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EVERY CENTRAL SIMPLE ALGEBRA IS BRAUER EQUIVALENT TO A HOPF SCHUR ALGEBRA

EHUD MEIR

ABSTRACT. We show that every central simple algebra A over a field k is Brauer equivalent to a quotient of a finite dimensional Hopf algebra over the same field. This shows that the natural generalization of the Schur group for Hopf algebras (which we call the Hopf Schur group) is in fact the entire Brauer group of k. If the characteristic of the field is zero, or if the algebra has a Galois splitting field with certain properties, we can take this Hopf algebra to be semisimple. We also show that if F is any finite separable extension of k, then F is a quotient of a finite dimensional commutative semisimple and cosemisimple Hopf algebra over k.

1. Introduction

Let k be a field. In [1], we asked what central simple k-algebras can we get (up to Brauer equivalence) as quotients of finite dimensional Hopf algebras over k. We called such algebras Hopf Schur algebras, and we defined the Hopf Schur group of k, $HS(k)$, as the subgroup of $Br(k)$ which contains all classes of Hopf Schur algebras. This is analogue to the definition of the Schur group, $S(k)$, which contains all classes of central simple algebras which are quotients of finite group algebras, and also to the projective Schur group, $PS(k)$, which contains all classes of central simple algebras which are quotients of finite dimensional twisted group algebras.

Since any group algebra is a Hopf algebra, clearly $S(k) < HS(k)$. The Schur group is a "small" subgroup of $Br(k)$. It is known by a theorem of Brauer (see [9]) that any element in $S(k)$ has a cyclotomic splitting field, and therefore if k contains all roots of unity then $S(k) = 0$, whereas $Br(k)$ may

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be large (e.g. $k = \mathbb{C}(x_1, x_2, \ldots, x_n)$, $n \geq 2$, see [6]). We refer the reader to [10] and to [16] for a comprehensive account on the Schur group. In [1] it was shown that $HS(k)$ might be much bigger than $S(k)$. The authors have proved that $PS(k) < HS(k)$. The projective Schur group is already a much bigger subgroup of the Brauer group. It was conjectured that $PS(k) = Br(k)$ and this is indeed the case for many interesting fields (e.g., number fields). It was disproved, however, by Aljadeff and Sonn in [2]. In [1], it was also proved that any product of cyclic algebras is in $HS(k)$, and there is a conjecture which says that this is all of the Brauer group.

In this paper, we will prove that $HS(k) = Br(k)$. In addition, we will give sufficient conditions for a central simple k-algebra to be Brauer equivalent to a quotient of a *semisimple* Hopf algebra. For that reason, we introduce the following definition:

Definition 1.1. Let A be a central simple k-algebra. Denote by m the order of $[A]$ in $Br(k)$. We will say that the algebra A is *good* if $char(k) = 0$, or if $char(k) = p$, $p \nmid m$, and A has a Galois splitting field L such that $p \nmid |L(\xi_m) : k|$, where ξ_m is a primitive mth root of unity.

The main result of this paper is the following:

Theorem 1.2. *Any k-central simple algebra A is Bruaer equivalent to a quotient of a finite dimensional Hopf algebra H (that is, $Br(k) = HS(k)$). If A is a good algebra, then A is Brauer equivalent to a quotient of a finite dimensional semisimple Hopf algebra.*

Since division algebras arise naturally as Endomorphism rings of simple representations, we have the following:

Corollary 1.3. *Let D be a k-central division algebra. Then there is a finite dimensional Hopf algebra H, and a simple representation V of H such that $End_H(V) \cong D$. If in addition D is good, then we can take H to be semisimple.*

Proof. Let D be a k-central division algebra. By the above theorem, we have a Hopf algebra H (seimsimple in case D is good), and an algebra map $\pi : H \twoheadrightarrow M_n(D^{op})$ for some n, where D^{op} is the algebra D with opposite multiplication. Then $V = (D^{op})^n$ is a representation of H via π, and it is easy to see that $End_H(V) \cong D$. □

Remark 1.4. In the proof above, we have used the fact that D is good if and only if D^{op} is good.

