COVERING TRANSFORMATIONS AND UNIVERSAL FIBRATIONS

BY
Daniel H. Gottlieb

0. Introduction

The purpose of this paper is to derive the well known formula for the covering transformation group of a covering space by using universal fibrations in the theory of Hurewicz fibrations.

A Hurewicz fibration $p : E \to B$ is a map which has the homotopy covering property for all spaces. We shall assume that B is connected. It is possible to classify the collection of Hurewicz fibrations over CW-complexes up to fibre homotopy equivalence, by means of the universal fibration.

Given any space F, there exists a universal Hurewicz fibration

$$p_\infty : E_\infty \to B_\infty ,$$

with fibre the homotopy type of F, such that any Hurewicz fibration with fibre the homotopy type of F and base space B, a CW-complex, is fibre homotopically equivalent to a Hurewicz fibration induced by a map from B to B_∞. In addition, the homotopy classes of maps from B to B_∞ are in one-to-one correspondence with the fibre homotopy equivalence classes of Hurewicz fibrations with fibre the homotopy type of F and base space B. Also, B_∞ is a CW-complex.

This was first proved for F a compact CW-complex by Stasheff in [8], and later for F a CW-complex by Allaud [1], and for any space F by Dold [2].

Let $p : E \to B$ be a Hurewicz fibration with fibre F. Then there is a map $k : B \to B_\infty$ such that the fibration induced by k is of the same fibre homotopy type as $p : E \to B$. We call k a *classifying map* of $p : E \to B$.

With every fibration $p : E \to B$, there is a group $\mathcal{L}(E)$ which depends on the fibre homotopy equivalences $f : E \to E$. Let $\{f\}$ be the equivalence class of all fibre homotopy equivalence $g : E \to E$ which are homotopic to f by a homotopy h_t such that h_t is a fibre homotopy equivalence for each t. Then define multiplication by $\{f\} \cdot \{g\} = \{f \circ g\}$. This multiplication defines a group $\mathcal{L}(E)$ called the group of *fibre homotopy equivalences*. This group was classified by the author, [5], in terms of B_∞ and the classifying map $k : B \to B_\infty$ corresponding to the fibre space E. We record the result below.

Theorem. *Let $L(B, B_\infty)$ be the space of continuous maps from B to B_∞ with the compact-open topology. Then $\mathcal{L}(E) \cong \pi_1(L(B, B_\infty); k)$.*

A covering map $p : \tilde{X} \to X$, where $\tilde{X}$ is a covering space, is the earliest example of a Hurewicz fibration. The group of covering transformations is precisely the group of self homotopy equivalences, $\mathcal{L}(\tilde{X})$. Let H be the sub-

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group of $\pi_1(X)$ given by $H = p_* \pi_1(\tilde{X})$. Then, for connected, locally path-wise connected spaces, $\mathcal{L}(\tilde{X}) \cong N\pi_{1(X)}(H)/H$ where $N_{\pi_1(X)}(H)$ is the normalizer of H in $\pi_1(X)$. From the above theorem, we see that

$$\pi_1(L(X, B_\infty); k) \cong N_{\pi_1(X)}(H)/H$$

when X is a CW-complex. Since these two sides look so different, it is natural to ask how the right side follows from the left side? The purpose of this paper is to answer that question.

The proof given here is longer than the usual proofs, see [7] for example, and differs from them in that our proof does not construct explicitly the covering transformations. On the other hand, our proof depends upon four lemmas of independent interest. Lemma 3 gives B_∞ for discrete fibres, F, and Lemma 4 computes the classifying maps for covering spaces. Lemma 2 is a homotopy theoretic result, special cases of which played an important role in the author's work, [3], [4], and hence in [5]. Lemma 1 is an algebraic result, I am told well known, however I couldn't find it in the literature in the form I needed.

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1. Proof of theorem

Let G be a group and let H be a subgroup of G. Then G operates on the set of right cosets of H, $\{H, Hx_1, \cdots\}$ by right multiplication. Let m be the index of H in G (m possibly infinite). Let $S(m)$ be the full permutation group on m letters. In fact, we regard $S(m)$ as the permutations on the set of right cosets of H in G. If $\sigma \in S(m)$, we denote by $(Hx)^\sigma$ the image of Hx under the action of σ. Let $x \in G$, then we define $R_x \in S(m)$ to be the permutation $Hx_1 \rightarrow (Hx_1)^{R_x} = Hx_1 x$. We call R_x right multiplication by x. The subgroup of $S(m)$ consisting of all the right multiplications by elements in G will be denoted by $\bar{G}$. Finally, $N_G(H)$ will denote the normalizer of H in G, i.e., the group of all $x \in G$ such that $xHx^{-1} = H$; and $C_G(H)$ will denote the centralizer of H in G, i.e., the group of all $x \in G$ such that $xh = hx$ for all $h \in H$.

