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AN EXTREMAL FUNCTION FOR THE MULTIPLIER ALGEBRA OF THE UNIVERSAL PICK SPACE

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ABSTRACT. Let H^2_m be the Hilbert function space on the unit ball in $\mathbb{C}^m$ defined by the kernel $k(z,w) = (1-\langle z,w\rangle)^{-1}$. For any weak zero set of the multiplier algebra of H^2_m, we study a natural extremal function, E. We investigate the properties of E and show, for example, that E tends to 0 at almost every boundary point. We also give several explicit examples of the extremal function and compare the behaviour of E to the behaviour of δ^* and g, the corresponding extremal function for H^∞ and the pluricomplex Green function, respectively.

1. Introduction

Let $m \geq 1$ be an integer and define $k : \mathbb{B}^m \times \mathbb{B}^m \to \mathbb{C}$ by $k(z,w) = (1-\langle z,w\rangle)^{-1}$, where $\mathbb{B}^m$ is the unit ball in $\mathbb{C}^n$ and $\langle z,w\rangle = \sum_{j=1}^m z_j \bar{w}_j$ is the standard inner product on $\mathbb{C}^m$. It is not difficult to check that k is a positive kernel on $\mathbb{B}^m$, i.e., that for any choice $\{\lambda_1, \dots, \lambda_n\}$ of a finite number of points in $\mathbb{B}^m$ the matrix

$$(k(\lambda_j, \lambda_k))_{j,k=1}^n = \left(\frac{1}{1-\langle \lambda_j, \lambda_k\rangle}\right)_{j,k=1}^n$$

is positive semi-definite. Let H^2_m denote the Hilbert function space on $\mathbb{B}^m$ defined by k. More explicitly, let $k_w(\cdot) = k(\cdot, w)$ and let H^2_m be the closed linear span of $\{k_w : w \in \mathbb{B}^m\}$ under the inner product $\langle \sum_j a_j k_{w_j}, \sum_k b_k k_{w_k}\rangle_{H^2_m} = \sum_{j,k} a_j \bar{b}_k k(w_k, w_j)$.

Recently, there has been a substantial amount of interest in the space H^2_m, the main reason being that k is a *complete Pick kernel* and furthermore that k has a certain universal property among complete Pick kernels. Let us quickly review these notions. If $\mathcal{H}$ is a (complex) Hilbert function space on X, recall

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that a function $\phi : X \to \mathbb{C}$ is a *multiplier* of $\mathcal{H}$ if $\phi f \in \mathcal{H}$ for all $f \in \mathcal{H}$. If ϕ is a multiplier of $\mathcal{H}$, the closed graph theorem implies that the operator $M_\phi : \mathcal{H} \to \mathcal{H}$ defined by $M_\phi f = \phi f$ is bounded. We define $\mathrm{Mult}(\mathcal{H})$ to be the set of multipliers of $\mathcal{H}$ equipped with the norm $\|\phi\|_{\mathrm{Mult}(\mathcal{H})} = \|M_\phi\|_{\mathrm{op}}$, where $\|\cdot\|_{\mathrm{op}}$ is the operator norm. With this definition, one can check that $\mathrm{Mult}(\mathcal{H})$ is a Banach algebra.

Given $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$ and $w_1, \ldots, w_n \in \mathbb{D}$, recall that Pick's classical theorem [10] gives necessary and sufficient conditions for the existence of a function $f \in H^\infty(\mathbb{D})$ with $\|f\|_{H^\infty} \leq 1$ such that $f(\lambda_j) = w_j$ for $1 \leq j \leq n$. More precisely, such a function f exists if and only if the Pick matrix

$$\left(\frac{1 - w_j \bar{w}_k}{1 - \lambda_j \overline{\lambda}_k} \right)_{j,k=1}^n$$

is positive semi-definite. One modern approach to Pick's theorem is to view $H^\infty(\mathbb{D})$ as the multiplier algebra of the Hardy space $H^2(\mathbb{D})$.