In the course of the proof of Theorem 1.2, we will consider, in Section 3, forms of the function algebra of a finite group (i.e., the dual of a finite group algebra). As a result, we will see that there might be an infinite number of non-isomorphic semisimple and cosemisimple Hopf algebras over k of a given dimension, if k is not algebraically closed. In [15], Stefan proved that for a given number n, there are only finitely many isomorphism classes of semisimple and cosemisimple Hopf algebras of dimension n over an algebraically closed field. We therefore conclude that the algebraic closeness assumption in Stefan's theorem is necessary (this was observed also by Caenepeel Dascalescu and Le Bruyn in [5]). Stefan's theorem is a weaker form of Kaplansky's tenth conjecture, which states that for a given number n there are only finitely many isomorphism classes of (not necessarily semisimple and cosemisimple) Hopf algebras of dimension n over an algebraically closed field. Kaplansky's tenth conjecture was disproved by Andruskiewitsch and Schneider (see [3]) by Beattie, Dascalescu and Grunenfelder (see [4]), and by Gelaki (see [7]) (and so, also the semisimplicity and the cosemisimplicity of the Hopf algebra is necessary in Stefan's theorem).

The main idea behind the proof of Theorem 1.2 will be the following observation: if L/k is a Galois extension of fields, then a Hopf algebra over k is equivalent to a Hopf algebra over L together with a certain "Hopf-semilinear" action. This generalizes the idea from descent theory, that an algebra over k is the same thing as an algebra over L together with a certain "semilinear" action. In descent theory, this idea gives a description of the relative Brauer group $Br(L/k)$ (that is, of all Brauer classes in $Br(k)$ which split by L). In our setting, this idea gives us a tool to construct a variety of nonisomorphic Hopf algebras over k (which become isomorphic over L). We will be able to show in Section 5 that by choosing the "right" Hopf algebra and the "right" Hopf-semilinear action, we will be able to "catch" all the Brauer equivalence classes in $Br(L/k)$. Since L is arbitrary, this will prove that $HS(k)=Br(k)$.

We will begin by defining what are Hopf-semilinear actions in Section 2, where we will also describe the relevant parts from descent theory which will be in use. In Section 3, we use descent theory in order to describe all the k-forms of the Hopf algebra $L[T]$ of functions on some finite group T (by this we mean all the k-Hopf algebras which become isomorphic to $L[T]$ over L). Another tool which we will need will be semidirect products of Hopf algebras. This will be described in Section 4. In Section 5, we will prove Theorem 1.2.

2. Galois descent

We shall need to use a very small portion of the descent theory in here. The reader is referred to the second chapter of [8] for more comprehensive treatment. Let L/k be a Galois extension with a Galois group G.

Definition 2.1. Let V be a vector space over L. A G-semilinear action on V is an action of G on V as a k-vector space such that $g(x\cdot v)=g(x)\cdot g(v)$, for every $g\in G$, $x\in L$ and $v\in V$.

We shall simply say "semilinear action" instead of "G-semilinear action" if the group is clear from the context. The typical example we should have in mind for a semilinear action is the following: suppose that $\hat{V}$ is a vector space over k. Then $V = L \otimes_k \hat{V}$ is a vector space over L, and we have a semilinear action given by $g \cdot (x \otimes v) = g(x) \otimes v$. In descent theory, it is proved that every semilinear action is of this form (this is Speiser lemma, see [8, p. 27]). In this paper, we will be interested in semilinear actions which also commute with the Hopf structure.

Definition 2.2. Let H be a Hopf algebra over L. A G-semilinear action on H is said to be *Hopf-semilinear* if it commutes with the Hopf structure of H. That is, for every $g \in G$ and $a, b \in H$ we have $g(ab) = g(a)g(b)$, and similar equations hold for the coproduct, counit, unit and antipode.

We will need to use the following lemma, whose proof is based on Speiser Lemma together with general arguments.

Lemma 2.3. *Let H be a Hopf algebra over L. Suppose that H has a G-Hopf-semilinear action. Then the subspace of invariant elements H^G is a Hopf algebra over k (the structure maps are just the restrictions of the structure maps of H), and $H^G \otimes_k L \cong H$ as L-Hopf algebras.*

Remark 2.4. The algebra H^G is called a k-form of the L-Hopf algebra H.

We thus see that in order to construct Hopf algebras over k, we can construct Hopf algebras over L together with semilinear actions. In descent theory, all the semilinear actions of a given Hopf algebra are classified by a certain nonabelian cohomology group. In our case, it would be easier to find semilinear actions directly. Forms of Hopf algebras were also considered by Radford, Taft and Wilson in [14], by Pareigis in [12], by Parker in [13], and by Caenepeel, Dascalescu and Le Bruyn in [5].

