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ENUMERATING QUASIPLATONIC CYCLIC GROUP ACTIONS

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ABSTRACT. It is an open problem to determine the number of topologically distinct ways that a finite group can act upon a compact oriented surface X of genus $g(X) \geq 2$. We provide an explicit answer to this problem for special classes of cyclic groups and illustrate our results with detailed examples.

1. Introduction. A consequence of a resolution to the Nielsen realization problem, see [**10**], is that there is a one-to-one correspondence between conjugacy classes of finite subgroups of the mapping class group $\mathcal{M}_\sigma$ of a compact oriented surface of genus σ and the topological equivalence classes of finite groups of homeomorphisms which can act on such a surface. This correspondence has motivated a detailed study of classes of topological group actions, and, in particular, an attempt to classify or enumerate the different ways a group G can act topologically on a surface X of genus $\sigma \geq 2$, see for example, [**2, 3, 6, 7**] where Abelian groups are considered, and [**13, 14**] for other examples. In general, the problem of enumerating classes of topological group actions for arbitrary σ is highly computational and depends very much upon how G acts on X as well as the general structure of G. Indeed, the known results even for very simple classes of groups such as Abelian groups are very technical, and so a general classification for arbitrary groups seems unlikely. Moreover, for an arbitrary G and σ, even answering the simple question, "does G act on a surface of genus σ," is typically non-trivial. These observations motivate a study of classes of topological group actions of structurally simple groups acting in a relatively simple way where it may be possible to derive extremely explicit enumeration formulas as a stepping stone to studying more complicated group actions.

The technical enumeration formulas derived in [**3**] for elementary Abelian group actions of low rank illustrate how difficult it is to obtain

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explicit enumeration formulas for even structurally simple groups such as Abelian groups. Therefore, we restrict our attention to the simplest group structure possible–cyclic groups. To restrict the action of G on X, we specify that G is a so-called quasiplatonic group, that is, G has exactly three fixed points and the quotient space X/G has genus 0 (see Section 2 for a formal definition). By restricting our attention to groups acting in such a way, it is possible to derive general enumeration formulas for the number of different actions, provided we know enough structural information about G such as its character table, see [**9, 15**]. Of course, determining whether or not an arbitrary G acts on a surface X of genus σ is a very difficult problem. However, for cyclic groups, necessary and sufficient conditions are provided for the existence of such an action, see [**8**]. Therefore, it seems that it should be possible to determine explicit enumeration formulas for cyclic quasiplatonic groups, that is, given a cyclic group G and a genus $\sigma \geq 2$, a method to determine the total number of distinct topological group actions of G on a surface X of genus σ such that G is quasiplatonic.

There are a number of motivating reasons for our work. First, as previously remarked, though some general classification results and enumeration formulas are already known for classes of groups such as Abelian groups, they are not always explicit and often extremely technical. By restricting to quasiplatonic cyclic groups, we are able to provide very simple enumeration formulas by invoking already known results and a little elementary number theory. Secondly, many quasiplatonic groups are maximal as finite subgroups of $\mathcal{M}_\sigma$, and, for those which are not, there are computational methods to determine precisely which ones are not maximal, see for example, [**4**]. Thus, an enumeration method for the number of different classes of cyclic group actions which are quasiplatonic groups can be used to provide a lower bound on the number of conjugacy classes of maximal finite cyclic subgroups, or more generally, the number of conjugacy classes of maximal finite subgroups of $\mathcal{M}_\sigma$, thus providing insight into the general structure of $\mathcal{M}_\sigma$. Finally, though the groups we consider and their actions are extremely elementary, we hope that the techniques we have used may be generalized to other classes of groups. Indeed, it seems that it would be straightforward to derive explicit formulas for quasiplatonic Abelian group actions and perhaps some simple semi-direct products, and this could lead to new methods to the more general problem.

Our paper is structured as follows. In Section 2 we develop the necessary preliminary results. Following this, in Section 3, we use some elementary number theory to derive explicit formulas for solutions to certain congruences which will be used in our enumeration formulas. Next, we shall derive all enumeration formulas in Section 4. We finish in Section 5 by providing applications of our results and a number of explicit examples.

2. Preliminaries. Let G be a finite group. The group G is said to act topologically (in an orientation-preserving manner) on surface X of genus $\sigma \geq 2$, if there is an injection

$$\varepsilon : G \hookrightarrow \mathrm{Homeo}^+(X)$$

into the group of orientation-preserving homeomorphisms (we shall identify G with its image under ε). Two actions $\varepsilon_1, \varepsilon_2$ are said to be *topologically equivalent* if there is a homeomorphism h of S and an automorphism ω of G such that

$$\varepsilon_2(\omega(y)) = h \circ \varepsilon_1(y) \circ h^{-1}$$

for all $y \in G$. This is equivalent to saying that the images $\varepsilon_1(G)$ and $\varepsilon_2(G)$ are conjugate in $\mathrm{Homeo}^+(X)$.

Fuchsian groups provide us with a way to describe topological group actions. Specifically, a surface X of genus $\sigma \geq 2$ is topologically equivalent to a quotient of the upper half plane $\mathbf{H}/\Lambda$ where Λ is any torsion-free Fuchsian group isomorphic to the fundamental group of X called a surface group for X. A finite group G acts on X if and only if $G = \Gamma/\Lambda$ for some Fuchsian group Γ containing such a Λ as a normal subgroup of index $|G|$. The structure of Γ is completely determined by the ramification data of the quotient map $\pi_G \colon X \to X/G$, which must satisfy the Riemann-Hurwitz formula. Specifically, if the quotient map π_G branches over r points with ramification indices m_i for $1 \leq i \leq r$ and the quotient space X/G has genus g, then a presentation for Γ is:

$$\Gamma = \left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_r \mid c_1^{m_1}, \dots, c_r^{m_r}, \prod_{i=1}^{r} c_i \prod_{j=1}^{g} [a_j, b_j] \right\rangle$$

where

$$\sigma = 1 + |G|(g-1) + \frac{|G|}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right).$$

Such a group is described by the tuple $(g; m_1, \dots, m_r)$ called the *signature* of Γ (we also say that G has signature $(g; m_1, \dots, m_r)$, and we call the numbers $m_1, \dots, m_r$ the periods of the signature). In the special case that $g = 0$ and $r = 3$, we call Γ a triangle group. The following is a formal definition of the central focus of our work.