For the abstract formulation of Pick's theorem, assume that $\mathcal{H}$ is a Hilbert function space on X with reproducing kernel K and assume that $\lambda_1, \ldots, \lambda_n \in X$ and $w_1, \ldots, w_n \in \mathbb{C}$. The Pick problem is to give necessary and sufficient conditions on the λ_j's and the w_j's for the existence of $\phi \in \mathrm{Mult}(\mathcal{H})$ with $\|\phi\|_{\mathrm{Mult}(\mathcal{H})} \leq 1$ and $\phi(\lambda_j) = w_j$ for $1 \leq j \leq n$. It is not too difficult to verify that a necessary condition is that the matrix

$$\big((1 - w_j \bar{w}_k) K(\lambda_j, \lambda_k) \big)_{j,k=1}^n$$

is positive semi-definite. If this condition is also sufficient we say that K is a *Pick kernel*. If the corresponding necessary condition for the matrix-valued version of Pick interpolation is sufficient for all matrix sizes, we say that K is a *complete Pick kernel*. Pick's classical theorem can now be formulated as saying that the *Szegő kernel* $S(z, w) = (1 - z\bar{w})^{-1}$, i.e., the reproducing kernel for $H^2(\mathbb{D})$, is a Pick kernel. (In fact, the Szegő kernel is a complete Pick kernel.) The kernel k in this paper is just the Szegő kernel for $m = 1$ and our space H^2_1 is just the Hardy space $H^2(\mathbb{D})$. For $m \geq 2$, H^2_m is not the usual Hardy space in the ball of $\mathbb{C}^m$ but a proper subspace of it.

For every integer m, the kernel k is a complete Pick kernel on $\mathbb{B}^m$, and conversely, if $\mathcal{H}$ is a Hilbert function space with reproducing kernel K, and K is an irreducible complete Pick kernel, then $\mathcal{H}$ can be isometrically embedded in δH^2_m for some m and some non-vanishing function δ. This was proven by Agler and McCarthy in [1].

In this paper, we will define a certain extremal function for the multiplier algebra of H^2_m and study its properties. This extremal function is a natural analogue of the Carathéodory function δ^* which has been studied in connection with the pluricomplex Green function. (See, for example, Edigarian and Zwonek [7] and Wikström [12], [13].)

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2. H^2_m and its multiplier algebra

It is straight-forward to check that the monomials $\{z^\alpha\}_{\alpha\in\mathbb{N}^m}$ are mutually orthogonal in H^2_m, and from a power series expansion of k we see that $\|z^\alpha\|^2_{H^2_m} = \alpha!/|\alpha|!$. (Here, as usual, if $\alpha = (\alpha_1, \dots, \alpha_m)$ is a multi-index, $z^\alpha = z_1^{\alpha_1}\cdots z_m^{\alpha_m}$, $|\alpha| = \alpha_1 + \cdots + \alpha_m$ and $\alpha! = \alpha_1!\cdots\alpha_m!$.) From this it follows that

$$H^2_m = \left\{ f = \sum_\alpha c_\alpha z^\alpha : \|f\|^2_{H^2_m} = \sum_\alpha \frac{|c_\alpha|^2\alpha!}{|\alpha|!} < \infty \right\}.$$

It is also possible to give integral representations of the norm in H^2_m. (See Alpay and Kaptanoğlu [3].) If ϕ is a multiplier of H^2_m, then ϕ must be holomorphic since $\phi = \phi\cdot 1$ and $1 \in H^2_m$. By general Hilbert function space theory it also follows that ϕ must be bounded, so $\mathrm{Mult}(H^2_m) \subset H^\infty(\mathbb{B}^m)$. If $m > 1$, this inclusion is proper. Interestingly enough, the Mult-norm and the H^2_m-norm agree on monomials.

Proposition 2.1. *Let $\phi(z) = z^\alpha$. Then $\|\phi\|^2_{\mathrm{Mult}(H^2_m)} = \|\phi\|^2_{H^2_m} = \alpha!/|\alpha|!$.*

Proof. For simplicity of notation, assume that $m = 2$. (The argument can be adapted to work for every m.) Let $\alpha = (\alpha_1, \alpha_2)$. First note that if $j = (j_1, j_2)$ is a multiindex, then

$$\frac{(|j| + |\alpha|)!}{(j+\alpha)!} = \binom{j_1 + j_2 + \alpha_1 + \alpha_2}{j_1 + \alpha_1} \geq \binom{j_1 + j_2}{j_1}\binom{\alpha_1 + \alpha_2}{\alpha_1} = \frac{|j|!}{j!}\frac{|\alpha|!}{\alpha!}.$$

