Illinois Journal of Mathematics
Volume 58, Number 2, Summer 2014, Pages 585–591
S 0019-2082

HÖLDER CONTINUOUS SOBOLEV MAPPINGS AND THE LUSIN N PROPERTY

ALEKSANDRA ZAPADINSKAYA

ABSTRACT. We give a new proof for the result of J. Malý and O. Martio, stating that Hölder continuous mappings in $W^{1,n}$ satisfy the Lusin N property. We further generalize this result to a metric setting.

1. Introduction

In this note, we study the Lusin N property for Sobolev mappings. We say that a mapping $f:\mathbb{R}^n \to \mathbb{R}^m$ satisfies the Lusin N property, if every set of zero Lebesgue n-measure has a zero n-dimensional Hausdorff measure image under f. The validity of the Lusin N property for a Sobolev mapping $f:\mathbb{R}^n \to \mathbb{R}^n$ enables the application of the change of variable formula for integration and of the area formula [7]. This fact makes the study of the Lusin N property important.

Let us consider a Sobolev mapping $f \in W^{1,p}(\Omega;\mathbb{R}^m)$, defined in a domain $\Omega \subset \mathbb{R}^n$, where $2 \le n \le m$. It is well known that $p > n$ implies the Lusin N property [9]. On the other hand, this property may fail in the case $p = n$ (see examples in [12] and [8, Section 5]). However, additional assumptions on f, such as monotonicity or Hölder continuity, may guarantee the Lusin N property even when $p = n$ [11], [12], [10], [8].

We consider the latter case, that is we assume that our mapping $f \in W^{1,n}(\Omega;\mathbb{R}^m)$ satisfies a Hölder continuity condition

$$\big|f(x) - f(y)\big| \le C_0|x-y|^\gamma \tag{1}$$

for all $x, y \in \Omega$, where $C_0 > 0$ and $0 < \gamma < 1$. We give a new shorter proof for the result in [8, Theorem C], where it was established for $n = m$.

Received June 18, 2014; received in final form July 2, 2014.
2010 *Mathematics Subject Classification.* 46E35, 26B35, 28A75, 28A78.

Theorem 1. *Let Ω be a domain in $\mathbb{R}^n$ and $f \in W^{1,n}(\Omega;\mathbb{R}^m)$, $m \geq n \geq 2$, a Hölder continuous mapping. Then $\mathcal{H}^n(f(E)) = 0$ for each $E \subset \Omega$, such that $|E| = 0$.*

We do not know, whether this Hölder continuity assumption is sharp. However, given any $\alpha \in]0,(n-1)/n[$, there exists a mapping $f \in W^{1,n}(\mathbb{R}^n;\mathbb{R}^n)$, violating the Lusin N property and having modulus of continuity no worse than

$$\psi(t) = C_0 \exp\left(-c \log^\alpha\left(\frac{1}{t}\right)\right) \tag{2}$$

with some $C_0, c > 0$ [5, Example 1.3]. Note that the modulus of continuity (1) we assume is (2) with $c = \gamma$ and $\alpha = 1$.

The method we use has its origins in [6], where quasiconformal mappings were considered. First applications of those ideas to more general non-injective Sobolev mappings can be found in [4].

Our proof gives a direct generalization to a metric setting, providing a new result, stated as follows (see Section 3).

Theorem 2. *Let $Q \geq 1$ and let (X, dist, μ) be an Ahlfors Q-regular metric measure space, which supports the Q-Poincaré inequality for continuous functions. Suppose that $f \in N^{1,Q}(X;V)$, with some Banach space V, is a Hölder continuous mapping. Then $\mathcal{H}^Q(f(E)) = 0$ for each $E \subset X$, such that $\mu(E) = 0$.*

To demonstrate the elegance of the proof in the Euclidean case, we give separate proofs for the two theorems.