Definition 2.1. Suppose that X is a surface and $G = \Gamma/\Lambda$ where Λ is a surface group for X and Γ is a triangle group. Then we call G a quasiplatonic group.

Remark. Typically, when studying quasiplatonic groups, we are considering the actions as conformal actions rather than simply topological actions, and under these circumstances, we call X a quasiplatonic surface. For our purposes, since we are only considering topological actions, we avoid using this terminology for the surface as there will always exist at least one quasiplatonic topological group action on any such surface, see [**1, 11**].

Group actions are usually described through the use of surface kernel epimorphisms. Specifically, we have the following.

Theorem 2.2. *A finite group G acts on a surface S of genus $\sigma \geq 2$ with signature $(g; m_1, \dots, m_r)$ if and only if there exists a Fuchsian group with signature $(g; m_1, \dots, m_r)$ and an epimorphism $\rho\colon \Gamma \to G$ preserving the orders of the elements of finite order* (*called a surface kernel epimorphism*) *such that*

$$\sigma = 1 + |G|(g-1) + \frac{|G|}{2}\sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right).$$

A useful way to describe surface kernel epimorphisms is through the use of generating vectors defined as follows (see [**2**]).

Definition 2.3. A vector of group elements $(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \eta_1, \dots, \eta_r)$ in a finite group G is called a $(g; m_1, \dots, m_r)$-generating vector for G if all of the following hold:

(i) $G = \langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g, \eta_1, \dots, \eta_r \rangle$.

(ii) $\Pi_{i=1}^{g}[\alpha_i, \beta_i] \cdot \Pi_{j=1}^{r} \eta_j = 1$ (where $[\ ,\ ]$ denotes the commutator).

(iii) $O(\eta_i) = m_i$ (where $O(.)$ denotes group order).

Clearly, any $(g; m_1, \dots, m_r)$-generating vector $\mathcal{V}$ for G defines a unique surface kernel epimorphism from a fixed Γ with signature $(g; m_1, \dots, m_r)$ onto G, called the surface kernel epimorphism of $\mathcal{V}$. Conversely, any surface kernel epimorphism $\rho\colon \Gamma \to G$ uniquely defines a generating vector called the *generating vector of* ρ. Thus, topological group actions can be described through the utilization of generating vectors of finite groups. The exact correspondence is given in the following.

Theorem 2.4. *Two equivalence classes of* $(g; m_1, \dots, m_r)$*-generating vectors of the finite group* G *define the same topological equivalence class of* G*-actions if and only if the generating vectors lie in the same* $\mathrm{Aut}\,(G) \times \mathrm{Aut}\,(\Gamma)$*-class where the action of* $\mathrm{Aut}\,(G) \times \mathrm{Aut}\,(\Gamma)$ *on a generating vector* $\mathcal{V}$ *is defined by the action on a surface kernel epimorphism* ρ *with generating vector* $\mathcal{V}$ *given by* $(\alpha, \gamma) \cdot \rho = \alpha \circ \rho \circ \gamma^{-1}$ *for* $\alpha \in \mathrm{Aut}\,(G)$*,* $\gamma \in \mathrm{Aut}\,(\Gamma)$*.*

Proof. See [**2**, Proposition 2.2]. $\square$

Thus, given a generating vector $\mathcal{V}$ for a group G, it defines a topological action, namely, the action of Γ/Λ on $X = \mathbf{H}/\Lambda$ where Λ is the kernel of the surface kernel epimorphism of $\mathcal{V}$ where Γ is a Fuchsian group with signature $(g; m_1, \dots, m_r)$ (note that Γ can be any subgroup of $\psi(2, \mathbf{R})$ with signature $(g; m_1, \dots, m_r)$). Conversely, given a group acting topologically on X with signature $(g; m_1, \dots, m_r)$, it defines a $(g; m_1, \dots, m_r)$-generating vector up to $\mathrm{Aut}\,(G) \times \mathrm{Aut}\,(\Gamma)$ equivalence. Specifically, it defines the class containing the generating vector of $\rho\colon \Gamma \to G$ where ρ is the surface kernel epimorphism from Γ with signature $(g; m_1, \dots, m_r)$ and kernel Λ such that S is topologically equivalent to $\mathbf{H}/\Lambda$ and G is topologically equivalent to Γ/Λ. We call any generating vector from this $\mathrm{Aut}\,(G) \times \mathrm{Aut}\,(\Gamma)$ class a *generating vector of* G.

One of the main technical difficulties that arises when trying to enumerate classes of topological group actions using generating vectors is trying to describe the action of the group $\mathrm{Aut}\,(\Gamma)$. In the special case when Γ is a triangle group, however, $\mathrm{Aut}\,(\Gamma)$ is very easy to describe. Specifically, $\mathrm{Aut}\,(\Gamma)$ is trivial, cyclic of order 2 or isomorphic to the symmetric group on three letters depending upon whether none, two or all three of the periods are equal, see [**5**]. Thus, for quasiplantonic groups, provided we know that a group G acts on a surface of genus σ with signature $(0; n_1, n_2, n_3)$, we can derive very specific formulas for the number of topologically distinct actions of a group G with signature $(0; n_1, n_2, n_3)$ dependent upon the number of $(0; n_1, n_2, n_3)$-generating vectors for G, see [**15**].