This can be seen from considering the natural expansions of

$$(x+1)^{j_1+j_2+\alpha_1+\alpha_2} = (x+1)^{j_1+j_2}(x+1)^{\alpha_1+\alpha_2},$$

and comparing the coefficients of $x^{j_1+\alpha_1}$. Now, if $f \in H^2_m$, $f = \sum c_{j_1j_2} z_1^{j_1} z_2^{j_2}$, then

$$\|\phi f\|^2_{H^2_m} = \sum_{j_1,j_2=0}^{\infty} \frac{|c_{j_1j_2}|^2}{\binom{j_1+j_2+\alpha_1+\alpha_2}{j_1+\alpha_1}} \leq \sum_{j_1,j_2=0}^{\infty} \frac{|c_{j_1j_2}|^2}{\binom{j_1+j_2}{j_1}\binom{\alpha_1+\alpha_2}{\alpha_1}} = \frac{\alpha!}{|\alpha|!}\|f\|^2_{H^2_m}.$$

Hence $\|\phi\|^2_{\mathrm{Mult}(H^2_m)} \leq \alpha!/|\alpha|!$. On the other hand, since $1 \in H^2_m$ and $\|1\| = 1$, $\|\phi\|^2_{\mathrm{Mult}(H^2_m)} \geq \|\phi\|^2_{H^2_m} = \alpha!/|\alpha|!$. □

As far as the author is aware, there is no explicit description of $\mathrm{Mult}(H^2_m)$, but the following result gives a characterization of the multipliers of H^2_m that can be used to deduce some properties of the multiplier algebra, even though the condition is far from easy to verify for a given $\phi \in H^\infty(\mathbb{B}^m)$.

THEOREM 2.2. *Assume that ϕ is a holomorphic function on $\mathbb{B}^m$. Then ϕ is a multiplier of H^2_m with $\|\phi\|_{\mathrm{Mult}(H^2_m)} \leq 1$ if and only if*

$$K(z,w) = \frac{1-\phi(z)\overline{\phi(w)}}{1-\langle z,w\rangle}$$

is a positive kernel on $\mathbb{B}^m$.

For a proof of Theorem 2.2, see, for example, [2, Corollary 2.37].

REMARK. Recall that a sesqui-holomorphic kernel $K(z,w)$ is positive if and only if there is a Hilbert space $\mathcal{H}$ and a holomorphic function $H : X \to \mathcal{H}$ such that $K(z,w) = \langle H(z), H(w)\rangle$. (See, for example, Agler-McCarthy [2, Theorem 2.53].)

As a consequence of this characterization of the multiplier algebra of H^2_m we can prove that the unit ball of $\mathrm{Mult}(H^2_m)$ is biholomorphically invariant. Let $\mathrm{Aut}(\mathbb{B}^m)$ denote the group of biholomorphic self-mappings of $\mathbb{B}^m$ and let $\mathrm{ball}(\mathrm{Mult}(H^2_m)) = \{\phi \in \mathrm{Mult}(H^2_m) : \|\phi\|_{\mathrm{Mult}(H^2_m)} \leq 1\}$.

THEOREM 2.3. *Assume that $\phi \in \mathrm{ball}(\mathrm{Mult}(H^2_m))$ and that $T \in \mathrm{Aut}(\mathbb{B}^m)$. Then $\phi \circ T \in \mathrm{ball}(\mathrm{Mult}(H^2_m))$.*

Proof. Let $\phi \in \mathrm{ball}(\mathrm{Mult}(H^2_m))$. Recall that $\mathrm{Aut}(B^m)$ is generated by unitary mappings of $\mathbb{C}^m$ and mappings of the form

$$T_a(z) = \left(\frac{a-z_1}{1-\bar{a}z_1}, \frac{(1-|a|^2)^{1/2}z_2}{1-\bar{a}z_1}, \dots, \frac{(1-|a|^2)^{1/2}z_m}{1-\bar{a}z_1}\right),$$

where $a \in \mathbb{D}$. (See, for example, Rudin [11] for a proof of this fact.) It is clear that $\|f\|_{H^2_m} = \|f\circ U\|_{H^2_m}$ for all unitaries U and all $f \in H^2_m$ and hence $\|f\cdot\phi\circ U\|_{H^2_m} = \|f\circ U^{-1}\cdot\phi\|_{H^2_m} \leq \|f\circ U^{-1}\|_{H^2_m} = \|f\|_{H^2_m}$, so $\phi\circ U \in \mathrm{ball}(\mathrm{Mult}(H^2_m))$. To finish the proof, it is enough to show that $\phi\circ T_a \in \mathrm{ball}(\mathrm{Mult}(H^2_m))$ for every $a \in \mathbb{D}$. Note that

