Illinois Journal of Mathematics
Volume 47, Number 1/2, Spring/Summer 2003, Pages 207–222
S 0019-2082

REINHOLD BAER AND HIS INFLUENCE ON THE THEORY OF ABELIAN GROUPS

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Abstract. Reinhold Baer's contributions to and influence on the theory of abelian groups are surveyed, concentrating on his publications on the theory of p-groups, torsion-free and mixed groups, subgroup lattices, endomorphism rings, the group Ext, as well as injective modules. His papers on these subjects were instrumental in the development of this branch of group theory.

About 20 years ago when I wrote a survey on Reinhold Baer's work on abelian groups (see [F2]), I did not explore his tremendous influence on this fast developing area. Today we have a better perspective to assess the impact of his contributions, although it seems nearly impossible to give a full account of his influence on abelian group theory and adjacent areas. Indeed, his ideas on abelian groups spread over several branches of mathematics, the methods he developed permeated into various fields, and the open problems he posed gave impetus to research to an enormous extent. In my survey, I intend to concentrate on abelian group theory proper, and I shall only briefly touch upon module-theoretic aspects.

Reinhold Baer's influence on the theory of abelian groups has three major components:

1. his publications;
2. his doctoral students; and
3. his talks at various universities as well as individual discussions.

I plan to restrict myself to his publications, but it seems appropriate to list his students who worked on discrete or topological abelian groups, with the dates of graduation:

Ross Beaumont (1940)
Wolfgang Liebert (1965)
Peter Grosse (1966)
Peter Plaumann (1966)

Received September 10, 2002.
2000 *Mathematics Subject Classification.* Primary 01A70. Secondary 20K10, 20K20, 20K21, 20K30, 20K40.

Kai Faltings (1967)
Rüdiger Göbel (1967)
Petrus Gräbe (1967)
Jutta Hausen (1967)

Let us see first what the state of abelian group theory was when Reinhold Baer entered the scene in 1934. There were three major areas of development.

The theory of infinite abelian groups started with Friedrich Levi. In his Habilitationsschrift (1917) [L] he investigated countable abelian groups, in particular, torsion-free groups of finite rank, and mixed groups. He actually constructed an indecomposable torsion-free group of finite rank and a non-splitting mixed group. Since his results were not published in a journal, his contributions were nearly forgotten at that time. Levi was a good friend of Reinhold Baer and a multiple coauthor, so it is fairly certain that Baer was aware of these results.

The other pioneer in abelian groups was Heinz Prüfer. His seminal papers, [P1] and [P2], were published in the early 1920's. He laid the foundations of the theory of p-groups, also restricting himself to the countable case. His results, especially those on direct sums of cyclic groups, served as the basis of the structure theory of countable abelian p-groups developed by Ulm [U] in 1933. (Baer started to work on abelian groups before the publication of L. Zippin's paper [Z], in which the existence question was solved.)

In these years, another most powerful theory was developed: the well-known duality theory of Pontryagin for locally compact abelian groups [Po]. In this theory, the discrete and compact abelian groups play a dual role, so problems concerning compact groups gave rise to several exciting questions on discrete groups. One of these was the existence problem of (directly) indecomposable abelian groups of arbitrary finite rank. (This problem led the prominent Russian algebraists Kurosh [Ku] and Malcev [M] to the classification theory of these groups, based on Derry's work [D] on modules over the ring of p-adic integers.)

This was the state of affairs in the theory of abelian groups when Reinhold Baer became interested in the subject. I speculate that his interest was kindled when, working on the manuscript [1] on non-commutative groups, he came across a theorem of Prüfer, and he noticed that he could get rid of the countability condition. He continued his research on problems on p-groups, and a few years later he extended his study to other areas of abelian groups. He found the area attractive—as he once noted: it was comparatively rich in structure theorems.

This was a turbulent period of his life, he being forced to move with his family from Germany first to England, and then to America. In spite of the inherent difficulties, in a period of five-to-six years he changed the entire theory: he introduced new ideas and created new methods, stimulating increasing interest in the area.

While Ulm used ingenious matrix-theoretical methods, and the Kurosh-Malcev theory is totally based on matrices, Baer turned away from matrix theory. His approach was more conceptual, more genuinely group-theoretical. This was a drastic change, a very fortunate one, which became permanent in the theory.