LEMMA 1. $C_{S(m)}(\bar{G}) \cong N_G(H)/H$.

Proof. Let $c \in C_{S(m)}(\bar{G})$. Then

$$(Hx)^c = (H^{R_x})^c = (H^c)^{R_x} \quad \text{for all} \quad x \in G.$$

Define $c_1 \in G$ to be an element such that $Hc_1 = H^c$. Then $(Hx)^c = (Hc_1)^{R_x} = Hc_1 x$. Thus every $c \in C_{S(m)}(\bar{G})$ is determined by $H^c = Hc_1$. Now let $h \in H$. Then $H^c = (Hh)^c = (H^{R_h})^c = (H^c)^{R_h} = (Hc_1)^{R_h} = Hc_1 h$. So $Hc_1 = Hc_1 h$. Hence $c_1 hc_1^{-1} \in H$. This holds for all $h \in H$. Therefore $c_1 Hc_1^{-1} \subseteq H$. Similarly, $c_1^{-1}Hc_1 \subseteq H$, so $c_1 \in N_G(H)$.

Conversely, if $c_1 \in N_G(H)$, then the permutation σ which sends Hx_i to

$Hc_1 x_i$ is an element of $C_{S(m)}(\bar{G})$ because

$$(Hx)^{R_y\sigma} = (Hxy)^\sigma = Hc_1 xy = (Hc_1 x)^{R_y} = (Hx)^{\sigma R_y}.$$

If $c = c' \in C_{S(m)}(\bar{G})$, then $Hc_1 = Hc_1'$. Hence any element of $N_G(H)$ determines an element in $C_{S(m)}(\bar{G})$, and any two elements of $N_G(H)$ determine the same permutation if and only if they are in the same coset of H. Hence $C_{S(m)}(\bar{G}) \cong N_G(H)/H$ by the mapping $c \to Hc_1^{-1}$.

LEMMA 2. *Let X be a CW-complex and let Y have the homotopy type of a $K(\pi, 1)$. Then $\pi_1(L(X, Y); k) \cong$ centralizer of $k_*(\pi_1(X))$ in π.*

Proof. Any element of $\pi_1(L(X, Y); k)$ is associated with a map

$$f : X \times S^1 \to Y$$

such that f restricted to $X \times *$ is just $k : X \to Y$.

We shall show first that for every such map f, f restricted to $* \times S^1$ represents an element in the centralizer of $k_*(\pi_1(X))$ and conversely any map

$$k \vee \sigma : X \vee S^1 \to Y,$$

where $\sigma : S^1 \to Y$ represents an element in the centralizer of $k_*\pi_1(X)$, can be extended to an $f : X \times S^1 \to Y$.

According to [6, p. 198], $k \vee \sigma : X \vee S^1 \to Y$ can be extended to an $f : X \times S^1 \to Y$ if and only if there exists a homomorphism

$$h : \pi_1(X \times S^1) \to \pi_1(Y)$$

which makes the following diagram commutative.

$$\begin{tikzcd} \pi_1(X \times S^1) \arrow[r, dotted, "h"] & \pi_1(Y) \cong \pi \\ \pi_1(X \vee S^1) \arrow[u, "i_*"] \arrow[ur, "(k \vee \sigma)_*"'] & \end{tikzcd}$$

Here i_* is induced by the usual inclusion $i : X \vee S^1 \to X \times S^1$. Now $\pi_1(S^1) \cong Z$, the additive group of the integers and

$$\pi_1(X \times S^1) \cong \pi_1(X) \oplus \pi_1(S^1) \quad \text{and} \quad \pi_1(X \vee S^1) \cong \pi_1(X) * \pi_1(S^1)$$

where $*$ is the free product. Thus the diagram above becomes

$$\begin{tikzcd} \pi_1(X) \oplus Z \arrow[r, dotted, "h"] & \pi \\ \pi_1(X) * Z \arrow[u, "i_*"] \arrow[ur, "(k \vee \sigma)_*"'] & \end{tikzcd}$$

It is easily seen that the required homomorphism exists if and only if $\sigma_* : Z \to \pi$ carries Z into $C_\pi(k_* \, \pi_1(X))$.

