E}/_E$ has an $\mathcal{E}$-indexed left adjoint.*

Proof: Following [3], Theorem 4, it is enough to show that $(f/_E)^*$ preserves exponentials of the form $\alpha^{\Delta X}$ for $\alpha\colon A \longrightarrow E$ in $\mathcal{E}/_E$, X in $\mathcal{E}$, and Δ the pullback functor $\mathcal{E} \longrightarrow \mathcal{E}/_E$. But $\alpha^{\Delta X} = (C \longrightarrow E)$ for

$$C = \sum_{e \in E} A_e^B = \{\gamma \in A^B \mid \forall b_1, b_2 \in B.\alpha(\gamma(b_1)) = \alpha(\gamma(b_2))\},$$

which is preserved by assumption. □

5 Completeness Here we construct minimal models of $\mathcal{L}^\lambda_{\omega\omega}$-theories in a similar way as was done in [6] or Palmgren [16]. Let $T \subset \mathcal{L}^\lambda_{\omega\omega}(\mathbb{S})$ be a set of axioms. We define a syntactic site $\mathbf{Syn}(T)$ as follows.

1. *Objects* are pairs $([\varphi(x), X)$ where X is a (derived) type, x is a variable of type X, and $[\varphi(x)]$ is an equivalence class of $\mathcal{L}^\lambda_{\omega\omega}$-formulas. Two formulas $\varphi_1(x_1)$ and $\varphi_2(x_2)$ are equivalent if
$$T \vdash \forall x(\varphi_1(x) \longleftrightarrow \varphi_2(x)),$$
where x is a new variable.
2. *Arrows* from $([\varphi(x), X)$ to $([\psi(y)], Y)$ are triples $([\sigma(x,y)], X, Y)$ such that $[\sigma(x,y)]$ is an equivalence class of $\mathcal{L}^\lambda_{\omega\omega}$-formulas and, moreover, σ is provably functional:
$$T \vdash \forall x \forall y(\sigma(x,y) \longrightarrow \varphi(x) \wedge \psi(y)),$$
$$T \vdash \forall x(\varphi(x) \longrightarrow \exists y \sigma(x,y)),$$
$$T \vdash \forall x \forall y \forall z(\sigma(x,y) \wedge \sigma(x,z) \longrightarrow y = z).$$
Here we used the same names for the variables occurring in φ, ψ, and σ, indicating that we do not care about possibly renaming the variables.
3. We say that a finite family of arrows $([\sigma_i(x_i, y)], X_i, Y)\colon ([\varphi_i(x_i)], X_i) \longrightarrow ([\psi(y)], Y)$ is a *cover* if
$$T \vdash \forall y(\psi(y) \longrightarrow \bigvee\nolimits_I \exists x_i \sigma_i(x_i, y)).$$

It is easy to show that $\mathbf{Syn}(T)$ has *all* finite limits and *some* exponentials (namely, those of the form $([z=z], Z)^{([y=y],Y)} = ([w=w], Z^Y)$), and the topology is *subcanonical*. But the category $\mathbf{Syn}(T)$ fails to be cartesian closed. Still, there is a canonical interpretation of our language in this category, and this interpretation yields a conservative model of T in $\mathbf{Syn}(T)$.

Write $\mathfrak{B}^\lambda(T)$ for the topos of sheaves on $\mathbf{Syn}(T)$, equipped with the finite cover topology. The Yoneda embedding $\mathbf{y}\colon \mathbf{Syn}(T) \longrightarrow \mathfrak{B}^\lambda(T)$ provides an interpretation U of the underlying language as follows:

$$A^{(U)} = \mathbf{y}([x=x], A),$$

for each basic sort A. The above-mentioned properties of $\mathbf{Syn}(T)$ and the fact that $\mathbf{y}$ preserves exponentials imply that

$$Y^{(U)} \cong \mathbf{y}([y=y], Y)$$

for any derived type Y. Constants and relations are interpreted as follows:

$$\begin{array}{lcl} c^{(U)}\colon 1 \longrightarrow Y^{(U)} & := & \mathbf{y}([c=y],\varnothing,Y)\colon \mathbf{y}([\top],\varnothing) \longrightarrow \mathbf{y}([y=y],Y) \\ f^{(U)}\colon Y^{(U)} \longrightarrow Z^{(U)} & := & \mathbf{y}([f(y)=z],Y,Z)\colon \mathbf{y}([y=y],Y) \longrightarrow \mathbf{y}([z=z],Z) \\ R^{(U)} \rightarrowtail Y^{(U)} & := & \mathbf{y}([R(y)],Y) \rightarrowtail \mathbf{y}([y=y],Y). \end{array}$$