As previously remarked, in general, determining whether a given group acts on a surface with a given signature is a difficult problem, especially when considering arbitrarily large genus. In our case, however, we are restricting to cyclic groups, and for such groups, necessary and sufficient conditions for the existence of a cyclic group G acting on a surface X of genus σ with signature $(g; n_1, \dots, n_r)$ are provided in [**8**]. Therefore, we can combine the enumeration formulas from [**15**] with the existence conditions provided in [**8**] to provide enumeration formulas for the number of classes of topological actions of a cyclic group G on a surface of genus $\sigma \geq 2$ with signature $(0; n_1, n_2, n_3)$ dependent upon the number of generating vectors of G with signature $(0; n_1, n_2, n_3)$. To state these formulas, we need the following definitions.

Definition 2.5. Suppose (x, y, z) is a generating vector for a quasiplatonic group G. Then we define the following permutations:

- $i_1 : x \to y, y \to x, z \to z$,
- $i_2 : x \to x, y \to z, z \to y$,
- $i_3 : x \to z, y \to y, z \to x$,
- $j : x \to y, y \to z, z \to x$.

Definition 2.6. Fix a signature $(0; n_1, n_2, n_3)$, and let G be a cyclic group of order M. The we define the following sets:

(i) V_G is the set of generating vectors of G with signature $(0; n_1, n_2, n_3)$ for which none of the permutations i_1, i_2, i_3 and j extend to an automorphism of G.

(ii) $V_{G,i}$ is the set of generating vectors of G with signature $(0; n_1, n_2, n_3)$ for which i_1, i_2 or i_3 extends to an automorphism of G but j does not.

(iii) $V_{G,j}$ is the set of generating vectors of G with signature $(0; n_1, n_2, n_3)$ for which j extends to an automorphism of G but i_1, i_2 and i_3 do not.

(iv) $V_{G,i,j}$ is the set of generating vectors of G with signature $(0; n_1, n_2, n_3)$ for which i_1, i_2, i_3 and j extend to automorphisms of G.

We can now state the two main results we shall be utilizing.

Theorem 2.7. *Fix a signature* $(0; n_1, n_2, n_3)$, *and let* $m = \mathrm{lcm}\,(n_1, n_2, n_3)$. *Then the cyclic group* G *of order* m *acts on a surface* X *of genus* σ *with signature* $(0; n_1, n_2, n_3)$ *if and only if the following conditions are met*:

(i) $m = \mathrm{lcm}\,(n_1, n_2) = \mathrm{lcm}\,(n_1, n_3) = \mathrm{lcm}\,(n_2, n_3)$;

(ii) *if* m *is even, then exactly two of the periods* n_i *must be divisible by the maximum power of* 2 *that divides* m;

(iii) *the Riemann-Hurwitz formula is satisfied*:

$$\sigma = 1 + \frac{m}{2}(1 - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}).$$

Proof. This is Harvey's theorem, [**8**], modified to the case where G is quasiplatonic. □

Theorem 2.8. *Suppose the group* G *acts on a compact oriented surface* X *with genus* σ *with signature* $(0; n_1, n_2, n_3)$. *Then the number of distinct topological* G *actions on* X *with signature* $(0; n_1, n_2, n_3)$, T, *can be calculated as follows*:

(i) *If* $n_1 < n_2 < n_3$, *then*

$$T = \frac{|V_G|}{|\mathrm{Aut}\,(G)|}.$$

(ii) *If* $n_1 < n_2 = n_3$, *then*

$$T = \frac{|V_G|}{2|\mathrm{Aut}\,(G)|} + \frac{|V_{G,i}|}{|\mathrm{Aut}\,(G)|}.$$

(iii) *If* $n_1 = n_2 = n_3$, *then*

$$T = \frac{|V_G|}{6|\mathrm{Aut}\,(G)|} + \frac{|V_{G,i}|}{3|\mathrm{Aut}\,(G)|} + \frac{|V_{G,j}|}{2|\mathrm{Aut}\,(G)|} + \frac{|V_{G,i,j}|}{|\mathrm{Aut}\,(G)|}.$$

Proof. This is the main result of [**15**]. □

Thus, to derive explicit formulas, we need to determine the number of generating vectors for which the maps i_1, i_2, i_3 or j do not extend to automorphisms of G. This is the primary focus of our work in Sections 3 and 4.

3. Solutions to congruences. Before we consider general enumeration formulas, we need to use some elementary number theory to determine explicit values of two different congruences which will appear in our formulas. These are the following:

Definition 3.1. Let $\mathbf{N}$ denote the positive integers and $\mathbf{O}$ the odd positive integers. We define $\tau_1 : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ where $\tau_1(m,n)$ represents the number of noncongruent nonzero solutions x to $x^2 + 2x \equiv 0 \bmod m$ where $\gcd(x,m) = m/n$. Similarly, we define $\tau_2 : \mathbf{O} \to \mathbf{N}$ where $\tau_2(m)$ represents the number of noncongruent solutions x to $x^2 + x + 1 \equiv 0 \bmod m$.

Our main goal in this section is to derive explicit formulae for τ_1 and τ_2. The general approach we take is to first derive explicit formulas in the case where m is a power of a prime, and then extend the results by showing τ_1 and τ_2 are multiplicative over their prime power decompositions. Since the proofs are very similar and rely only on elementary number theory, we only provide the details for the more complicated of the two congruences, τ_2.