Undoubtedly, the years 1934–1940 were the pinnacle of Reinhold Baer's abelian period. He lectured for a term in Princeton on infinite abelian groups [7]. Curiously enough, even in these years abelian groups were not his exclusive interest: he split his time between commutative and non-commutative groups. The non-commutative interest became overwhelming and soon non-commutative problems took over. Many years later, he returned to abelian groups a couple of times, but not with the same impact on the theory. Yet his interest in abelian groups never ceased. As a matter of fact, in the late 1940's, he contemplated writing a monograph on abelian groups; he even completed a portion of the manuscript and had a contract with a publisher, but he changed his mind, and instead he wrote a book on linear algebra and projective geometry (1952). Fortunately, some of his ideas on abelian groups were incorporated in Kaplansky's little red book "Infinite Abelian Groups", published in 1954.

I intend to survey his most relevant contributions to abelian group theory in seven main areas. I will not mention topics like partition problems, which currently do not lie in the main stream of abelian group theory.

For unexplained terminology on abelian groups, see, e.g., [F]. The author wishes to express his thanks to Rüdiger Göbel for his comments.

1. The theory of p-groups

Baer's first result on abelian groups was included in a 1934 paper [1] dealing with a generalization of the center: he investigated the set of elements that normalize every subgroup of the group. He removes the countability hypothesis from an important theorem of Prüfer by showing that a bounded abelian group of arbitrary cardinality is the direct sum of cyclic groups. His proof became a standard argument: first, he decomposes the socle into summands whose elements have fixed heights, and then he lifts the vector space bases to generators of the group.

In 1935 he published two papers on abelian groups in the Oxford Journal, [3] and [4]. The first paper is the first non-trivial application of the Ulm-Zippin theory of countable p-groups. To start with, Baer identifies the countable indecomposable p-groups as cyclic groups $\mathbb{Z}(p^k)$ of prime power order and Prüfer's quasicyclic groups $\mathbb{Z}(p^\infty)$. The main result points out that a countable p-group has the property that its direct decompositions admit isomorphic refinements if and only if it is a direct sum of cyclic and quasicyclic groups.

Paper [4] investigates decompositions of abelian groups and modules (in his terminology: abelian groups with operators) into fully invariant summands, using the efficient technique of idempotents.

Paper [5], published in the same year, deals with the question when two elements in a p-group A can be carried into each other by an automorphism of A. He associates with an element $a \in A$ a finite sequence

$$I(a) = (s(a), s(pa), \dots, s(p^{n(a)-1}a))$$

where $p^{n(a)}$ denotes the order of a, $h(a)$ stands for the height of a and $s(a) = n(a) + h(a)$. Baer shows that the elements $a, b \in A$ can be carried into each other by an automorphism of A if and only if

$$I(a) = I(b).$$

This $I(a)$ is the original version of the *height-sequence* (or *indicator*), used later on in a modified form by I. Kaplansky [Ka1]. Although Baer assumed that the group A is a direct sum of cyclics, his arguments extend at once to p-groups without elements of infinite height. In terms of $I(a)$ he describes the structure of a minimal direct summand of A that contains the given element a, and classifies the characteristic subgroups in a form somewhat different from the generally accepted version due to Kaplansky.

The difference between the Baer and Kaplansky versions of indicator is minor, merely technical in character. If a belongs to a cyclic summand of order p^n, then $I(a)$ is always n in Baer's version, while it increases by 1 in Kaplansky's version until it reaches n. If a is not contained in a cyclic summand, then $I(a)$ serves to detect the orders of the cyclic summands in a minimal summand containing a. When Luigi Salce and I needed a generalized form of the indicator in torsion modules over arbitrary valuation domains, we found the original version by Baer more adequate in the general case (it is easier to measure the deviation from constant behavior than from a regular increase): see [FS1].

In a 1937 paper [8] he investigates the automorphism groups of p-groups. He succeeds in describing the center of the automorphism groups. For $p > 2$, the center consists of multiplications by p-adic units whenever the group is unbounded; otherwise, one has to pass modulo the bound. The case $p = 2$ is exceptional. He concludes that the automorphism group of a p-group is abelian exactly if the group is cyclic, quasicyclic, or of the form $\mathbb{Z}(2) \oplus \mathbb{Z}(2^\infty)$. This generalizes results by Châtelet [Ch] from the finite case.