3. Hopf-semilinear actions on function algebras

Let L, k and G be as before, and let T be any finite group. We consider the function algebra (which is also the dual of the group algebra of T, $L[T] = (LT)^*$). This is the L-algebra of all the functions from T to L. This algebra has a basis of simple idempotents $\{e_t\}_{t \in T}$, where $e_t(s) = \delta_{t,s}$ for $t, s \in T$. This algebras also has a Hopf structure given by $\Delta(e_t) = \sum_{rs=t} e_r \otimes e_s$. In this section, we will describe the Hopf-semilinear actions of G on $L[T]$ and how the corresponding invariant Hopf algebras look like.

Lemma 3.1. *There is a one to one correspondence between Hopf-semilinear actions on $L[T]$ and homomorphism $G \to Aut(T)$ (i.e. actions of G on T).*

Remark 3.2. The reader who is familiar with descent theory will notice that this correspondence is exactly the correspondence between k-forms of $L[T]$ and $H^1(G, Aut(T))$, where G acts trivially on $Aut(T)$.

Proof of Lemma 3.1. The correspondence is given in the following way: for $\phi: G \to Aut(T)$ we have the semilinear action $g_\phi \cdot xe_t = g(x)e_{\phi(g)(t)}$ for $g \in G$, $x \in L$ and $t \in T$. Any Hopf-semilinear action is of this form due to the following reason: since the action is by algebra automorphisms, every $g \in G$ permutes the set of simple idempotents $\{e_t\}_{t\in T}$, and so acts on T. The fact that the action preserves the coalgebra structure means that this permutation is an automorphism of T as a group. □

REMARK 3.3. In Section 3 of [1], we have described explicitly a specific form of a specific function algebra. Let us describe the corresponding Hopf-semilinear action. We have an abelian Galois extension L/k with an (abelian) Galois group G. We have constructed a k-form H of the L-Hopf algebra $L[Z_2 \ltimes G]$ (the action of $\mathbb{Z}_2 = \langle\sigma\rangle$ is by inversion) which is isomorphic as an algebra to $L \oplus k[G]$. The form H corresponds to a semilinear action, which corresponds, by Lemma 3.1, to a homomorphism $\phi: G \to Aut(\mathbb{Z}_2 \ltimes G)$. For $g \in G$, the homomorphism $\phi(g)$ is the homomorphism which sends σ to σg and fixes G pointwise.

Let us describe, for a given $\phi: G \to Aut(T)$, the structure of the algebra of invariants $(L[T])^G$. Let $a = \sum_{t\in T} a_t e_t \in L[T]^G$. It is easy to see that the fact that a is invariant is equivalent to the fact that $g(a_t) = a_{\phi(g)(t)}$ for every $g \in G$ and $t \in T$. In particular $a_t \in L^{stab(t)}$, where by $stab(t)$ we denote the stabilizer of t in G with respect to the action ϕ. If we fix representatives of the different orbits $t_1, \dots, t_m$, then we have an isomorphism of algebras

$$L^{stab(t_1)} \oplus L^{stab(t_2)} \oplus \cdots \oplus L^{stab(t_m)} \to \big(L[T]\big)^G$$

given by

$$x \mapsto \sum_{g\in G/stab(t_i)} g(x)e_{\phi(g)t_i}$$

for $x \in L^{stab(t_i)}$. Notice in particular that all the fields $L^{stab(t_i)}$ are quotient of the k-Hopf algebra $(L[T])^G$. Can we get any subfield of L in this way? the answer is yes. Using Galois correspondence, any subfield of L is of the form L^H, for some $H < G$. Therefore we need to prove that for every $H < G$, we have a group T and an action of G on T such that T contains an element t such that $stab(t) = H$. Let us take $T = \mathbb{Z}_2 G/H$, the vector space over $\mathbb{Z}_2$ with the coset space G/H as a basis. The group G acts from the left on G/H and therefore also on T. It is clear that if we take $t = H$ (the trivial coset), then $stab(t) = H$ as required. Since L was an arbitrary Galois extension of k, and any finite separable extension of k is contained in its Galois closure, we have (almost) proved the following:

THEOREM 3.4. *Let F/k be any finite separable extension. Then there is a semisimple cosemisimple commutative Hopf algebra H over k such that F is a quotient of H.*