Theorem 3.2. *Suppose that*

$$m = \prod_{i=1}^{l} p_i^{k_i},$$

for odd primes $p_1, \dots, p_l$ where the k_i are positive integers. Then, for each i,

(i) *if $p_i^{k_i} = 3$, then $\tau_2(p_i^{k_i}) = 1$;*

(ii) *if 3 divides $\phi(p_i^{k_i})$, then $\tau_2(p_i^{k_i}) = 2$;*

(iii) *and $\tau_2(p_i^{k_i}) = 0$ otherwise.*

Further, $\tau_2(m) = \prod_{i=1}^{l} \tau_2(p_i^{k_i})$.

Proof. First suppose $m = p^k$ for some odd prime p, and suppose a is a solution to this congruence. Then, since

$$0 \equiv a0 \equiv a(a^2 + a + 1) \equiv a^3 + a^2 + a \bmod p^k,$$

we know that $a^3 \equiv 1 \bmod p^k$. So, either $a = 1$, or a has order 3 modulo p^k. But, $a = 1$ is a solution if and only if $p^k = 3$, and (i) follows.

Now let us suppose that $p^k \neq 3$. We know there are elements of order 3 modulo p^k if and only 3 divides $\phi(p_i^{k_i})$, and in this case it follows that there are $\phi(3) = 2$ of them. (iii) follows from these observations.

To prove (ii), we need only show that any element of order 3 modulo p^k satisfies $x^2 + x + 1 \equiv 0 \bmod p^k$. Suppose a is an element of order 3 modulo p^k. Then,

$$0 \equiv a^3 - 1 \equiv (a - 1)(a^2 + a + 1) \bmod p^k.$$

To show $a^2 + a + 1 \equiv 0 \bmod p^k$, we need only show that $a - 1$ is not a zero divisor, since we know $a \neq 1$. We proceed by contradiction and suppose that $a-1$ is a zero divisor. For some integer h, $a-1=ph$, which tells us that $a=ph+1$. Then,

$$0 \equiv a^2+a+1 \equiv (ph+1)^2+(ph+1)+1 \equiv p(ph^2+3h)+3 \bmod p^k,$$

which forces $p=3$. This tells us that we have

$$0 \equiv 9(h^2+h)+3 \bmod 3^k,$$

which will force $p^k=3$, which is a contradiction. So, (ii) follows.

Next, we show that τ_2 is multiplicative. Let m and n be positive integers such that $\gcd(m,n)=1$. We will construct a one-to-one correspondence between solutions (a,b) for the system of congruences $(x^2+x+1) \equiv 0 \bmod m$ and $(y^2+y+1) \equiv 0 \bmod n$ and solutions c for $(z^2+z+1) \equiv 0 \bmod mn$. First, suppose $(a^2+a+1) \equiv 0 \bmod m$ and $(b^2+b+1) \equiv 0 \bmod n$. Consider the unique solution c to the system of linear congruences $z \equiv a \bmod m$ and $z \equiv b \bmod n$, given by the Chinese remainder theorem. Then, since $(c^2+c+1) \equiv 0 \bmod m$, $(c^2+c+1) \equiv 0 \bmod n$ and $\gcd(m,n)=1$, it follows that $(c^2+c+1) \equiv 0 \bmod mn$. This gives us one direction of the one-to-one correspondence. Now, instead, suppose that we have a solution c for $(z^2+z+1) \equiv 0 \bmod mn$. Then, it is clear that $(c^2+c+1) \equiv 0 \bmod m$ and $(c^2+c+1) \equiv 0 \bmod n$. We need only reduce c modulo m and n, respectively, to get values a and b that are congruent to c modulo m and n, respectively, where $0<a<m$ and $0<b<n$. This completes our one-to-one correspondence and shows that $\tau_2(mn)=\tau_2(m)\tau_2(n)$. □

The following result can be derived in a similar way for τ_1.

Theorem 3.3. *Suppose $m \geq 3$ is an integer and that n is a divisor of m with $1<n<m$. Let $p_1, p_2, \ldots, p_l$ be the distinct odd primes that divide m. Write m and n in terms of these primes (and 2):*

$$m = 2^{k_0}\left(\prod_{i=1}^{l} p_i^{k_i}\right), \qquad n = 2^{h_0}\left(\prod_{i=1}^{l} p_i^{h_i}\right),$$

where the k_i are positive integers when $i \neq 0$, and the h_i and k_0 are nonnegative integers. Then,

(i) $\tau_1(m,n) = 0$ *if either of the following hold*:

(a) $h_0 \neq 0, 1$, or $k_0 - 1$

(b) $h_i \neq 0$ or k_i for any $i > 0$;

(ii) *if none of the conditions of* (i) *hold, then* $\tau_1(m,n) = 1$ *if* $h_0 = 0$ *or* 1;

(iii) *and* $\tau_1(m,n) = 2$ *for all other cases.*

4. Enumerating actions. We are now ready to derive the enumeration formulas for classes of cyclic group actions. As suggested in Section 2, there are three different cases which need to be considered, depending upon whether none, two or all three of the periods of the signature are the same. As we derive each formula, we shall present explicit examples to illustrate. We begin by deriving the total number of generating vectors given a cyclic group G and signature $(0; n_1, n_2, n_3)$.

Lemma 4.1. *Consider a cyclic group G of order m, and fix a signature $(0; n_1, n_2, n_3)$. Let $p_1, p_2, \dots, p_l$ be the distinct primes that divide m. Write m and the periods in terms of these primes*:

$$m = \prod_{i=1}^{l} p_i^{k_i}, \qquad n_1 = \prod_{i=1}^{l} p_i^{r_i}, \qquad n_2 = \prod_{i=1}^{l} p_i^{s_i}, \qquad n_3 = \prod_{i=1}^{l} p_i^{t_i},$$

where the k_i are positive integers, and the r_i, s_i and t_i are nonnegative integers. Then we have the following:

(i) *There exists an integer $w \leq l$ so that, if $1 \leq i \leq w$, then r_i, s_i, and t_i are all equal to k_i, and, if $w < i \leq l$, then exactly one of r_i, s_i, and t_i is less than k_i. In the latter case, let h_i represent this smaller value.*

(ii) *If $w \neq 0$, the total number of generating vectors with signature* $(0; n_1, n_2, n_3)$ *is*

$$\phi(m)\phi(\gcd(n_1, n_2, n_3))\left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\right),$$

where ϕ represents Euler's phi-function. If $w = 0$, then the total number of generating vectors with signature $(0; n_1, n_2, n_3)$ *is*

$$\phi(m)\phi(\gcd(n_1, n_2, n_3)).$$

Proof. The existence of w, the reordering of the p_i's and the existence of the h_i's is a result of Theorem 2.7. To find the number of generating vectors that meet the conditions of Theorem 2.8, we outline a process to construct valid generating vectors for this signature. During each step of the process, we shall count the number of different choices made.