2. Proof of Theorem 1

We start by introducing our basic notation. Given a set $A \subset \mathbb{R}^n$, we denote its n-dimensional Lebesgue measure by $|A|$. If $|A| < \infty$ and f is a Lebesgue integrable mapping, we denote the average $\frac{1}{|A|}\int_A f$ of f over the set A by $⨍_A f$ or f_A. Next, $A + a$ with $A \subset \mathbb{R}^n$ and $a > 0$ stands for the set $\{x \in \mathbb{R}^n : \mathrm{dist}(x, A) < a\}$. By $\mathrm{diam}(A)$ and χ_A, we denote the diameter and the characteristic function of the set A, respectively. Given a point $x \in \mathbb{R}^m$ and a non-negative number r, $B(x,r)$ denotes an open ball centred in x and having radius r. If $B = B(x,r)$ is a ball and a is a positive number, the notation aB stands for the ball $B(x, ar)$. We write $\mathcal{H}^s_\delta(A)$ with $s > 0$ and $0 < \delta \leq \infty$ for the s-dimensional Hausdorff content of a set A, while $\mathcal{H}^s(A)$ denotes its s-dimensional Hausdorff measure. Finally, C denotes a positive constant, which may depend on data (n, m and the modulus of continuity of f) and differ from occurrence to occurrence.

We need also a weighted Hausdorff content of a set A given by

$$\lambda_\infty^s(A) = \inf\left\{\sum_{i=1}^\infty c_i(\operatorname{diam} U_i)^s : c_i \geq 0 \text{ and } \chi_A \leq \sum_{i=1}^\infty c_i \chi_{U_i}\right\}$$

for $s > 0$. It is known that there exists a constant $\beta > 0$, such that $\mathcal{H}_\infty^s(E) \leq \beta\lambda_\infty^s(E)$ for all sets E (see, for instance, [3], Theorem 8.6 and Theorem 9.7).

Proof of Theorem 1. We denote the modulus of continuity of f by $\psi(t) = C_0 t^\gamma$. We may clearly assume that E is bounded and $\overline{E} \subset \Omega$. Let us consider a dyadic decomposition of $\mathbb{R}^n$. We denote by $\mathcal{Q}_i = \{Q_{i,1}, Q_{i,2}, \dots\}$ the collection of cubes of generation $i \in \mathbb{N}$ with edge length 2^{-i}, such that $\mathbb{R}^n = \bigcup_{j=1}^\infty Q_{i,j}$. For each $i, j \in \mathbb{N}$, there exist 2^n cubes, denoted by $Q_{i,j}^1, \dots, Q_{i,j}^{2^n} \in \mathcal{Q}_{i+1}$, such that $Q_{i,j} = \bigcup_{q=1}^{2^n} Q_{i,j}^q$. Similarly, when $i \geq 2$, the unique cube $Q \in \mathcal{Q}_{i-1}$, such that $Q_{i,j} \subset Q$ is denoted by $\hat{Q}_{i,j}$.

Once $Q_{i,j}$ is such that $\hat{Q}_{i,j} \subset \Omega$, we define $f_{i,j} = f_{Q_{i,j}} \in \mathbb{R}^m$ and

$$r_{i,j} = \max\Big\{|f_{i,j} - f_{\hat{Q}_{i,j}}|, \max_{q=1,\dots,2^n} |f_{i,j} - f_{Q_{i,j}^q}|\Big\}. \tag{3}$$

We obtain the following estimate for $q \in \{1, \dots, 2^n\}$, using the Poincaré and Jensen inequalities

$$\begin{aligned} |f_{i,j} - f_{Q_{i,j}^q}| &\leq \fint_{Q_{i,j}^q} |f - f_{i,j}| \leq 2^n \fint_{Q_{i,j}} |f - f_{i,j}| \\ &\leq C \operatorname{diam} Q_{i,j} \fint_{Q_{i,j}} |Df| \\ &\leq C\left(\int_{Q_{i,j}} |Df|^n\right)^{1/n}. \end{aligned} \tag{4}$$

Similar computations give $|f_{i,j} - f_{\hat{Q}_{i,j}}| \leq C(\int_{\hat{Q}_{i,j}} |Df|^n)^{1/n}$. Thus, $r_{i,j}^n \leq C\int_{\hat{Q}_{i,j}} |Df|^n$. For each $r_{i,j} > 0$, we need a family of balls

$$\mathcal{B}_{i,j} = \big\{B_{i,j}^k = B\big(f_{i,j}, r_{i,j}/2^k\big) : k = 0, 1, \dots\big\}.$$

Fix an arbitrary $\varepsilon > 0$ and a $\delta \in\,]0, \varepsilon]$, such that $E + \delta \subset \Omega$ and

$$\int_{E+\delta} |Df|^n < \varepsilon.$$

Additionally, we choose a number $i_0 \in \mathbb{N}$, which satisfies $\sqrt{n}2^{-i_0+1} < \delta$. We restrict the families $\mathcal{Q}_i$, $i = i_0, \dots$, so that $\hat{Q} \subset E + \delta$, whenever $Q \in \mathcal{Q}_i$.

