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CURVATURE BOUNDS VIA RICCI SMOOTHING

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ABSTRACT. We give a proof of the fact that the upper and the lower sectional curvature bounds of a complete manifold vary at a bounded rate under the Ricci flow.

Let (M^n, g) be a complete Riemannian manifold with $|\sec(M)| \leq 1$. Consider the Ricci flow of g given by

$$\frac{\partial}{\partial t} g = -2 \operatorname{Ric}(g). \tag{1}$$

It is known (see [Ham82], [Shi89]) that (1) has a solution on $[0, T]$ for some $T > 0$. It is also known (see [BMOR84], [Shi89]) that the solution smoothes out the metric. Namely, g_t satisfies

$$e^{-c(n)t} g \leq g_t \leq e^{c(n)t} g, \quad |\nabla - \nabla_t| \leq c(n)t, \quad |\nabla^m R_{ijkl}(t)| \leq c(n, m, t) \tag{2}$$

Moreover, by [Shi89], the sectional curvature of $g(t)$ satisfies

$$|K_{g_t}| \leq C(n, T). \tag{3}$$

This result proved to be a very useful technical tool in many situations and in particular in the theory of convergence with two-sided curvature bounds (see [CFG92], [Ron96], [PT99], etc). However, it turns out that in applications to convergence with two-sided curvature bounds, in addition to the above properties, it is often convenient to know that $\sup K_{g_t}$ and $\inf K_{g_t}$ also vary at the bounded rate and, in particular, that the upper and the lower curvature bounds for g_t are almost the same as those for g for sufficiently small t. For example, it is very useful to know that if g_0 has pinched positive [Ron96] or negative ([Kan89], [BK]) curvature, then g_t has almost the same pinching.

This fact has apparently been known to some experts and it was used without a proof by various people (see, e.g., [Kan89]). A careful proof was given in [Ron96] in the case of a compact manifold M. To the best of our knowledge, no proof exists in the literature in the case of a noncompact manifold M. The purpose of this note is to rectify this situation. To this end we prove:

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Proposition. *In the above situation one has*

$$\inf K_g - C(n,T)t \le K_{g_t} \le \sup K_g + C(n,T)t.$$

Proof. Throughout the proof we will denote by C various constants depending only on n, T. The proof in [Ron96] relies on the maximum principle applied to the evolution equation for the curvature tensor Rm, which can be computed to have the form [Shi89]

$$\frac{\partial}{\partial t} R_{ijkl} = \Delta R_{ijkl} + P(\mathrm{Rm}), \tag{4}$$

where $P(\mathrm{Rm})$ is a quadratic polynomial in Rm. However, in the noncompact case the maximum principle can not be applied directly. We will use a local version of the maximum principle often employed in [Shi89]. Let $\chi\colon \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

(1) $\chi \ge 0$ and is nonincreasing,
(2) $\chi(x) = \begin{cases} 1 \text{ for } x \le 1, \\ \text{nonincreasing for } 1 \le x \le 2, \\ 0 \text{ for } x \ge 2, \end{cases}$
(3) $|\chi''(x)| \le 8$,
(4) $|(\chi'(x))^2/\chi(x)| \le 16$.

Fix $z \in M$ and let $d_z(x,t) = d_{g_t}(x,z)$ be the distance with respect to g_t. Put $\xi_z(x,t) = \chi(d_z(x,t))$. Using the properties of χ we obtain

(i) $0 \le \xi_z \le 1$,
(ii) $|\nabla \xi_z| \le C$,
(iii) $\Delta \xi_z \ge C$ in the barrier sense,
(iv) $|\nabla \xi_z|^2/|\xi_z| \le C$,
(v) $|\partial \xi_z(x,t)/\partial t| \le C$..

To see (iii), we compute

$$\Delta \xi_z = \chi''(d_z)|\nabla d_z|^2 + \chi'(d_z)\Delta d_z \ge C$$

because $\chi' \le 0$ and $\Delta d_z \le C$ for $d_z \ge 1$ by the Laplace comparison for spaces with sec ≥ -1. Finally, (v) holds by the evolution equation of the metric (1) and the estimate (3).

Assume for now that $\sup K_{g_t} \ge 0$ for all $t \in [0,T]$. Let $\bar{A}(t) = \sup K_{g_t}$ and $\bar{A}_z(t) = \max_{(x,\sigma)}\{K_{g_t}(x,\sigma)\xi_z(x,t)\}$, where $x \in M$, σ is a 2-plane at x. Clearly $\bar{A}(t) = \sup_z \bar{A}_z(t)$.

We want to show that $\bar{A}'_z(t) \le C$ independent of z, t. Fix $t_0 \in [0,T]$ and let $\phi_z(x,\sigma,t) = K_{g_t}(x,\sigma)\xi_z(x,t)$. By a standard argument, it is enough to check that $\frac{\partial \phi_z}{\partial t}(x_0,\sigma_0,t_0) \le C$ for any point of maximum of $\phi_z(\cdot,t_0)$.

Let U, V be a basis of σ_0 orthonormal with respect to g_{t_0}. Extend U, V to constant vector fields in normal coordinates at x_0 with respect to g_{t_0}.