Let u be a generator of G. Observe that $G = C_{p_1^{k_1}} \times C_{p_2^{k_2}} \times \cdots \times C_{p_l^{k_l}}$, where $C_{p_i^{k_i}}$ is the cyclic group of order $p_i^{k_i}$. For each i, there exists an element $u_i \in G$ such that $u = \prod_{i=1}^{l} u_i$ and u_i generates $C_{p_i^{k_i}}$. We will use these generators to construct a vector (x, y, z) where x_i, y_i and z_i will be powers of u_i for each i, and $x = \prod_{i=1}^{l} x_i$, $y = \prod_{i=1}^{l} y_i$, and $z = \prod_{i=1}^{l} z_i$.

Fix i. Suppose exactly one of r_i, s_i, and t_i is less than k_i. Then, there are $\phi(p_i^{h_i})$ choices of a_i such that $u_i^{a_i}$ is an element in $C_{p_i^{k_i}}$ of order $p_i^{h_i}$. For any such choice of a_i, we know that $u_i^{-(a_i+1)}$ has order $p_i^{k_i}$. Now, assign $u_i^{a_i}$ as the one of x_i, y_i and z_i whose order is $p_i^{h_i}$. Then, let the first of the two remaining from x_i, y_i and z_i be u_i, and let the last be $u_i^{-(a_i+1)}$. We observe that the other possible assignments of x_i, y_i and z_i will be counted when our initial choice of generator is an element v such that $v_i = u_i^{-(a_i+1)}$ and a parameter b_i is chosen so that $v_i^{b_i} = u_i^{a_i}$. Also note that assigning x_i, y_i and z_i in this manner prevents multiple countings of generating vectors. The important thing to remember is that there were $\phi(p_i^{h_i})$ choices for a_i, and therefore $\phi(p_i^{h_i})$ choices for the elements x_i, y_i and z_i.

The only other case to consider is when r_i, s_i, and t_i are all equal to k_i. Now we must choose a_i such that both $u_i^{a_i}$ and $u_i^{-(a_i+1)}$ have order $p_i^{k_i}$. So, p_i cannot divide a_i or $-(a_i + 1)$. There are $\phi(p_i^{k_i})$ choices of a_i where p_i cannot divide a_i. But, for $1/(p_i - 1)$ of these choices, p_i will divide $-(a_i + 1)$. So, there are $((p_i - 2)/(p_i - 1))\phi(p_i^{k_i})$ choices for a_i. Now, let u_i, $u_i^{a_i}$ and $u_i^{-(a_i+1)}$ be x_i, y_i and z_i, respectively. As in the first case, assigning in this manner prevents multiple countings of generating vectors. The important thing to remember is that there were $((p_i - 2)/(p_i - 1))\phi(p_i^{k_i})$ choices for a_i, and, therefore, $((p_i - 2)/(p_i - 1))\phi(p_i^{k_i})$ choices for the elements x_i, y_i and z_i.

Now, let $x = \prod_{i=1}^{l} x_i$, $y = \prod_{i=1}^{l} y_i$ and $z = \prod_{i=1}^{l} z_i$. By construction, (x, y, z) is a valid generating vector. Further, all generating vectors satisfying Theorem 2.8 can be constructed in this manner. We have also seen that the number of such vectors is

$$\phi(m)\bigg(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\phi(p_i^{k_i})\bigg)\bigg(\prod_{i=w+1}^{l} \phi(p_i^{h_i})\bigg) = \phi(m)\phi(\gcd(n_1, n_2, n_3))\bigg(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\bigg),$$

since there were $\phi(m)$ choices for our generator u of G and because we found the number of choices for a_i in each case. In the event that $w = 0$, then $\prod_{i=1}^{w}(p_i - 2)/(p_i - 1) = 1$. $\square$

We now consider the case where each period of the signature is distinct.

Theorem 4.2. *Consider a cyclic group G of order m, and fix a signature $(0; n_1, n_2, n_3)$ where all the n_i are distinct. Let $p_1, p_2, \ldots, p_l$ be the distinct primes that divide m. Write m and the periods in terms of these primes*:

$$m = \prod_{i=1}^{l} p_i^{k_i}, \qquad n_1 = \prod_{i=1}^{l} p_i^{r_i}, \qquad n_2 = \prod_{i=1}^{l} p_i^{s_i}, \qquad n_3 = \prod_{i=1}^{l} p_i^{t_i},$$

where the k_i are positive integers, and the r_i, s_i and t_i are nonnegative integers, and let $w \le l$ and h_i where $w < i \le l$ are the integers specified in Lemma 4.1 (i). *Then, if $w \neq 0$, the number of nonequivalent generating vectors T with signature $(0; n_1, n_2, n_3)$ is*

$$T = \phi(\gcd(n_1, n_2, n_3))\bigg(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\bigg).$$