$$\begin{aligned} 1-\langle T_a(z), T_a(w)\rangle &= 1-\frac{(a-z_1)(\bar{a}-\bar{w}_1)+(1-|a|^2)(z_2\bar{w}_2+\cdots+z_m\bar{w}_m)}{(1-\bar{a}z_1)(1-a\bar{w}_1)} \\ &= \frac{(1-|a|^2)(1-\langle z,w\rangle)}{(1-\bar{a}z_1)(1-a\bar{w}_1)}. \end{aligned}$$

Let $\psi = \phi\circ T_a$. By Theorem 2.2, $\psi \in \mathrm{ball}\,\mathrm{Mult}(H^2_m)$ if and only if

$$K_\psi(z,w) = \frac{1-\psi(z)\overline{\psi(w)}}{1-\langle z,w\rangle}$$

is a positive kernel. But

$$\begin{aligned} K_\psi(T_a(z), T_a(w)) &= \frac{1-\phi(z)\overline{\phi(w)}}{1-\langle T_a(z), T_a(w)\rangle} \\ &= \frac{1-\phi(z)\overline{\phi(w)}}{1-\langle z, w\rangle} \frac{(1-\bar{a}z_1)(1-a\bar{w}_1)}{1-|a|^2} \\ &= (1-|a|^2)^{-1} \langle H(z)(1-\bar{a}z_1), H(w)(1-\bar{a}w_1)\rangle \end{aligned}$$

for some auxiliary Hilbert space $\mathcal{H}$ and some holomorphic $H : \mathbb{B}^m \to \mathcal{H}$ using the remark following Theorem 2.2. Hence K_ψ is a positive kernel, and $\psi \in \mathrm{ball}(\mathrm{Mult}(H^2_m))$. □

3. The extremal function

DEFINITION 3.1. Let $\mathcal{F}$ be a set of functions on X and let $A \subset X, A \neq X$. If there is a function $f \in \mathcal{F}$ such that $f^{-1}(0) = A$, we say that A is a *zero set* for $\mathcal{F}$. If A is the intersection of zero sets, we say that A is a *weak zero set* for $\mathcal{F}$.

DEFINITION 3.2. Let A be a weak zero set for $\mathrm{Mult}(H^2_m)$. We define the ($\mathrm{Mult}(H^2_m)$-)*extremal function* for A as

$$E(z, A) = \sup\{\log \mathrm{Re}\, \phi(z) : \phi \in \mathrm{Mult}(H^2_m), \|\phi\|_{\mathrm{Mult}(H^2_m)} \leq 1, \phi|_A = 0\}.$$

If $A = \{w\}$ is a singleton, we usually write $E(z, w)$ instead of $E(z, \{w\})$. Similarly, if A is a weak zero set for H^2_m, we define the (H^2_m-)*extremal function* for A as

$$F(z, A) = \sup\{\log \mathrm{Re}\, f(z) : f \in H^2_m, \|f\|_{H^2_m} \leq 1, f|A = 0\}.$$

In this paper we will be mostly concerned with the $\mathrm{Mult}(H^2_m)$-extremal function, but we will shortly see that E and F are closely related.

DEFINITION 3.3. Let $z \in \mathbb{B}^m$ and let A be a weak zero set of B^m. If $\phi \in \mathrm{ball}(\mathrm{Mult}(H^2_m))$ satisfies that $\phi|A = 0$ and $\log \mathrm{Re}\, \phi(z) = E(z, A)$, we say that ϕ is *E-extremal* (with respect to z and A). Similarly, if $f \in H^m$ with $\|f\| \leq 1, f|A = 0$ and $\log \mathrm{Re}\, f(z) = F(z, A)$, we say that f is *F-extremal*.

Note that if $m = 1$, and A consists of a single point, then $E(z, w)$ is just the (negative) Green function for the unit disc $E(z, w) = g(z, w) = \log \left|\frac{z-w}{1-z\bar{w}}\right|$. Also, if we replace $\mathrm{ball}(\mathrm{Mult}(H^2_m))$ with $\mathrm{ball}(H^\infty(B^m))$ in the definition of E, we obtain the *Carathéodory function* δ^*. Later on, we will compare E to δ^* and to the *pluricomplex Green function* g, so for completeness let us define these functions here as well.