Let $x \in E$. We choose a sequence of cubes $(Q_i(x))_{i=i_0}^\infty$, such that $Q_i(x) \in \mathcal{Q}_i$ and $x \in Q_i(x)$. We have $Q_i(x) = Q_{i,j(i,x)}$ for a suitable index $j(i,x) \in \mathbb{N}$. This sequence defines a sequence of centres $f_i(x) = f_{i,j(i,x)}$, $i = i_0, \dots$, on the image side, which converges to $f(x)$; indeed,

$$(5) \qquad \big|f(x) - f_i(x)\big| \leq \fint_{Q_i(x)} \big|f(x) - f(y)\big|\, dy \leq \psi\big(\operatorname{diam} Q_i(x)\big) \to 0,$$

when i goes to infinity. Finally, we put $r_i(x) = r_{i,j(i,x)}$. Note that (3) implies

$$(6) \qquad r_i(x) \geq \max\big\{\big|f_i(x) - f_{i-1}(x)\big|, \big|f_i(x) - f_{i+1}(x)\big|\big\}$$

for each $i = i_0 + 1, \dots$.

We neglect the set $E_0 = \{x \in E : f_i(x) = f(x) \text{ for each } i = i_0, i_0 + 1, \dots\}$, because its image under f is countable. For a point $x \in E \setminus E_0$, we define a large number $l_0(x) \in \mathbb{N}$ so that there are some of $f_i(x)$ with $i \geq i_0$ outside the ball $B(f(x), 2^{-l_0(x)+1})$. Denoting $E_l = \{x \in E \setminus E_0 : l_0(x) \leq l\}$, we have $E \setminus E_0 = \bigcup_{l \in \mathbb{N}} E_l$ and $f(E \setminus E_0) = \bigcup_l f(E_l)$.

Fix $l_1 \in \mathbb{N}$ and consider the set E_{l_1}. Let $x \in E_{l_1}$ and $l = 4l_1$. We find the smallest number $J \in \mathbb{N}$, $J \geq i_0$, such that $f_j(x) \in B(f(x), 2^{-l})$ for all $j > J$. We have by (5)

$$2^{-l} \leq \big|f(x) - f_J(x)\big| \leq \psi\big(\operatorname{diam} Q_J(x)\big) = C_0\sqrt{n^\gamma}2^{-J\gamma},$$

which implies $l \geq J\gamma - \log_2(C_0\sqrt{n^\gamma}) \geq J\gamma/2$, if i_0 is initially picked so that $i_0 \geq \frac{2}{\gamma}\log_2(C_0\sqrt{n^\gamma})$. On the other hand, if we denote by N the number of integers k in the set $\{l_1, \dots, l\}$, such that the annulus $A_k(x) = B(f(x), 2^{-k+1}) \setminus B(f(x), 2^{-k})$ contains more than $8/\gamma$ centres $f_i(x)$, $i = i_0, \dots, J$, we obtain $8N/\gamma \leq J \leq 2l/\gamma$, hence $N \leq l_1$. Thus, there exist at least $l - l_1 + 1 - N \geq 2l_1$ annuli $A_k(x)$, $k = l_1, \dots, l$, which contain at most $8/\gamma$ centres $f_i(x)$, $i = i_0, \dots, J$.

Let $A_k(x)$ be one such annulus. If it contains at least one centre $f_i(x)$ for some $i = i_0, \dots, J$, then (6) and the fact that $l_1 \geq l_0(x)$ yield