If $w = 0$, then the number of nonequivalent generating vectors T with signature $(0; n_1, n_2, n_3)$ is

$$T = \phi(\gcd(n_1, n_2, n_3)).$$

Proof. By Theorem 2.8, we know that $T = |V_G|/|\mathrm{Aut}\,(G)|$. Since G is a cyclic group, we know that $|\mathrm{Aut}\,(G)| = \phi(m)$. So, we only need to find $|V_G|$ to find T. But, by Lemma 4.1, we know that

$$|V_G| = \phi(m)\phi(\gcd(n_1, n_2, n_3))\left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\right).$$

It follows that

$$T = \phi(\gcd(n_1, n_2, n_3))\left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\right). \qquad \square$$

Example 4.3. Consider the cyclic group G of order 105 with signature $(0; 15, 21, 35)$ acting on a surface of genus 46. Theorem 4.2 tells us that

$$T = \phi(\gcd(n_1, n_2, n_3)) = \phi(\gcd(15, 21, 35)) = \phi(1) = 1.$$

We now look at the case where exactly two of the periods must be identical. By Theorem 2.7, we know that the two identical periods must be equal to the order of G.

Theorem 4.4. *Consider a cyclic group G of order m, and fix a signature $(0; n, m, m)$ where $n \neq m$. Let $p_1, p_2, \ldots, p_l$ be the distinct primes that divide m. Write m and n in terms of these primes*:

$$m = \prod_{i=1}^{l} p_i^{k_i}, \qquad n = \prod_{i=1}^{l} p_i^{h_i},$$

where the k_i are positive integers, and the h_i are nonnegative integers. Let $w \leq l$ be the integer as specified in Lemma 4.1 (i) *modified to this case, i.e., for $i > w$, we have $h_i < k_i$ and, for $i \leq w$, $h_i = k_i$. When $w \neq 0$, the number of nonequivalent generating vectors T with signature $(0; n, m, m)$ is*

$$T = \frac{1}{2}\left(\tau_1(m, n) + \phi(n)\left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\right)\right).$$

When $w = 0$, the number of nonequivalent generating vectors T with signature $(0; n, m, m)$ *is*

$$T = \frac{1}{2}\left(\tau_1(m,n) + \phi(n)\right).$$

Proof. By Theorem 2.8, we know that $T = |V_G|/2|\mathrm{Aut}\,(G)| + |V_{G,i}|/|\mathrm{Aut}\,(G)|$. Since G is a cyclic group, we know that $|\mathrm{Aut}\,(G)| = \phi(m)$. So, we only need to find $|V_G|$ and $|V_{G,i}|$ to find T. We note that i_1 and i_3 cannot be extended to automorphisms. This leaves two cases: when i_2 is an automorphism of G, and when it is not an automorphism of G. Choose a generator $x \in G$, and suppose we choose a such that we have a generating vector $(x^a, x^{-(a+1)}, x)$. Further, let us suppose that i_2 does extend to an automorphism. That is, the map that sends $x \to x^{-(a+1)}$, $x^{-(a+1)} \to x$, and $x^a \to x^a$ extends to an automorphism. Observe that

$$x^a = i_2(x^a) = (i_2(x))^a = (x^{-(a+1)})^a = x^{-a^2-a},$$

which tells us that $a^2 + 2a \equiv 0 \bmod m$. Recall that $\tau_1(m,n)$ is the number of noncongruent solutions x to $x^2 + 2x \equiv 0 \bmod m$ where $\gcd(x,m) = m/n$. Then, i_2 extends to an automorphism if and only if a is such a solution. So, $|V_{G,i}| = \phi(m)\tau_1(m,n)$. We also know from Lemma 4.1 that we can reorder the p_i's and find an integer $w \leq l$ so that, if $1 \leq i \leq w$, then $k_i = h_i$, and, if $w < i \leq l$, then $h_i < k_i$, and that

$$|V_G| + |V_{G,i}| = \phi(m)\phi(\gcd(n,m))\left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\right).$$

So,

$$|V_G| = \phi(m)\phi(n)\left(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\right) - \phi(m)\tau_1(m,n).$$

Thus,

$$\begin{aligned} T &= \frac{|V_G|}{2|\mathrm{Aut}\,(G)|} + \frac{|V_{G,i}|}{|\mathrm{Aut}\,(G)|} \\ &= \frac{\phi(m)\phi(n)\left(\prod_{i=1}^{w}(p_i-2)/(p_i-1)\right) - \phi(m)\tau_1(m,n)}{2\phi(m)} \end{aligned}$$

$$+ \frac{\phi(m)\tau_1(m,n)}{\phi(m)}$$
$$= \frac{1}{2}\Bigg(\tau_1(m,n) + \phi(n)\Bigg(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\Bigg)\Bigg). \qquad \square$$

Example 4.5. Consider the cyclic group G of order 120 with signature $(0;12,120,120)$ acting on a surface of genus 55. Our earlier work tells us that $\tau_1(120,12) = 2$. By Theorem 4.4, we see that

$$T = \frac{1}{2}\Bigg(\tau_1(m,n) + \phi(n)\Bigg(\prod_{i=1}^{w} \frac{p_i - 2}{p_i - 1}\Bigg)\Bigg) = \frac{1}{2}\Bigg(2 + \phi(12)\Bigg(\frac{3-2}{3-1}\Bigg)\Bigg) = 2.$$

Example 4.6. Consider the cyclic group G of order p^k for some prime $p \neq 2$ with signature $(0;p^h,p^k,p^k)$, where $1 \leq h < k$, acting on a surface of genus $(p^k - p^{k-h})/2$. Theorem 4.4 tells us that

$$T = \frac{1}{2}\left(\tau_1(p^k,p^h) + \phi(p^h)\right) = \frac{1}{2}\phi(p^h).$$

The last case to consider is the case where all of the periods are equal. It follows from Theorem 2.7 that the periods must all be the order of the group.