In his paper [6] on mixed groups, he proves something interesting about p-groups: if a p-group is bounded modulo its subgroup of elements of infinite heights, then this subgroup is a summand of the group (and hence divisible).

The theory of p-groups was revitalized by L. Ya. Kulikov [K2] in 1945. He came up with drastically new ideas to deal with the uncountable case. One of Kulikov's most important discoveries was that every p-group contains a *basic subgroup*, and such a subgroup is unique up to isomorphism. Actually, in his paper [8] (see pages 100–101) Reinhold Baer constructed a basic subgroup of an arbitrary p-group. However he failed to recognize its general significance. Once when we were conversing on the role of basic subgroups, he called my attention to this construction, making a gesture indicating that he missed something very important, but he could not do anything about it. (As a matter of fact, though Kulikov was undoubtedly aware of the importance of basic subgroups which he required in developing the theory of torsion-complete p-groups, it was actually T. Szele who recognized their real significance for the theory of p-groups.)

2. Mixed groups

In 1936, Baer published a paper [6] initiating the theory of mixed abelian groups. At that stage of development, deep structural questions on mixed groups would have been unthinkable. Of central interest was the splitting problem: to find criteria under which the group splits into the direct sum of a torsion and a torsion-free group. In today's terminology, one was looking for criteria for the pair F, T under which

$$\mathrm{Ext}(F, T) = 0$$

holds for a torsion T and a torsion-free F. No satisfactory necessary and sufficient condition is known even today, but Baer succeeded in finding necessary conditions as well as sufficient conditions for the splitting. His main result provides a full description of those torsion groups that are summands in every mixed abelian group in which they are the torsion subgroups: these are precisely the direct sums of a torsion divisible group and a bounded group. (Sufficiency was proved simultaneously and independently by the Russian S.V. Fomin [Fo].) He also gives an explicit example of a non-splitting abelian group (the first such example was buried in Levi's unpublished Habilitationsschrift).

The dual problem (concerning torsion-free groups) turned out to be more difficult and much more exciting; see Baer groups in Section 4.

Baer's approach to the splitting problem required a thorough study of the divisibility properties of the elements and their multiples by integers. He was convinced that one ought to have a device recording the divisibility properties of elements in the given group, this yielding essential information about the way in which an element is located in the group. At that time he had not as yet developed the characteristic of elements in torsion-free groups, so he could not introduce the common generalization of the indicator in p-groups and the characteristic in torsion-free groups. But it was certainly in his spirit when Rotman [R] defined the *height matrix* of elements in mixed groups. This is an $\omega \times \omega$-type matrix that gives information about the way an element is embedded in a mixed group. These matrices play a distinguished role in the current theory of mixed groups, in particular, in the theory developed by the New Mexico school of abelian groups. (They came up with the revolutionary idea of viewing a mixed group as an extension of a free abelian group by a torsion group.) The height-matrices of the free generators provide crucial information about the groups themselves.

3. Injective modules

In his paper on mixed groups [6], Baer starts by proving that divisible subgroups are always summands. He claims that this has been a known result, presumably a kind of folklore, but not available in the literature. He goes on by showing that every torsion-free group can be embedded as a subgroup in a minimal divisible group.

Obviously, this was the prelude to one of his most influential results [13], published in 1940. He works with abelian groups with operators in a ring (i.e., modules). He considers modules that are summands in every containing module (these are of course exactly the *injective* modules), and shows that this property can be recognized by using only the (one-sided) ideals of the ring. This widely used criterion is generally known as the *Baer criterion for injectivity.*

He then makes use of this criterion to establish the embeddability of a module in an injective module via a clever transfinite induction. In addition, he proves the existence of what we now call the *injective hull.* (The final touch for injectivity was given later by Eckmann and Schopf [ES].) He also points out that two essential submodules of an injective module are isomorphic if and only if they can be carried into each other by an automorphism of the injective module.

It is impossible to overstate the importance of this paper in view of the tremendous role injective modules play in contemporary mathematics.

4. The theory of torsion-free groups

I would like to explain why his Duke Math. J. paper [9] on torsion-free abelian groups is his most relevant contribution to the general theory of abelian groups.