$$\sum_{f_i(x) \in A_k(x)} 2r_i(x) > 2^{-k}.$$

Thus, there must be at least one $i \geq i_0$, such that $f_i(x) \in A_k(x)$ and $r_i(x) \geq 2^{-k-4}\gamma$. We have $f(x) \in \frac{32}{\gamma}B^0_{i,j(i,x)} = B(f_i(x), 32r_i(x)/\gamma)$. Whenever there are no $f_i(x) \in A_k(x)$, we take the smallest $i = i_0, \dots, J+1$ such that $f_i(x) \in B(f(x), 2^{-k})$. By (6) and $l_1 \geq l_0(x)$, we necessarily have $r_i(x) > 2^{-k}$. We pick $B^p_{i,j(i,x)} \ni f(x)$ so that $2^{-k} < r_i(x)2^{-p} \leq 2^{-k+1}$. Note that when x is fixed and we choose balls for different k, each ball is taken no more than twice. That is

$$2\sum_{i=i_0}^{2l/\gamma+1}\ \sum_{Q_{i,j} \in \mathcal{Q}_i}\ \sum_{k=0}^{\infty} \chi_{\frac{32}{\gamma}B^k_{i,j}}(y) \geq 2l_1$$

for each $y \in f(E_{l_1})$. In other words, the collection of inflated balls $\frac{32}{\gamma} B^k_{i,j}$ covers the set $f(E_{l_1})$ with l_1 layers. We conclude

$$\begin{aligned}(7)\quad \mathcal{H}^n_\infty\big(f(E_{l_1})\big) &\le \beta\lambda^n_\infty\big(f(E_{l_1})\big) \le \frac{\beta}{l_1}\sum_{i=i_0}^{2l/\gamma+1}\sum_{Q_{i,j}\in\mathcal{Q}_i}\sum_{k=0}^{\infty}\frac{2^{6n}}{\gamma^n}\frac{r^n_{i,j}}{2^{kn}}\\ &\le \frac{C\beta}{l_1}\sum_{i=i_0}^{2l/\gamma+1}\sum_{Q_{i,j}\in\mathcal{Q}_i} r^n_{i,j} \le \frac{C\beta}{l_1}\sum_{i=i_0}^{2l/\gamma+1}\sum_{Q_{i,j}\in\mathcal{Q}_i}\int_{\hat{Q}_{i,j}}|Df|^n\\ &\le \frac{C\beta}{l_1}\sum_{i=i_0}^{2l/\gamma+1}\int_{E+\delta}|Df|^n \le \frac{C\beta(\frac{2l}{\gamma}+1)}{l_1}\varepsilon \le C\beta\varepsilon.\end{aligned}$$

Since the sets E_l are nested, we obtain $\mathcal{H}^n_\infty(f(E\setminus E_0)) < C\beta\varepsilon$. By the arbitrariness of ε, we have $\mathcal{H}^n_\infty(f(E\setminus E_0)) = 0$, which yields $\mathcal{H}^n(f(E)) = 0$. □

3. Metric setting

For this section, we preserve the notation $f_A = ⨍_A f$, $\operatorname{diam}(A)$, χ_A, $B(x,r)$, aB, $\mathcal{H}^s_\delta(A)$, $\mathcal{H}^s(A)$, $\lambda^s_\infty(A)$, defined suitably. Recall that $\mathcal{H}^s_\infty(A) \le \beta\lambda^s_\infty(A)$.

By Ahlfors regularity in the statement of Theorem 2, we mean that a metric space (X,dist) is equipped with a Borel regular measure μ, such that $c_1 r^Q \le \mu(B) \le c_2 r^Q$, for all open balls $B \subset X$ of radius $r \in \,]0,\operatorname{diam} X[$ and some constants $Q, c_1, c_2 > 0$. Additionally, we assume that $(X,\operatorname{dist},\mu)$ supports Q-Poincaré inequality for continuous functions (see [2, Section 4]):

$$⨍_B \|f - f_B\|\, d\mu \le C_P(\operatorname{diam} B)\left(⨍_{\sigma B}\rho^Q\, d\mu\right)^{1/Q}$$

for all balls $B \subset X$, all continuous integrable functions f, defined in the ball σB and taking values in some Banach space V, all Q-weak V-upper gradients ρ of f, and with constants $C_P \ge 0$, $\sigma \ge 1$, independent of B, f and ρ. Let V be a Banach space. The mapping f in the statement of Theorem 2 is in the Sobolev class $N^{1,Q}(X;V)$ (see [2, Section 3]) and is Hölder continuous with modulus $\psi(t) = C_0 t^\gamma$. We fix some Q-weak V-upper gradient ρ of f.