Theorem 4.7. *Consider a cyclic group G of order m, and fix a signature $(0;m,m,m)$. Write m in its prime factorization*:

$$m = \prod_{i=1}^{l} p_i^{k_i}.$$

The number of nonequivalent generating vectors T with signature $(0;m,m,m)$ is

$$T = \frac{3 + 2\tau_2(m) + \phi(m)(\prod_{i=1}^{l}(p_i - 2)/(p_i - 1))}{6},$$

where ϕ represents Euler's phi-function.

Proof. By Theorem 2.8, we know that $T = |V_G|/(6|\mathrm{Aut}\,(G)|) + |V_{G,i}|/(3|\mathrm{Aut}\,(G)|) + |V_{G,j}|/(2|\mathrm{Aut}\,(G)|) + |V_{G,i,j}|/|\mathrm{Aut}\,(G)|$. Since G is a cyclic group, we know that $|\mathrm{Aut}\,(G)| = \phi(m)$. We only need to find $|V_G|$, $|V_{G,i}|$, $|V_{G,j}|$ and $|V_{G,i,j}|$ to find T. We begin by finding when i_1, i_2 or i_3 can be extended to automorphisms. Since, for signatures of this form, a vector where i_2 or i_3 extends to an automorphism is equivalent to a vector where i_1 extends to an automorphism, we will first just consider i_1. Choose a generator $x \in G$, and suppose we choose a such that we have a generating vector $(x, x^{-(a+1)}, x^a)$. Further, let us suppose that i_1 does extend to an automorphism. That is, the map that sends $x \to x^{-(a+1)}$, $x^{-(a+1)} \to x$ and $x^a \to x^a$ extends to an automorphism. Observe that

$$x^a = i_1(x^a) = (i_1(x))^a = (x^{-(a+1)})^a = x^{-a^2-a},$$

which tells us that $a^2 + 2a \equiv 0 \bmod m$. We know that $\gcd\,(a, m) = 1$ since $|x^a| = m$. So, m cannot divide a, but m must divide $a + 2$ since m divides $a^2 + 2a$. Thus, $a \equiv -2 \bmod m$. Thus, the vector in question is (x, x, x^{-2}). Note that, in this case, j cannot extend to an automorphism. This tells us that $|V_{G,i}| = 3\phi(m)$ (since we now take into account i_2 and i_3) and that $|V_{G,i,j}| = 0$.

We now ask ourselves when j can extend to an automorphism. Choose a generator $x \in G$, and suppose we choose a such that we have a generating vector $(x, x^{-(a+1)}, x^a)$. Further, let us suppose that j does extend to an automorphism. That is, the map that sends $x \to x^a$, $x^a \to x^{-(a+1)}$ and $x^{-(a+1)} \to x$ extends to an automorphism. Observe that

$$x^{-(a+1)} = j(x^a) = (j(x))^a = (x^a)^a = x^{a^2},$$

which tells us that $a^2 + a + 1 \equiv 0 \bmod m$. Note that any solution to this congruence will be a value that is coprime to m, that is, any such a will satisfy $|x^a| = m$. Recall that the number of solutions for a is $\tau_2(m)$. So, $|V_{G,j}| = \phi(m)\tau_2(m)$.

By Lemma 4.1, we know that $|V_G| + |V_{G,i}| + |V_{G,j}| + |V_{G,i,j}| = \phi(m)\phi(\gcd\,(m, m, m))(\prod_{i=1}^{w}(p_i - 2)/(p_i - 1)) = \phi(m)^2 \prod_{i=1}^{l}(p_i - 2)/(p_i - 1)$. Solving for $|V_G|$, we get that

$$|V_G| = -3\phi(m) - \phi(m)\tau_2(m) + \phi(m)^2 \prod_{i=1}^{l} \frac{p_i - 2}{p_i - 1}.$$

We now put all of the pieces together to see

$$\begin{aligned}
T &= \frac{|V_G|}{6|\mathrm{Aut}\,(G)|} + \frac{|V_{G,i}|}{3|\mathrm{Aut}\,(G)|} + \frac{|V_{G,j}|}{2|\mathrm{Aut}\,(G)|} + \frac{|V_{G,i,j}|}{|\mathrm{Aut}\,(G)|} \\
&= \frac{-3\phi(m) - \phi(m)\tau_2(m) + \phi(m)^2 \prod_{i=1}^{l}((p_i-2)/(p_i-1))}{6\phi(m)} \\
&\quad + \frac{3\phi(m)}{3\phi(m)} + \frac{\phi(m)\tau_2(m)}{2\phi(m)} + \frac{0}{\phi(m)} \\
&= \frac{-3 - \tau_2(m) + \phi(m) \prod_{i=1}^{l}((p_i-2)/(p_i-1))}{6} \\
&\quad + \frac{6}{6} + \frac{3\tau_2(m)}{6} \\
&= \frac{3 + 2\tau_2(m) + \phi(m) \prod_{i=1}^{l}((p_i-2)/(p_i-1))}{6}. \qquad \square
\end{aligned}$$

Example 4.8. Consider the cyclic group G of order 21 with signature $(0; 21, 21, 21)$ acting on a surface of genus 10. Theorem 4.7 tells us that

$$\begin{aligned}
T &= \frac{3 + 2\tau_2(m) + \phi(m) \prod_{i=1}^{l}((p_i-2)/(p_i-1))}{6} \\
&= \frac{3 + 4 + 12 \times (5/12)}{6} = 2.
\end{aligned}$$

This was a somewhat simplistic example where, in each generating vector, either i_1, i_2 or j extended to an automorphism. (The proof of this was omitted.) This need not be the case. In fact, it is possible that neither i_1, i_2, i_3 nor j will extend to automorphisms for the vast majority of generating vectors. The following is an example of this situation.