In this paper, he goes back to the rudiments and builds systematically a theory which brings the investigations at once to an advanced level. It is a gold mine of fruitful ideas and is certainly the most important paper in the theory of torsion-free groups. It is seldom the case that a single paper which starts a new area goes this far in developing the theory.

His starting point is a device recording the divisibility properties of elements. He uses Steinitz numbers; we now use the equivalent notion of *height-sequences* or *characteristic*. This is a sequence attached to an element a of a torsion-free group A,

$$\chi(a) = (h_2(a), h_3(a), \dots, h_p(a), \dots),$$

where $h_p(a)$ is the *height* of a at the prime p, a non-negative integer or the symbol ∞, indicating the divisibility of a in A by powers of p. He recognizes that the equivalence classes of height-sequences—called *types*—provide complete sets of invariants for torsion-free groups of rank 1.

The type is perhaps the most relevant concept in the theory of torsion-free groups even today. Baer used types to define certain fully invariant subgroups in torsion-free groups—these appear explicitly or implicitly in almost all publications on the subject. These subgroups enabled him to classify the *completely decomposable* (in his terminology: completely reducible) groups: the direct sums of rank 1 groups. These fully invariant subgroups play an equally important role in the classification of the so-called Butler groups, whose theory is now in the center of interest of many abelian group theorists.

Also in this paper he proves *inter alia* that finite rank summands of completely decomposable groups are again completely decomposable. He handles the infinite rank case only when the types satisfy the maximum condition—he uses a sophisticated induction argument. The countable case was settled 15 years later by Kulikov [K3], and the final touch was given by Kaplansky [Ka2] in 1958. This is the famous Baer-Kulikov-Kaplansky theorem: summands of completely decomposable groups are again completely decomposable.

Baer's lemma on the splitting property of rank 1 factor groups has been a continuous source of inspiration for generalizations.

Another most influential notion introduced in this paper is the *separability* of a torsion-free group in the sense that every finite set of elements is included in a completely decomposable direct summand. He uses his favorite torsion-free group as a model: the countable product of the group of integers, often called the *Baer-Specker group*. He shows that countable separable groups are completely decomposable, and presents a clever argument to refute the complete decomposability of the Baer-Specker group.

The homogeneous separable groups deserve special attention (*homogeneous* means that every non-zero element has the same type). They are characterized by the useful property that their finite rank pure subgroups are summands.

There is a huge literature dealing with various aspects of separability of abelian groups and modules. I find it especially interesting to inquire what special features these separable torsion-free groups may have; see, e.g., the volume of Eklof-Mekler [EM].

Baer introduced the class of *vector groups*: these are cartesian products of rank 1 groups. He describes cases when they are separable, or even completely decomposable. Vector groups are permanent topics in the recent literature.

As we have pointed out earlier, the first example of an indecomposable torsion-free group of rank > 1 was given by F. Levi, but it was forgotten. Pontryagin succeeded in constructing a different example of rank 2 in order to verify the existence of indecomposable, connected compact abelian groups of dimension > 1. Kurosh and Malcev developed a theory based on a complicated equivalence relation between infinite sets of matrices, see [Ku] and [M]. One of the main purposes was to construct indecomposable abelian groups of any finite rank.

Baer solves the indecomposability problem with one stroke, using a brilliant idea: pure subgroups of the additive group of the p-adic integers, for any prime p, are all indecomposable. He thus established the existence of indecomposable groups up to the continuum.

It seems that at that time not even Baer himself realized what an important result he had proved. When he mentions this property of the p-adic integers in his paper [9] (p. 107), it is just a simple remark. Indeed, he does not phrase it as a theorem and does not emphasize that this yields indecomposable groups of infinite rank.

I think that the result concerning the indecomposable subgroups of the p-adic integers is a prime example to illustrate Reinhold Baer's extraordinary intuition with groups: he always pinpoints the relevant property that makes everything work. The example of the p-adic integers not only provides a solution for indecomposable abelian groups, but the idea behind it has a much wider range: it works for torsion-free modules over arbitrary commutative domains (for valuation domains this was pointed out by Vámos [V]). Indeed, if $J \neq 0$ is an irreducible ideal of the domain R, then

$$\mathrm{Hom}_R(Q/R, E(R/J))$$

is an indecomposable torsion-free R-module, and so is every submodule with torsion-free cokernel. Here Q denotes the field of quotients of R as an R-module, while $E(*)$ designates the injective hull. The proof is an easy consequence of the Matlis category equivalence.