The core of this section is the following proposition.

Proposition 5.1 *For each $\mathcal{L}^{\lambda}_{\omega\omega}(\mathbb{S})$-formula $\psi(y\colon Y)$ there is a canonical isomorphism $\mathbf{y}([\psi(y)],Y) \cong \{y \mid \psi(y)\}^{(U)}$.*

Proof: This is a long induction over the complexity of ψ. Roughly speaking, $\mathbf{Syn}(T)$ is a Heyting category, and the Yoneda embedding preserves the first-order structure (see [6] for details). Moreover, since the topology is subcanonical, $\mathbf{y}$ preserves exponentials which happen to exist. □

As a corollary we derive the major result, namely, completeness with respect to models in Grothendieck toposes.

Theorem 5.2 *U is a conservative model of T. For a closed formula τ we have $U \models \tau$ if and only if $T \vdash \tau$. In particular, $\mathcal{L}^{\lambda}_{\omega\omega}$ is complete with respect to models in Grothendieck toposes (and therefore complete with respect to Heyting categories with exponentials).*

Proof: The first part is immediate from Proposition 5.1, the rest is trivial. □

Using a recent covering theorem for toposes with enough points, we can strengthen the theorem the following way.

Corollary 5.3 *For each consistent set of axioms $T \subset \mathcal{L}^{\lambda}_{\omega\omega}(\mathbb{S})$ there exists a topological space X and an OX-valued model M of T (a Heyting-valued model of T which takes its truth values in the complete Heyting algebra OX of open sets of X) such that $M \models \tau$ if and only if $T \vdash \tau$ for each closed formula τ.*

Proof: Given T the site $\mathbf{Syn}(T)$ is coherent and therefore $\mathfrak{B}^{\lambda}(T)$ has enough points. By Theorem 13.5 of Butz [5] (see as well Butz and Moerdijk [7]) there exists a connected, locally connected geometric morphism

$$m\colon \mathrm{Sh}(X) \longrightarrow \mathfrak{B}^{\lambda}(T)$$

for X a topological space. By Lemma 4.1, $M = m^{*}U$ is a model of T in $\mathrm{Sh}(X)$, which is conservative since m is a surjective geometric morphism. The corollary follows since models in $\mathrm{Sh}(X)$ correspond to OX-valued models (see Fourman and Scott [8] for details). □

What are the points of the topological space X? Classical second-order logic is complete with respect to models which are called nowadays Henkin models, see Henkin [10]. Combining Henkin's proof and the standard proof of Heyting-valued completeness for first-order intuitionistic logic, one shows that our logic $\mathcal{L}^{\lambda}_{\omega\omega}$ (but in fact, full intuitionistic second-order logic) is complete with respect to Heyting-valued Henkin models. Fixing a set of enough Heyting-valued Henkin models S_T, points of

X are pairs (M, α) where M is in S_T and α is an enumeration of M, similar as in [2], Appendix. The enumerations are used to define the topology.

Before we end this section let us mention that the model U in $\mathfrak{B}^\lambda(T)$ is minimal in the following sense.