Proof of Theorem 2. The proof of Theorem 2 is a direct generalization of the proof in the previous section, so we just outline the main differences. Let us fix $\varepsilon > 0$ and take an open set $\Omega \supset E$, such that

$$\int_\Omega \rho^Q < \varepsilon.$$

Pick $i' \in \mathbb{N}$ so that $10\sigma 2^{-i'} < \operatorname{diam} X$. This choice ensures that the radii of all balls, to which we are going to apply the doubling condition, are smaller than $\operatorname{diam} X$. We consider the decomposition $E = \bigcup_{i\ge i'} E_i$, where $E_i = \{x \in E : B(x, 5\sigma 2^{-i}) \subset \Omega\}$. Next, we fix an integer $i_0 \ge i'$ and consider the set E_{i_0}.

Applying the covering theorem [1, Theorem 1.16], we obtain finite collections of balls $\mathcal{Q}_i$, $i=i_0,\dots$, such that $E_{i_0}\subset\bigcup_{B\in\mathcal{Q}_i}B$, each $B\in\mathcal{Q}_i$ is centred in E_{i_0} and has radius 2^{-i}, and $\{\frac{1}{5}B : B\in\mathcal{Q}_i\}$ is a disjoint family for each $i=i_0,\dots$.

Let $x\in E_{i_0}$. There exists a sequence of balls $(B_i(x))_{i=i_0}^{\infty}$, such that $x\in B_i\in\mathcal{Q}_i$. We denote $f_i(x)=f_{B_i(x)}$. As in (5), we have $\|f_i(x)-f(x)\|\le\psi(2\cdot 2^{-i})\to 0$, when $i\to\infty$. Moreover, similarly to (4), we obtain

$$\max\big\{\big\|f_i(x)-f_{i-1}(x)\big\|,\big\|f_i(x)-f_{i+1}(x)\big\|\big\}\le A\bigg(\int_{5\sigma B_i(x)}\rho^Q\bigg)^{1/Q}$$

for $i=i_0+1,\dots$, where the constant $A>0$ depends on c_1, c_2, Q, C_P and σ. We put $r_B=A(\int_{5\sigma B}\rho^Q)^{1/Q}$ for each $B\in\mathcal{Q}_i$ and each $i=i_0,\dots$, and consider the collection $\mathcal{B}_B=\{B_B^k=B(f_B,r_B/2^k) : k=0,1,\dots\}$.

As in the previous section, we decompose $E_{i_0}=\bigcup_l E_{i_0,l}$ according to the number $l_0(x)$ and fix some E_{i_0,l_1} and $l=4l_1$. Analogous argument implies

$$2\sum_{i=i_0}^{2l/\gamma+1}\sum_{B\in\mathcal{Q}_i}\sum_{k=0}^{\infty}\chi_{\frac{32}{\gamma}B_B^k}(y)\ge 2l_1$$

for each $y\in f(E_{i_0,l_1})$. Since the families $\{\frac{1}{5}B : B\in\mathcal{Q}_i\}$ are disjoint, the doubling condition for the measure μ gives the boundedness of the overlap

$$\sum_{B\in\mathcal{Q}_i}\chi_{5\sigma B}(y)\le C$$

for each $i=i_0,\dots$, $y\in\Omega$ and some constant C, which depends on c_1, c_2, Q and σ. We finally obtain similarly to (7)

$$\mathcal{H}^Q_\infty\big(f(E_{i_0,l_1})\big)\le\frac{C}{l_1}\sum_{i=i_0}^{2l/\gamma+1}\sum_{B\in\mathcal{Q}_i}\int_{5\sigma B}\rho^Q\le\frac{C}{l_1}\sum_{i=i_0}^{2l/\gamma+1}\int_\Omega\rho^Q\le C\varepsilon,$$

where the constant $C>0$, differing from occurrence to occurrence, depends only on β, γ, c_1, c_2, Q, C_P and σ. Thus, we conclude $\mathcal{H}^Q_\infty(f(E_{i_0}))\le C\varepsilon$ and $\mathcal{H}^Q_\infty(f(E))\le C\varepsilon$, since the involved sequences of sets are nested. □

Acknowledgments. The author thanks Pekka Koskela for inspiring this note and the referees for the helpful comments.

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Aleksandra Zapadinskaya, Department of Mathematics, University of Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy

E-mail address: azapadinskaya@mail.dm.unipi.it