Proof. The only thing that requires a proof is the cosemisimplicity. The function algebra $k[T]$ is cosemisimple if and only if $char(k) \nmid |T|$ (this is Maschke's Theorem applied to the dual Hopf algebra). If $char(k) \neq 2$, we can take $T = \mathbb{Z}_2 G/H$ as above, and if $char(k) = 2$, we can take $T = \mathbb{Z}_3 G/H$. □

Now assume that we have an infinite number of nonisomorphic Galois extensions of k with the same Galois group G which satisfies the condition $char(k) \nmid |G|$ (e.g. $k = \mathbb{Q}$ and $G = \mathbb{Z}_2$). Then any Galois extension L of k with a Galois group G is a quotient of a twisted form of $k[\mathbb{Z}_2 G]$. The algebra $k[\mathbb{Z}_2 G]$ (and each of its forms) is semisimple and cosemisimple (if $char(k) = 2$ we can take $k[\mathbb{Z}_3 G]$ as in the proof of the theorem above). It is easy to see that by considering all these forms, we get an infinite number of nonisomorpic commutative semisimple and cosemisimple Hopf algebras of dimension $2^{|G|}$, as was claimed in Section 1. This generalizes the example given in Section 2 of [5] which shows that over a non algebraically closed field a group algebra of an abelian group can have infinitely many nonisomorphic forms.

4. A semidirect product

In this section, we will construct some specific semidirect products of Hopf algebras over k. Let N and T be groups such that N acts on T by group automorphisms (that is, we have a homomorphism $\psi : N \to Aut(T)$). We will construct a Hopf algebra $k[T] \rtimes kN$ which we call the semidirect product of kN and $k[T]$. As a coalgebra, $k[T] \rtimes kN$ is $k[T] \otimes_k kN$. The product in $k[T] \rtimes kN$ is given by the rule

$$e_{t_1} \otimes n_1 \cdot e_{t_2} \otimes n_2 = \delta_{t_1, \psi(n_1)(t_2)} e_{t_1} \otimes n_1 n_2.$$

In other words, $k[T]$ and kN are subalgebras of $k[T] \rtimes kN$, and $n \in N$ acts by conjugation on e_t via ψ. The algebra $k[T] \rtimes kN$ is a Hopf algebra. It is the bicrossed product of the Hopf algebras kN and $k[T]$. For the definition of bicrossed products in general, see Chapter IX.2 of [11].

Remark 4.1. It can be seen that $k[T] \rtimes kN$ is isomorphic as an algebra to a direct sum of the form $\bigoplus_i M_{n_i}(kH_i)$, where H_i are subgroups of N which arise as stabilizers of element in T. Therefore, if $char(k) \nmid |N|$, then the semidirect product $k[T] \rtimes kN$ is semisimple, and thus also every form of it. We will need this observation later, in order to deal with semisimplicity questions.

5. A proof of Theorem 1.2

In this section, we will show that $HS(k) = Br(k)$. Let A be a k-central simple algebra. The algebra A splits by some Galois extension L/k. Let $G = GalS(L/k)$. By Galois descent, we know that A is equivalent to a crossed product algebra $L_t^{\alpha} G$, where $\alpha \in H^2(G, L^*)$, and the action on L^* is the Galois action. This algebra has an L-basis $\{U_\sigma\}_{\sigma\in G}$, and the multiplication is given by the rule:

$$xU_g yU_h = x\sigma(y)\alpha(g,h)U_{gh},$$

where $g,h\in G$ and $x,y\in L$. We will show that A is (up to Brauer equivalence) a quotient of a Hopf algebra. We begin with the following definition:

DEFINITION 5.1. We say that the cocycle α is *finite* if all its values are roots of unity.

Note that this definition depends on the particular cocycle α, and not just on its cohomology class $[\alpha]$. We will prove that A is Brauer equivalent to a quotient of a Hopf algebra in the following way: we will first show that A is Brauer equivalent to a product of cyclic algebras with a crossed product algebra in which the cocycle is finite, and then we will prove that a crossed product algebra with a finite cocycle is a quotient of a Hopf algebra. In [1], we have proved that any cyclic algebra is a quotient of a Hopf algebra, and therefore $[A]$ (the Brauer class of A) is in $HS(k)$.

REMARK 5.2. In case $char(k)=0$, A is equivalent to a crossed product algebra with a finite cocycle, and we do not need to use the result from [1].

The following lemma seems to be well known. We have included it here nevertheless, as it makes our construction more explicit.