Now we must show that f and $g : X \times S^1 \to Y$ represent the same element of $\pi_1(L(X, Y); k)$ if and only if $f \mid S^1$ and $g \mid S^1$ represent the same element of $\pi_1(Y) \cong \pi$. Thus we must ask, under what conditions can we extend a map

$$G : K = (X \times S^1 \times 0) \cup (X \times S^1 \times 1) \cup (X \times * \times I) \to Y$$

to

$$H : X \times S^1 \times I \to Y$$

where G is defined by

$$\begin{aligned} G(x, s, 0) &= f(x, s) \quad \text{for} \quad x \in X, s \in S^1 \quad \text{and} \quad t \in I \\ G(x, s, 1) &= g(x, s) \\ G(X, *, t) &= k(x). \end{aligned}$$

Observe $\pi_1(K) \cong \pi_1(X \times (S^1 \vee S^1))$. Now the appropriate diagram is

$$\begin{tikzcd} \pi_1(X) \oplus Z \arrow[r, dotted, "h"] & \pi \\ \pi_1(X) \oplus (Z * Z) \arrow[u, "i_*"] \arrow[ur, "G_*"'] & \end{tikzcd}$$

It is easy to see that i_* carries the generators of either Z in $\pi_1(X) \oplus (Z * Z)$ onto the generators of Z in $\pi_1(X) \oplus Z$. Thus, if h exists, G_* carries a generator of each factor of Z in $\pi_1(X) \oplus (Z * Z)$ onto the same element of π. Conversely, if G_* carries the above generators onto the same element of π, then it is clear that the required h exists. This proves Lemma 2.

The next two lemmas are more concerned with covering spaces than the last two.

It will be convenient to assume that the fibre F is a locally compact CW-complex from now on. Let L_*^* consist of the space of all maps carrying F into E which are homotopy equivalences for *some* $p^{-1}(x)$. We give L_*^* the compact open topology. We choose a point $* \in F$ and define the evaluation map $\omega : L_*^* \to E_\infty$ by $\omega(f) = f(*)$. Also, we define a map $\Phi : L_*^* \to B_\infty$ by letting $\Phi(f) = x$, where $f : F \to p^{-1}(x)$. It turns out, see [1] or [5], that Φ is a Serre fibration, i.e., Φ has the covering homotopy property for finite polyhedra. The fibre $\Phi^{-1}(x_0) = F^F$, where F^F will denote the space of homotopy equivalences from $F \to F$ and $x_0 \in B_\infty$ is a base point. We will let $\omega : F^F \to F$ be the evaluation map. Then we have the commutative diagram:

$$\begin{tikzcd} F \arrow[d] & F^F \arrow[l, "\omega"'] \arrow[d] \\ E_\infty \arrow[d, "p"] & L_*^* \arrow[l, "\omega"'] \arrow[d, "\Phi"] \\ B_\infty & B_\infty \arrow[l, "1"'] \end{tikzcd}$$

A second fact is true of L_*^*, namely $\pi_i(L_*^*) = 0$ for all i. Then from the homotopy exact sequence for $\Phi : L_*^* \to B_\infty$, we see that the boundary homomorphism $d'_\infty : \pi_i(B_\infty) \to \pi_{i-1}(F^F)$ is an isomorphism for all i.

LEMMA 3. *If F is a discrete set of points containing m (possibly infinite) points, then the classifying space B_∞ for fibre spaces with fibre F has the homotopy type of a $K(\pi, 1)$ where $\pi = S(m)$, the symmetric group on m letters.*

Proof. We know that $\pi_i(B_\infty) \cong \pi_{i-1}(F^F)$, where F^F is the space of homotopy equivalences of F into F. But a homotopy equivalence of F into F is just a permutation of the elements of F. Thus F^F consists of the space of $m!$ discrete points. Hence $\pi_i(F^F) = 0$ if $i > 0$ and $\pi_0(F^F) \cong S(m)$. Hence $\pi_i(B_\infty) = 0$ if $i > 1$ and $\pi_1(B_\infty) \cong S(m)$. (The isomorphism is true when $i = 1$ because of the theorem mentioned in the introduction.)