Proposition 5.4 *For any model M of T in a Grothendieck topos $\mathcal{F}$ there is a unique (up to isomorphism) geometric morphism $\chi_M : \mathcal{F} \longrightarrow \mathfrak{B}^\lambda(T)$ such that for each $\mathcal{L}^\lambda_{\omega\omega}(\mathbb{S})$-formula $\psi(y{:}\,Y)$*

$$\{y \mid \psi(y)\}^{(M)} \cong \chi_M^*\{y \mid \psi(y)\}^{(U)}. \tag{1}$$

Thereby, we get a fully faithful functor

$$\chi : \underline{\mathrm{Mod}}^\lambda(T, \mathcal{F}) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{F}, \mathfrak{B}^\lambda(T)),$$

natural in locally connected geometric morphisms $\mathcal{F}' \longrightarrow \mathcal{F}$.

Proof: Soundness of $\vdash$ implies that $H_M : \mathbf{Syn}(T) \longrightarrow \mathcal{F}$, defined on objects by

$$([\psi(y)], Y) \mapsto \{y \mid \psi(y)\}^{(M)}$$

is a well-defined functor. This functor preserves finite limits and covers, therefore induces by Diaconescu's theorem a geometric morphism $\chi_M : \mathcal{F} \longrightarrow \mathfrak{B}^\lambda(T)$ satisfying (1).

By the remark following definition 3.2, $\mathcal{L}^\lambda_{\omega\omega}$-homomorphisms $h : M \longrightarrow M'$ correspond exactly to natural transformations $H_M \longrightarrow H_{M'}$, which shows that $\chi_{(-)}$ extends to a fully faithful functor $\underline{\mathrm{Mod}}^\lambda(T, \mathcal{F}) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{F}, \mathfrak{B}^\lambda(T))$, which is clearly natural in locally connected geometric morphisms. □

As a final remark we mention that given M in $\mathcal{F}$, the geometric morphism χ_M is, in general, *not* open, hence, in general, *not* locally connected.

6 Concluding remarks Our main goal was to study the logic $\mathcal{L}^\lambda_{\omega\omega}$, but there are as well the infinitary variants $\mathcal{L}^\lambda_{\kappa\omega}$, where one allows disjunctions and conjunctions over sets of formulas of cardinality less than or equal to κ. In that case one has to use cartesian closed κ-Heyting categories as the natural categories where models live. The calculus of Section 2 extends immediately to these infinitary logics, and the completeness results of Section 5 remain true, although the complete Heyting algebra of Corollary 5.3 does not have to come from a topological space: given a theory $T \subset \mathcal{L}^\lambda_{\kappa\omega}(\mathbb{S})$, the site $\mathbf{Syn}_\kappa(T)$, defined similarly as above using formulas from $\mathcal{L}^\lambda_{\kappa\omega}$, is not coherent and Theorem 13.5 of [5] does not apply. Instead, one has to appeal to the covering theorem of Joyal and Moerdijk [12]. As noted in the introduction, the logics $\mathcal{L}^\lambda_{\kappa\omega}$ are the strongest logics we know for which a Heyting-valued completeness theorem holds. Such a statement for full (intuitionistic) second-order logic is certainly wrong: second-order logic is even not complete with respect to models in arbitrary Grothendieck toposes.

Finally we should admit that there is something wrong with the syntax of our logic: we should not just extend first-order logic by quantification over function types, but by quantification over *definable* function types, that is, we should allow expressions such as

$$\forall f\colon \{x \mid \varphi(x)\}^{\{y \mid \psi(y)\}}(\quad \cdots \quad)$$

where (recursively) φ and ψ are formulas of our language. Write $\mathcal{L}^{\lambda+}_{\kappa\omega}$ for this logic. Given a theory $T \subset \mathcal{L}^{\lambda+}_{\kappa\omega}(\mathbb{S})$ we can construct as before a syntactic site $\mathbf{Syn}^{+}_{\kappa}(T)$, which will now be a cartesian-closed κ-Heyting category. In fact, it has the obvious universal property in the category of all cartesian-closed κ-Heyting categories. Therefore, a presentation using $\mathcal{L}^{\lambda+}_{\kappa\omega}$ would parallel [6] much more. But there are good reasons why we did not choose this way: even though we know intuitively very well how to handle the syntax of $\mathcal{L}^{\lambda+}_{\kappa\omega}$, the formal presentation is clumsy. Any formula defines a type, so that there is no distinction between formulas and types, in particular, there are many identifications and subtypes.

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