LEMMA 5.3. *Let $\alpha\in H^2(G,L^*)$. Denote the order of α by m. If $char(k)=0$ or if $char(k)=p$ and $p\nmid m$, then the crossed product algebra $L_t^\alpha G$ is Brauer equivalent to a crossed product algebra $K_t^\beta N$ where β is a finite cocycle.*

Proof. Since the order of α is m, there is a function $f:G\to L^*$ which satisfies

$$\alpha^m(g_1,g_2)=\partial f(g_1,g_2)=f(g_1)g_1\big(f(g_2)\big)f^{-1}(g_1g_2),$$

for $g_1,g_2\in G$. Let K be a Galois extension of L which contains, for every $g\in G$, an element r_g which satisfies $r_g^m=f(g)$ (the fact that we have such a Galois extension follows from the assumption on m and $char(k)$). If we denote by N the Galois group of K over k, we have an onto map $\pi:N\twoheadrightarrow G$. Define a two cocycle $\beta\in H^2(N,K^*)$ by

$$\beta(h_1,h_2)=\alpha\big(\pi(h_1),\pi(h_2)\big)r^{-1}_{\pi(h_1)}h_1\big(r^{-1}_{\pi(h_2)}\big)r_{\pi(h_1h_2)}.$$

A direct calculation shows that all the values of β are mth roots of unity, and so β is finite. The cocycle β is cohomologous to $\inf_G^N(\alpha)$. By Brauer theory, we thus know that the central simple algebras $L_t^\alpha G$ and $K_t^\beta N$ are Brauer equivalent, as required. □

REMARK 5.4. The field K in the lemma can be taken to be $L(\xi_m,\{r_g\}_{g\in G})$. Notice that if $ord([A])=m$ is prime to p, and if $p\nmid|L(\xi_m):k|$, then also $p\nmid|L(\xi_m,\{r_g\}_{g\in G}):k|$. Thus, the lemma implies that if A is a good algebra, then A is Brauer equivalent to a crossed product algebra $K_t^{\beta}G$ with a finite cocycle, such that $p \nmid |K:k|$. By the next proposition, this implies that A is Brauer equivalent to a quotient of a finite dimensional semisimple Hopf algebra.

Suppose now that k is a field of characteristic p, and that the order of α, m, is not prime to p. Write $m = rp^e$, where r is prime to p. Then $L_t^{\alpha}G$ is Brauer equivalent to a tensor product of the form $L_t^{\alpha_1}G \otimes_k L_t^{\alpha_2}G$ where $ord(\alpha_1) = r$ and $ord(\alpha_2) = p^e$. By a theorem of Teichmueller, if $char(k) = p$ then any central simple algebra whose order is p^e is Brauer equivalent to a product of cyclic algebras (see Chapter 9.1 of [8]), and Brauer classes of cyclic algebras are in $HS(k)$ (see [1]). By the lemma above, $L_t^{\alpha_1}G$ is Brauer equivalent to a crossed product algebra in which the cocycle is finite. The proof of the following proposition together with the above remark therefore finishes the proof of Theorem 1.2:

PROPOSITION 5.5. *Let $A = L_t^{\alpha}G$ be a crossed product algebra such that α is finite. Then A is a quotient of a Hopf algebra. If $char(k) \nmid |G|$, then A is a quotient of a semisimple Hopf algebra.*

Proof. Consider the subgroup of $L_t^{\alpha}G$ generated by the U_g's. Since α is finite, we get an extension of **finite** groups

$$1 \to \mu \to \widehat{G} \to G \to 1,$$

where μ is the **finite** subgroup of L^* generated by all elements of the form $\alpha(g_1, g_2)$, for $g_1, g_2 \in G$ (by assumption, they are all roots of unity). We thus know how to get the field L as a quotient of a Hopf algebra, and how to get the subalgebra generated by the U_g's as a quotient of a Hopf algebra (the group algebra $k\widehat{G}$). We now use the semidirect product construction in order to combine these two constructions into one Hopf algebra.

Let T be the group $\mathbb{Z}_2G$, the vector space over $\mathbb{Z}_2$ with basis G (multiplication in T is just addition of vectors). We have a natural action of G (and thus of $\widehat{G}$, using the map $\widehat{G} \to G$) on T by left multiplication. If we denote the action of $\widehat{G}$ on T by ψ, we can construct the semidirect product $k[T] \rtimes k\widehat{G}$ as explained in Section 4. Notice that by Remark 4.1 $k[T] \rtimes k\widehat{G}$ is semisimple if $char(k) \nmid |G|$ (it is easy to see by the fact that the order of α in $H^2(G, L^*)$ divides $|G|$, that the prime divisors of $|\mu|$ are also prime divisors of $|G|$). Now consider the induced L-Hopf algebra $X_L = L \otimes_k (k[T] \rtimes k\widehat{G})$. We have an action of G on T not only from the left but also from the right by multiplication. We define an action of G on X_L via

$$g \star (l \otimes e_t \otimes h) = g(l) \otimes e_{t \cdot g^{-1}} \otimes h.$$

We claim the following lemma.