The problem of Baer groups had been for 30 years one of the most outstanding problems in the theory of abelian groups. It asked for the characterization of groups B (called *Baer groups*) for which

$$\mathrm{Ext}(B, T) = 0 \quad \text{for all torsion groups } T.$$

Baer himself could prove only that countable Baer groups had to be free abelian groups, and he left open the uncountable case with the statement that they have to be $\aleph_1$-*free* (i.e., all countable subgroups are free). Later on, he showed that the Baer-Specker group is not a Baer group. Only in 1967 did P. Griffith [G] succeed in proving that all Baer groups are free abelian.

The question of Baer groups stimulated a large amount of research in module theory as well. It was Kaplansky [Ka3] who raised the question of Baer modules over integral domains. He noticed immediately that Baer modules are always flat and of projective dimension at most 1. Generalizing Griffith's method, R.P. Grimaldi [Gr] showed that Baer modules over Dedekind domains are projective. A totally different approach was required by the valuation and Prüfer domain cases: in joint papers 1988–2000 with P. Eklof, S. Shelah and L. Salce (see [EF], [EFS], [FS2]) we showed that Baer modules over valuation domains are free, and Baer modules over Prüfer domains are always projective. The problem for general domains is still open.

The problem of Baer groups had another remarkable ramification: the question whether or not there exists a single test module, i.e., a torsion module T^* such that $\mathrm{Ext}(B, T^*) = 0$ implies that B is a Baer module. It is not difficult to show that a module B has to be Baer whenever $\mathrm{Ext}(B, T) = 0$ for a suitable direct sum T of cyclic modules, but such a T depends on the size of B. Eklof [E] has shown that such a universal test module T exists for the Baer property in Gödel's constructible universe, but not in all models of ZFC. Actually, Eklof-Shelah [ESh] proved that the existence of a single test module over any domain for the Baer property is undecidable in ZFC+GCH. Cf. also Strüngmann [St].

5. Lattices of subgroups

Reinhold Baer's interest in the lattice $L(G)$ of subgroups of a group G goes back at least to the 1930's. At that time lattice theory was still in its infancy, but Baer had sharp eyes and immediately recognized that this lattice contains most valuable information about the group itself.

In a 1937 paper he became deeply involved in the study of $L(G)$, using it as a source of information about the group structure. Two years later in [11] he shows that the isomorphism of the lattices of subgroups of two abelian groups implies the isomorphism of the groups themselves: one needs only two independent elements of infinite order. Then every isomorphism between the subgroup lattices is induced by exactly two group isomorphisms.

Paper [10] is devoted to the *dualism* between two groups G and G': by this he meant an anti-isomorphism between the lattices $L(G)$ and $L(G')$ of their subgroups. For abelian groups this is equivalent to the existence of a bijective function

$$d : L(G) \to L(G')$$

such that $S \cong G'/dS$ for every $S \in L(G)$. He completely solves the problem of dualism in the abelian case by showing that such a dualism exists if and only if G is a torsion group with finite p-components and $G \cong G'$. He also points out that there might exist non-commutative groups dual to abelian groups.

The lattice of subgroups led Baer to his investigations on the foundation of projective geometry. He had a strong geometric background, and it is no wonder that he became interested in the algebraic aspects of projective geometry.

The point of departure was his conviction that projective geometry is basically the study of the lattice of all subspaces of a vector space, and the lattice of subgroups of an abelian group is a kind of generalized projective geometry. He tried to develop a unified treatment of abelian groups and projective spaces in his 1942 Transactions article. His efforts led him to a most influential book on the subject: Linear Algebra and Projective Geometry (1952).

6. Endomorphism rings

The interplay between groups and rings is the theme of his remarkable study of endomorphism rings. In one of his papers on p-groups, Baer emphasizes that endomorphism rings reflect particularly faithfully properties of the original group.

In a 1943 paper [15] he concentrates on bounded p-groups, and shows that

(i) these groups are completely determined by their endomorphism rings; moreover, each isomorphism between endomorphism rings is induced by some group-isomorphism; and

(ii) for bounded p-groups, the endomorphism rings can be characterized ring-theoretically.