Example 4.9. Consider the cyclic group G of order 91 with signature $(0; 91, 91, 91)$ acting on a surface of genus 45. Theorem 4.7 tells us that

$$\begin{aligned}
T &= \frac{3 + 2\tau_2(m) + \phi(m) \prod_{i=1}^{l}(p_i-2)/(p_i-1)}{6} \\
&= \frac{3 + 8 + 72 \times (55/72)}{6} = 11.
\end{aligned}$$

5. Examples and applications. The individual examples presented in Section 4 illustrate how our enumeration formulas can be used for a specific group and signature. However, with each of these examples, arguably, they could have been calculated by hand. Therefore, the real power behind our enumeration formulas is that it allows us to efficiently enumerate all possible actions of a specific group. The following two examples illustrate such calculations.

Example 5.1. In Table 1 we list all ways in which the cyclic group of order 315 can act as a quasiplatonic group on a compact oriented surface. That is, we list each valid signature, along with the appropriate genus and value of T.

Example 5.2. In Table 2 we list all ways in which the cyclic group of order 360 can act as a quasiplatonic group on a compact oriented surface. That is, we list each valid signature, along with the appropriate genus and value of T.

TABLE 1. Topological actions for the cyclic group of order 315.

Genus	Signature	T	Genus	Signature	T
105	$(0;3,315,315)$	1	148	$(0;35,45,63)$	1
124	$(0;5,63,315)$	1	150	$(0;35,45,315)$	3
126	$(0;5,315,315)$	2	150	$(0;21,315,315)$	5
132	$(0;7,45,315)$	1	151	$(0;45,63,105)$	2
135	$(0;7,315,315)$	3	152	$(0;45,63,315)$	1
136	$(0;9,35,315)$	1	153	$(0;45,105,315)$	6
139	$(0;9,105,315)$	2	153	$(0;35,315,315)$	8
140	$(0;9,315,315)$	2	154	$(0;45,315,315)$	5
140	$(0;15,21,315)$	2	154	$(0;63,105,315)$	10
145	$(0;15,63,315)$	2	155	$(0;63,315,315)$	8
147	$(0;21,45,315)$	2	156	$(0;105,315,315)$	15
147	$(0;15,315,315)$	3	157	$(0;315,315,315)$	8

TABLE 2. Topological actions for the cyclic group of order 360.

Genus	Signature	T	Genus	Signature	T
90	$(0; 2, 360, 360)$	1	171	$(0; 20, 360, 360)$	4
120	$(0; 3, 360, 360)$	1	172	$(0; 40, 72, 90)$	1
135	$(0; 4, 360, 360)$	2	172	$(0; 30, 72, 360)$	2
142	$(0; 5, 72, 360)$	1	172	$(0; 40, 45, 360)$	3
144	$(0; 5, 360, 360)$	2	172	$(0; 24, 180, 360)$	4
150	$(0; 6, 360, 360)$	1	173	$(0; 45, 72, 120)$	1
154	$(0; 8, 45, 360)$	1	173	$(0; 40, 72, 180)$	2
156	$(0; 9, 40, 360)$	1	174	$(0; 45, 72, 360)$	3
156	$(0; 8, 90, 360)$	1	174	$(0; 40, 90, 360)$	3
157	$(0; 8, 180, 360)$	2	174	$(0; 36, 120, 360)$	4
159	$(0; 9, 120, 360)$	2	174	$(0; 30, 360, 360)$	6
160	$(0; 10, 72, 360)$	1	175	$(0; 60, 72, 360)$	2
160	$(0; 9, 360, 360)$	2	175	$(0; 36, 360, 360)$	4
162	$(0; 10, 360, 360)$	2	175	$(0; 45, 120, 360)$	6
165	$(0; 12, 360, 360)$	2	175	$(0; 40, 180, 360)$	6
166	$(0; 18, 40, 360)$	1	176	$(0; 72, 90, 360)$	1
166	$(0; 15, 72, 360)$	2	176	$(0; 45, 360, 360)$	5
168	$(0; 15, 360, 360)$	3	176	$(0; 72, 120, 180)$	12
169	$(0; 18, 120, 360)$	2	177	$(0; 90, 120, 360)$	6
169	$(0; 20, 72, 360)$	2	177	$(0; 72, 180, 360)$	6
169	$(0; 24, 45, 360)$	2	177	$(0; 60, 360, 360)$	6
170	$(0; 40, 45, 72)$	1	178	$(0; 90, 360, 360)$	5
170	$(0; 18, 360, 360)$	2	178	$(0; 120, 180, 360)$	12
171	$(0; 36, 40, 360)$	2	179	$(0; 180, 360, 360)$	10
171	$(0; 24, 90, 360)$	2			

Finally, as remarked in the introduction, our enumeration formulas also count conjugacy classes of finite cyclic subgroups of the mapping class group $\mathcal{M}_\sigma$. Although, in general, determining when a given conjugacy class of $\mathcal{M}_\sigma$ defines a class of maximal finite subgroups is a very difficult problem, in this case, the results of [**4**] provide us with explicit conditions for when these actions are maximal. Specifically, a generating vector corresponds to a maximal action precisely when none of the permutations i_1, i_2, i_3 or j extend to an automorphism of G, and hence the set V_G is precisely the set of all generating vectors corresponding to maximal actions. Thus we have the following simple consequence of our results.

Corollary 5.3. *Suppose that* $\sigma = 1 + (M/2)(1 - (1/n_1) - (1/n_2) - (1/n_3))$, *and let* n *be the number of occurrences of* M *in the signature* $(0; n_1, n_2, n_3)$. *Then the number of conjugacy classes of maximal cyclic subgroups of* $\mathcal{M}_\sigma$ *of order* M *with signature* $(0; n_1, n_2, n_3)$ *is equal to* $|V_G|/(n!\phi(M))$.

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