LEMMA 5.6. *The action $\star$ is a Hopf-semilinear action of G on X_L.*

Proof. This is a straightforward verification. The crux of the proof is the fact that the two actions of G on T from the left and from the right commute with each other. □

Consider now the k-Hopf algebra $H = (X_L)^G$ (which is semisimple in case $char(k) \nmid |G|$), where we take invariants with respect to the $\star$ action. We claim that we have an onto map $H \twoheadrightarrow L_t^\alpha G$. To see why this is true, we first decompose H as an algebra (we do not care any more about the Hopf structure at this stage).

Denote by H_1 the intersection of $(X_L)^G$ with the L subspace spanned by all $1 \otimes e_t \otimes \hat{g}$, where $\hat{g} \in \widehat{G}$, and $t \in G$ (we can consider G as the subset of T which contains the basis elements). Denote by H_2 the intersection of $(X_L)^G$ with the L subspace spanned by all $1 \otimes e_t \otimes \hat{g}$, where $\hat{g} \in \widehat{G}$ and $t \notin G$. Since G, as a subset of T, is stable under the action of $\widehat{G}$ from the right and under the action of G from the left, we see that H_1 and H_2 are two sided ideals, and we have a decomposition of algebras $H = H_1 \oplus H_2$.

Since H_1 is a quotient of H, it will be enough to prove that $L_t^\alpha G$ is a quotient of the algebra H_1. In order to prove this, we give a neater description of H_1. The group $\widehat{G}$ acts on L via $\pi : \widehat{G} \to G$. We define the algebra B to be $L \otimes_k k\widehat{G}$ as a vector space, with the product

$$(l_1 \otimes \hat{g_1}) \cdot (l_2 \otimes \hat{g_2}) = l_1 \hat{g_1}(l_2) \otimes \hat{g_1}\hat{g_2}.$$

Thus, B is the semidirect product (of algebras) of L and $k\widehat{G}$. We define a linear map $\phi : B \to H_1$ by

$$l \otimes \hat{g} \mapsto \sum_{g \in G} g(l) \otimes e_{g^{-1}} \otimes \hat{g}.$$

The following lemma is quite easy to prove the following lemma.

Lemma 5.7. *The map ϕ is an isomorphism of algebras.*

By the lemma, it is enough to prove that we have an onto map $B \to L_t^\alpha G$. To do this, recall that the group $\widehat{G}$ was constructed as a subgroup of the group of invertible elements in $L_t^\alpha G$, and thus we have a natural algebra map $k\widehat{G} \to L_t^\alpha G$. We define the following linear map

$$\Psi : B \to L_t^\alpha G,$$
$$l \otimes \hat{g} \mapsto l\hat{g}.$$

We claim the following lemma.

Lemma 5.8. *The map Ψ is an onto algebra map.*

Proof. The fact that Ψ is onto is easily seen by the fact that $L_t^\alpha G$ is spanned over L by the U_g's. The fact that Ψ is an algebra map follows from a direct calculation. □

We thus see that $L_t^\alpha G$ is a quotient of a Hopf algebra. This finished the proof of Theorem 1.2. □

REMARK 5.9. The Hopf algebra H is actually a form of a group algebra (i.e., $H \otimes_k L$ is a group algebra). This is due to the following fact: the group T is Abelian, and therefore $k[T] \cong kT^\sharp$, where $T^\sharp$ is the character group of T. The semidirect product $k[T] \rtimes k\widehat{G}$ can therefore be seen to be the group algebra of the semidirect product $\widehat{G} \ltimes T^\sharp$. Since $L \otimes_k (k[T] \rtimes k\widehat{G})$ and $L \otimes_k H$ are isomorphic, we see that H is a form of a group algebra.

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EHUD MEIR, DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF COPENHAGEN, UNIVERSITETSPARKEN 5, DK-2100, DENMARK

E-mail address: `meirehud@gmail.com`