Occasionally he assumes that the groups contain two or three independent elements of maximal orders, but the results are valid without this hypothesis.

An interesting consequence of (i) is that in this case all the automorphisms of the endomorphism rings are inner.

Theorem (i) is the basis of Kaplansky's generalization, announced in 1952, but proved in [Ka1], to arbitrary p-groups; this remarkable theorem is quoted as the Baer-Kaplansky Theorem. The proof—which was basically designed by Baer—consists of sophisticated manipulations of primitive idempotent endomorphisms (that are abundant in such groups). Baer proved this theorem before Kulikov's crucial 1945 article on p-groups was published, so he was not aware of the large supply of summands an arbitrary p-group has, while Kaplansky could take advantage of this fact.

This theorem has served as a model of several analogous results on modules which possess a sufficient supply of summands.

The isomorphism problem for automorphism groups of p-groups is considerably more difficult. One has to work with involutions rather than idempotent endomorphisms, which give less information about the groups. The cases $p = 2$ and $p = 3$ turned out to be exceptional. Leptin [Le], Liebert [Li], and Schultz [Sc] dealt with this problem successfully.

Reinhold Baer's pioneering contributions to the theory of endomorphism rings stimulated a lot of research, primarily in two directions:

1. When does the endomorphism ring determine the group?
2. Which rings can be endomorphism rings?

The latter problem led to spectacular results, both in the torsion-free and torsion cases. For instance, Corner's celebrated theorem [C] states that every countable ring (with 1) with torsion-free additive group not containing $\mathbb{Q}$ is an endomorphism ring of a countable torsion-free abelian group. A widespread theory has been developed by A.L.S. Corner, M. Dugas, R. Göbel, S. Shelah and others (see [CG], [DG1], [DG2], [Go]), using extremely sophisticated algebraic and set-theoretic methods. This trend had a tremendous impact on other areas of algebra as well.

7. The group of extensions

The theory of group extensions was initiated by Otto Schreier [S] in 1926. He dealt with factor sets and concentrated on the equivalence of extensions. In his 1934 paper [2] Baer followed a drastically different approach. He considered extensions

$$1 \to N \to G \to C \to 1$$

of a group N by a group C, observing that each such extension determines a homomorphism (called *the coupling*)

$$\chi : C \to \text{Out } N$$

into the group of outer automorphisms of N which comes from conjugation by elements of G in N. This is used to construct extensions. A particular case is when the normal subgroup N is abelian: in this situation Baer describes the structure of Ext as induced by Hom. The so-called *Baer sum* in Ext is a most essential contribution of Reinhold Baer to Homological Algebra.

Group extensions were reoccurring topics in his research, but predominantly in non-commutative situations. In his 1949 paper [16] he breaks with the tradition and proposes a different equivalence relation between two extensions, G and H, of a group S (without fixing the factor group). He calls G and H of the *same extension type* if there exist homomorphisms

$$G \to H \quad \text{and} \quad H \to G,$$

both leaving S elementwise fixed (evidently, the factor groups G/S and H/S need not be isomorphic). The extension types are classified in the special case where G/S is a torsion group with trivial p-components for the primes p for which the p-socle $G[p]$ of G is infinite.

Later on (in [17]), he returned to Ext and focused on the group Ext in connection with the study of torsion groups. He was the first, or at least among the very first, mathematicians who recognized that the new and powerful theory of Homological Algebra provided a new machinery to deal with genuine group theoretical questions. He initiated the study of the properties of the group of extensions *qua* abelian groups, such as torsion-freeness, divisibility, etc. He also investigates those endomorphisms of $\mathrm{Ext}(F, T)$ that are induced by the endomorphisms of the constituent groups F and T, describing the extensions that are contained in the image or in the kernel of an induced endomorphism.

In particular, he obtains information about the group $\mathrm{Ext}(P, T)$ where P is the Baer-Specker group and T is a torsion group. He makes a remarkable observation: every p-group that is an epimorphic image of P is a direct sum of a divisible group and a bounded group. He is able to prove this without resorting to slender groups.