Definition 3.4. Let Ω be a domain in $\mathbb{C}^m$ and let A be a zero set for $H^\infty(\Omega)$. We define the *Carathéodory function*, δ^*, by

$$\delta^*(z, A) = \sup\{\log |f(z)| : f \in H^\infty(\Omega), \|f\|_{H^\infty} \leq 1, f|A = 0\}.$$

Definition 3.5. Let Ω be a domain in $\mathbb{C}^m$ and let ν be a non-negative function on Ω. We define the *pluricomplex Green function* with poles defined by ν by

$$g(z, \nu) = \sup\{u(z) : u \in \mathcal{PSH}(\Omega), u < 0, \nu_u \geq \nu\},$$

where ν_u denotes the *Lelong number* of u, i.e.,

$$\nu_u(x) = \lim_{r \to 0} \frac{\sup_{|\xi - x| = r} u(\xi)}{\log r}.$$

Note that if A is a zero set for H^∞, then $\delta^*(z, A) \leq g(z, \chi_A)$, and if $\Omega = \mathbb{B}^m$ and A is a zero set for $\mathrm{Mult}(H^2_m)$, then $E(z, A) \leq \delta^*(z, A)$. Let us move on to collect the basic properties of E and F.

Theorem 3.6. *The functions E and F have the following properties:*

1. *E is biholomorphically invariant; more precisely, if $T \in \mathrm{Aut}(\mathbb{B}^m)$, then $E(z, A) = E(T(z), T(A))$.*
2. *For every $z \in \mathbb{B}^m$ and every weak zero set A for H^2_m, there is a unique F-extremal function, which we will denote by EF^z_A, given by $EF^z_A = P_A k_z / \|P_A k_z\|$, where P_A is the orthogonal projection $H^2_m \to I_A$ and I_A is the space of functions in H^2_m that vanish on A. Hence $F(z, A) = \log \|P_A k_z\|$.*
3. *For every $z \in \mathbb{B}^m$ and every weak zero set A for $\mathrm{Mult}(H^2_m)$, there is a unique E-extremal function, which we will denote by EE^z_A, given by $EF^z_A = EE^z_A k_z / \|k_z\|$. Hence $F(z, A) = E(z, A) + \log \|k_z\|$. Furthermore, if A is finite, then EE^z_A is a rational function of degree at most $|A|$.*
4. *$E(\cdot, A)$ and $F(\cdot, A)$ are plurisubharmonic on $\mathbb{B}^m$ and continuous on $\mathbb{B}^m \setminus A$.*

Proof. (1) is a direct consequence of Theorem 2.3.

(2) A normal family argument proves the existence of an F-extremal function. If f and g are two F-extremal functions, then $(f + g)/2$ is also F-extremal. Note that an F-extremal function must have norm exactly 1. Since every point in the unit sphere of a Hilbert space is an extreme point (in the convex sense), $f = g$. A variational argument (see [2, Proposition 9.31] for details) shows that any function which is orthogonal to the F-extremal function must be orthogonal to $P_A k_z$. Hence, the F-extremal function must be the normalization of $P_A k_z$.

(3) Again, a normal family argument proves the existence of an E-extremal function ϕ. Let $\mathcal{N}$ be the closed linear span of $\{k_\zeta : \zeta \in A\}$ and k_z. The Pick

property of k implies that the linear operator on $\mathcal{N}$ that sends k_ζ to 0 for all $\zeta \in A$ and sends k_z to $\phi(z)k_z$ has norm 1. From this it follows from a computation ([2, Proposition 9.33] for details) that $F(z,A) = E(z,A) + \log\|k_z\|$ and hence that the function $\phi k_z/\|k_z\|$ must be F-extremal. But since the F-extremal function is unique, so is the E-extremal function.