Let $p : \tilde{X} \to X$ be a covering space. Then by Lemma 3, B_∞ has the homotopy type of $K(S(m), 1)$. Now there exists a classifying map

$$k : X \to B_\infty$$

corresponding to $p : \tilde{X} \to X$. Since B_∞ is a $K(\pi, 1)$, the homotopy class of k depends only on the homomorphism $k_* : \pi_1(X) \to S(m)$.

LEMMA 4. *For the covering space $p : \tilde{X} \to X$, the classifying homomorphism $k_* : \pi_1(X) \to S(m)$ is given by $\alpha \to$ right multiplication by α on the right cosets of $p_* \pi_1(\tilde{X})$. Here, as in Lemma 1, we regard $S(m)$ as the group of permutations on the set of right cosets of $H = p_* \pi_1(\tilde{X}) \subseteq \pi_1(X)$.*

Proof. The commutative diagram

$$\begin{tikzcd}
F \arrow[r, leftrightarrow, "1"] \arrow[d] & F \arrow[d] & F^F \arrow[l, "\omega"'] \arrow[d] \\
\tilde{X} \arrow[r] \arrow[d, "p"] & E_\infty \arrow[d, "p_\infty"] & L_*^* \arrow[l, "\omega"'] \arrow[d, "\Phi"] \\
X \arrow[r, "k"'] & B_\infty \arrow[r, leftrightarrow, "1"'] & B_\infty
\end{tikzcd}$$

gives rise to the commutative diagram

$$\begin{tikzcd}
\pi_1(X) \arrow[r, "k_*"'] \arrow[d, "d"] & \pi_1(B_\infty) \arrow[r, leftrightarrow, "1"'] \arrow[d, "d_\infty"] & \pi_1(B_\infty) \arrow[d, "d'_\infty"', "\wr\|"] \\
\pi_0(F) \arrow[r, leftrightarrow, "1"'] & \pi_0(F) & \pi_0(F^F) \arrow[l, "\omega_*"] \cong S(m)
\end{tikzcd}$$

Thus we have $\omega_* \circ h = d$ where $h = d'_\infty k_*$. Since $\tilde{X}$ is connected,

$$d : \pi_1(X) \to \pi_0(F)$$

is onto and in fact is given by

$$d : \alpha \to [p_* \pi_1(\tilde{X}, \tilde{x}_0)] \cdot \alpha = H\alpha$$

where the points of F are considered as right cosets of H. That is $\pi_0(F)$ is regarded as the set of right cosets of H. In addition,

$$\omega_* : S(m) \cong \pi_0(F^F) \to \pi_0(F)$$

by $\omega_*(\sigma) = H^\sigma$ where $\sigma \in S(m)$. Since $\omega_* \circ h = d$ and we know that h is a homomorphism from $\pi_1(X)$ into $S(m)$, we may determine h.

Let $\alpha \in \pi_1(X)$. Then $\omega_* h(\alpha) = H^{h(\alpha)}$. But $d(\alpha) = H\alpha$. Therefore $H^{h(\alpha)} = H\alpha$. For any arbitrary right coset $H\beta$, $(H\beta)^{h(\alpha)} = (H^{h(\beta)})^{h(\alpha)} = H^{h(\beta)h(\alpha)} = H^{h(\beta\alpha)} = H\beta\alpha$. So $h(\alpha)$ is just right multiplication by α. Hence k_* is given by $\alpha \to$ right multiplication by α on the set of right cosets of H, proving Lemma 4.

Now, by Theorem 1, Lemma 3, and Lemma 4

$$\mathcal{L}(\tilde{X}) \cong \pi_1(L(X, K(S(m), 1)); k)$$

where k_* is as in Lemma 4. Thus by Lemma 2,

$$\pi_1(L(X, K(S(m), 1)); k) \cong C_{S(m)}(k_* \pi_1(X)).$$

By Lemma 1, this is just $\{N_{\pi_1(X)}(p_* \pi_1(\tilde{X}))\}/p_* \pi_1(\tilde{X})$. Thus

$$\mathcal{L}(\tilde{X}) \cong \{N_{\pi_1(X)}(p_* \pi_1(\tilde{X}))\}/p_* \pi_1(\tilde{X}),$$

which we set out to prove.

It is interesting to observe, that given $\mathcal{L}(\tilde{X})$ for a covering space $\tilde{X}$, we may prove Lemma 1, a group theoretic result, by applying Theorem 1 and the other corollaries.

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Institute for Defense Analysis
Princeton, New Jersey