Without doubt this paper inspired the investigations of structural properties of the groups obtained *via* homological functors. One of the most interesting discoveries is perhaps that the groups Ext are always *cotorsion* groups, i.e., $E = \mathrm{Ext}(A, B)$ for any groups A, B satisfies $\mathrm{Ext}(F, E) = 0$ for every torsion-free group F.

Personal remarks

Let me finish my survey with a few personal remarks. I am among those fortunate algebraists who knew Reinhold Baer personally and who benefited from his generosity in sharing his views and ideas.

I first met Reinhold at the 1956 group theory meeting in Oberwolfach. This was my very first trip to the West, and due to administrative difficulties I arrived late from Budapest. Baer was among the first participants to welcome me, smiling and shaking hands happily. He rushed to make changes in the schedule so that I would have a chance to give a talk.

After the conference I stayed two extra days at the Institute (which was still in the old building) to have ample time to talk to Reinhold. We went hiking in the Schwarzwald, collecting mushrooms and talking about mathematics with the Neumanns, F. Levi and some young group theorists: B. Huppert and others; of course, Marianne accompanied us.

I was deeply impressed by his personality and his friendly attitude, not to mention the atmosphere he created around himself. He had been one of my mathematical heroes, and I began to admire him also as a marvelous person.

Subsequently, he invited me to several meetings he organized in Oberwolfach, also some of his “Arbeitstagung”s, and later to Frankfurt. The picture that remains in my memory is of him speaking with young mathematicians, eager to learn from the admired master.

In 1958 when I decided to write a monograph on abelian groups, I approached him with the request to read and comment on the draft of the manuscript. He immediately agreed and had nice words to encourage me. He finished reading the draft in a short time, and made several suggestions for improvements in details. He pointed out various inaccuracies and unclear formulations. He suggested a new characterization of basic subgroups, and the definition of the group of ring multiplications on a given abelian group. His most important suggestion was the inclusion of homological functors and their group properties. I am convinced that my book [F] was more useful because of his comments.

After I moved to the United States, I met him less often. He visited me at the University of Miami, and we went on a trip to the Everglades in the company of K. Rangaswamy. At that time I was preparing my second book on abelian groups, so I had plenty of questions to ask Baer. His colloquium talks were brilliantly presented, making lasting impressions on the audience. He convinced everybody that mathematics was a delightful intellectual topic, and it was a great deal of fun to talk about it. As he proceeded, he would address the audience, while writing on the blackboard the essential information in full, but occasionally leaving empty spaces of various sizes. A few minutes later, he would return to an empty space and fill every square inch with the missing information.

I also met him at the first Abelian Groups Conference in Rome in 1972; as usual, he attended all the lectures, listening carefully to the speakers, and making comments or suggestions. At that time, he was no longer doing research in abelian groups, but he was very well informed about the recent developments in the theory.

The last time I had extended conversations with him was when he visited Tulane in the early 1970's. A few years later, in 1977, I talked with him on the phone while traveling through Zürich—this was our last contact. Two years later, I was shocked to learn that he had passed away.

We abelian group theorists will remember Reinhold Baer as a great mathematician, a brilliant speaker and a generous leader. He had tremendous influence on the development of abelian groups, though his heart never belonged exclusively to the field. His ideas, his methods and his theorems breathed new life into the theory, and helped to make it a lively branch of algebra. It is safe to claim that without him the theory of abelian groups would be completely different today.

Baer's papers related to abelian groups

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[2] *Erweiterungen von Gruppen und ihren Isomorphismen,* Math. Z. **38** (1934), 375-416.

[3] *The decomposition of enumerable, primary, abelian groups into direct summands,* Quart. J. Math. Oxford **6** (1935), 217-221.

[4] *The decomposition of abelian groups into direct summands,* Quart. J. Math. Oxford **6** (1935), 222-232.

[5] *Types of elements and characteristic subgroups of abelian groups,* Proc. London Math. Soc. **39** (1935), 481-514.

[6] *The subgroup of elements of finite order of an abelian group,* Ann. Math. **37** (1936), 766-781.

[7] *Lectures on abelian groups by Reinhold Baer,* Spring Term 1936, The Institute for Advanced Study.

[8] *Primary abelian groups and their automorphisms,* Amer. J. Math. **59** (1937), 99-117.

[9] *Abelian groups without elements of finite order,* Duke Math. J. **3** (1937), 68-122.

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