Now assume that A is finite, say $A = \{w_1,\dots,w_n\}$. It is clear that $E(z,A) = \log c$, where c is the (unique) positive real number such that $\det P = 0$, where

$$P = \begin{pmatrix} \frac{1-c^2}{1-\|z\|^2} & \frac{1}{1-\langle z,w_1\rangle} & \cdots & \frac{1}{1-\langle z,w_n\rangle} \\ \frac{1}{1-\langle w_1,z\rangle} & \frac{1}{1-\langle w_1,w_1\rangle} & \cdots & \frac{1}{1-\langle w_1,w_n\rangle} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{1-\langle w_n,z\rangle} & \frac{1}{1-\langle w_n,w_1\rangle} & \cdots & \frac{1}{1-\langle w_n,w_n\rangle} \end{pmatrix}.$$

Choose $v = (v_0,\dots,v_n)^T \in \mathbb{C}^{n+1}$ such that $Pv = 0$. Take any $\zeta \in \mathbb{B}^m$. Since there exists a function $\phi \in \operatorname{ball}(\operatorname{Mult}(H^2_m))$ with $\phi(z) = c$ and $\phi(w_1) = \cdots = \phi(w_n) = 0$, the matrix

$$\tilde{P} = \begin{pmatrix} & & & & \frac{1-c\bar{\alpha}}{1-\langle z,\zeta\rangle} \\ & & & & \frac{1}{1-\langle w_1,\zeta\rangle} \\ & & P & & \vdots \\ & & & & \frac{1}{1-\langle w_n,\zeta\rangle} \\ \frac{1-\alpha\bar{c}}{1-\langle \zeta,z\rangle} & \frac{1}{1-\langle \zeta,w_1\rangle} & \cdots & \frac{1}{1-\langle \zeta,w_n\rangle} & \frac{1-|\alpha|^2}{1-\|\zeta\|^2} \end{pmatrix}$$

must be positive semi-definite for some choice of α (namely for $\alpha = \phi(\zeta)$). Take any $\eta \in \mathbb{C}$ and let $v_\eta = v \oplus \eta$. Hence

$$0 \le v_\eta^* \tilde{P} v_\eta = |\eta|^2 \frac{1-|\alpha|^2}{1-\|\zeta\|^2} + 2\operatorname{Re}\bar{\eta}\left(v_0 \frac{1-c\bar{\alpha}}{1-\langle z,\zeta\rangle} + \sum_{j=1}^n \frac{v_j}{1-\langle w_j,\zeta\rangle}\right)$$

for all η. Consequently,

$$v_0 \frac{1-c\bar{\alpha}}{1-\langle z,\zeta\rangle} + \sum_{j=1}^n \frac{v_j}{1-\langle w_j,\zeta\rangle} = 0, \tag{3.1}$$

and this equation determines α uniquely, since it is straight-forward to check that v_0 must be non-zero. Furthermore, we see from Equation (3.1) that $\alpha = \phi(\zeta)$ is a rational function in ζ of degree at most n.

(4) Take a sequence z_j in $\mathbb{B}^m \setminus A$ converging to $z \in \mathbb{B}^m \setminus A$ and consider $EE_A^{z_j}$. By passing to a subsequence, we may assume that $EE_A^{z_j}$ converges locally uniformly to some $\phi \in \mathrm{Mult}(H_m^2)$. Clearly $\phi|_A = 0$, so $E(z, A) \geq \log|\phi(z)| = \lim_j \log|EE_A^{z_j}(z_j)| = \lim_j E(z_j, A)$. Hence E is upper semicontinuous. On the other hand, on compact subsets of $\mathbb{B}^m \setminus A$, E is the supremum of a class of continuous functions, and consequently E is lower semicontinuous. Since E is the supremum of a class of plurisubharmonic functions and E is upper semicontinuous, E must be plurisubharmonic. Also, since

$$F(z, A) = E(z, A) + \log\|k_z\| = E(z, A) + \frac{1}{2}\log\frac{1}{1 - \|z\|^2},$$

F is also plurisubharmonic and continuous on $\mathbb{B}^m \setminus A$. □

The fact that the E-extremal function is unique gives another proof that there is no Pick kernel whose multiplier algebra is H^∞. More precisely:

Corollary 3.7. *There is no Pick kernel on $\mathbb{B}^m$ for $m \geq 2$ whose multiplier algebra is $H^\infty(\mathbb{B}^m)$.*

Proof. Recall that the extremal functions for δ^* in general are not unique, not even when A is a singleton. In fact, if $w = 0$ and $z = (\lambda, 0)$, then $f = z_1 + cz_2^2$ is extremal for δ^* and all c with $|c| < 1/2$.