Let

$$\Phi_z(x,t) = K_{g_t}(x,U,V)\xi_z(x) = \frac{\mathrm{Rm}(t)(U,V,U,V)}{|U\wedge V|^2_{g_t}}\xi_z(x).$$

It is easy to see (cf. [Ron96]) that

$$(5)\quad |U\wedge V(x_0)|_{g_t} \le C, \quad |\nabla |U\wedge V(x_0)|_{g_t}| \le C, \quad |\nabla^2 |U\wedge V(x_0)|_{g_t}| \le C.$$

By construction, $\Phi_z(x,t_0)$ has a local maximum at x_0 and we have

$$\frac{\partial \phi_z(x_0,\sigma_0,t_0)}{\partial t} = \frac{\partial \Phi_z(x_0,t_0)}{\partial t}.$$

Therefore $\nabla\Phi_z(x_0,t_0) = 0$ and $\Delta\Phi_z(x_0,t_0) \le 0$. We compute

$$(6)\quad \begin{aligned}\frac{\partial \Phi_z(x_0,t_0)}{\partial t} &= \Delta\Phi_z(x_0,t_0)\\ &\quad - \mathrm{Rm}(x_0,t_0)(U,V,U,V)\xi_z(x_0,t_0)\frac{\partial}{\partial t}\left(\frac{1}{|U\wedge V|^2}\right)\\ &\quad - 2\nabla\mathrm{Rm}(x_0,t_0)(U,V,U,V)\nabla\left(\frac{\xi_z(x_0,t_0)}{|U\wedge V|^2}\right)\\ &\quad - \mathrm{Rm}(x_0,t_0)(U,V,U,V)\Delta\left(\frac{\xi_z(x_0,t_0)}{|U\wedge V|^2}\right)\\ &\quad - \frac{P(\mathrm{Rm}(x_0,t_0))\xi_z(x_0,t_0)}{|U\wedge V|^2} - K_{g_t}(x,U,V)\frac{\partial \xi_z(x_0,t_0)}{\partial t}.\end{aligned}$$

We claim that the right-hand side is bounded above by C. The only terms that need explaining are the third and the fourth summands. Let

$$f(x) = \frac{\xi_z(x,t_0)}{|U\wedge V|^2}.$$

To see that the third term is bounded we observe that $\nabla\Phi_z(x_0,t_0) = 0$ yields

$$\begin{aligned}&\nabla\,\mathrm{Rm}(x_0,t_0)(U,V,U,V)f(x_0) + \mathrm{Rm}(x_0,t_0)(U,V,U,V)\nabla f(x_0) = 0,\\ &\nabla\,\mathrm{Rm}(x_0,t_0)(U,V,U,V) = -\frac{\nabla f(x_0)}{f(x_0)}\,\mathrm{Rm}(x_0,t_0)(U,V,U,V),\end{aligned}$$

and hence

$$|\nabla\,\mathrm{Rm}(x_0,t_0)(U,V,U,V)\nabla f(x_0)| \le C$$

by the property (iv) of ξ_z above. The fourth term is bounded because

$$\begin{aligned}\Delta f = \Delta\xi_z(x_0)\frac{1}{|U\wedge V|^2} &+ 2\nabla\xi_z(x_0)\nabla\left(\frac{1}{|U\wedge V|^2}\right)\\ &+ \xi_z(x_0)\Delta\left(\frac{1}{|U\wedge V|^2}\right) \ge C\end{aligned}$$

by (5) and the property (iii) of ξ_z. Thus by (6) we have

$$\frac{\partial \phi_z}{\partial t}(x_0,\sigma_0,t_0)=\frac{\partial \Phi_z(x_0,t_0)}{\partial t}\le C.$$

Therefore $\bar{A}_z'(t)\le C$ for all $z\in M, t\in[0,T]$ and hence $\bar{A}'(t)\le C$ for all $t\in[0,T]$. This concludes the proof in the case $\sup K_{g_t}\ge 0$. The general case can be easily reduced to this one by replacing the function $K_{g_{t_0}}(x,\sigma)$ by $K_{g_{t_0}}(x,\sigma)+C$. The argument for $\inf K_{g_t}$ is the same except that there we can actually always assume that $\inf K_{g_t}\le 0$ since otherwise the manifold M is compact and our statement is known by [Ron96]. □

Remark 1. By changing the cutoff function $\xi_z(\cdot)$ to $\chi(d(\cdot,z)/R)$ in the proof of Proposition we see that the same proof actually shows that the *local* maximum and minimum of the curvature vary linearly. Namely, under condition of the Proposition, for any $R>0$ there exists $C=C(T,R)$ such that for any $z\in M$ we have

$$\inf_{B(z,R)} K_g - C(n,R,T)t \le K_{g_t}|_{B(z,R)} \le \sup_{B(z,R)} K_g + C(n,R,T)t$$

However, as constructed, $C(n,R,T)\to\infty$ as $R\to 0$.

Remark 2. A slightly more careful examination of the proof of Proposition shows that the local rate of change of the curvature bounds is proportional to the local absolute curvature bounds, i.e.,

$$\bar{A}_z'(t)\le C(n,T)\cdot \sup_{x\in B(z,2)} |\operatorname{Rm}(x)|.$$

In particular, if (M^n,g) is asymptotically flat, then so is (M^n,g_t) and it has the same curvature decay rate as (M^n,g).

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