Note that the proof of uniqueness for the E-extremal function in Theorem 3.6 only uses the fact that k is a Pick kernel. Hence, there is no Pick kernel on $\mathbb{B}^m$ whose multiplier algebra is $H^\infty(\mathbb{B}^m)$. □

Theorem 3.8. *Let A be a weak zero set for H_m^2, and let $a \in \mathbb{B}^m \setminus A$. Then*

$$F(z, A \cup \{a\}) = \frac{1}{2}\log\left(\exp(F(z, A))^2 - |EF_A^a(z)|^2\right).$$

Consequently,

$$E(z, A \cup \{a\}) = \frac{1}{2}\log\left(\exp(E(z, A))^2 - (1 - \|z\|^2)|EF_A^a(z)|^2\right).$$

Proof. Let $A' = A \cup \{a\}$. Recall that if X is a weak zero set for H_m^2, I_X denotes the subspace $\{f \in H_m^2 : f|X = 0\}$. Using the reproducing kernel property, we see that I_X is the orthogonal complement of the closed linear hull of $\{k_\lambda : \lambda \in X\}$. Hence, for $f \in H_m^2$,

$$P_{A'}f = f - P_{A'}^\perp f = P_A f - \frac{\langle f, P_A k_a\rangle}{\|P_A k_a\|^2} P_A k_a.$$

In particular,

$$\begin{aligned}\|P_{A'}k_z\|^2 = \langle P_{A'}k_z, k_z\rangle &= \langle P_A k_z, k_z\rangle - \frac{\langle k_z, P_A k_a\rangle}{\|P_A k_a\|^2}\langle P_A k_a, k_z\rangle \\ &= \|P_A k_z\|^2 - |EF_A^a(z)|^2,\end{aligned}$$

since $EF^a_A = P_A k_a/\|P_A k_a\|$. Using Theorem 3.6, we obtain the formula for $F(z, A')$. The expression for $E(z, A')$ follows from the fact that $E(z, A') = F(z, A') - \log\|k_z\|$. □

Proposition 3.9. *If A and B are weak zero sets of $\mathbb{B}^m$, then $E(z, A\cup B) \geq E(z, A) + E(z, B)$ with equality if and only if $EE^z_{A\cup B} = EE^z_A EE^z_B$.*

Proof. Take $z \in \mathbb{B}^m$. If $z \in A\cup B$, then clearly $E(z, A\cup B) = E(z, A) + E(z, B) = -\infty$. Otherwise, $\phi = EE^z_A EE^z_B$ vanishes on $A\cup B$, so $E(z, A\cup B) \geq \log|\phi(z)| = E(z, A) + E(z, B)$. Furthermore, if $E(z, A\cup B) = E(z, A) + E(z, B)$, then $\phi = EE^z_A EE^z_B$ is E-extremal for $(z, A\cup B)$. Conversely, if $EE^z_{A\cup B}$ can be factorized as $EE^z_{A\cup B} = EE^z_A EE^z_B$, it is clear that $E(z, A\cup B) = E(z, A) + E(z, B)$. □

Proposition 3.10. *If $A \subset B$ are weak zero sets of $\mathbb{B}^m$, then $E(z, A) \geq E(z, B)$ with equality if and only if $EE^z_A|B = 0$.*

Proof. Since $EE^z_B|A = 0$, $E(z, A) \geq E(z, B)$. Assume that $E(z, A) = E(z, B)$. Then EE^z_B must be E-extremal for (z, A), so the E-extremal function for (z, A) vanishes on B. Conversely, if $EE^z_A|_B = 0$, then $E(z, B) \geq= E(z, A)$, so $E(z, A) = E(z, B)$. □

Theorem 3.11. *Let A be a weak zero set for $\mathrm{Mult}(H^2_m)$. Then for almost all $p \in \partial\mathbb{B}^m$, $\lim_{z\to p} E(z, A) = 0$. (Here, the limit is to be taken in a Korányi region; see, for example, [11].)*

Proof. Let $\mathcal{M} = \{f \in H^2_m : f|A = 0\}$. Then $\mathcal{M}$ is a $\mathrm{Mult}(H^2_m)$-invariant subspace of H^2_m. (Note that $\mathcal{M}$ is non-empty, since $\mathrm{Mult}(H^2_m) \subset H^2_m$.) By a theorem of Arveson [4], there is a sequence $\{\phi_j\} \subset \mathrm{Mult}(H^2_m)\cap\mathcal{M}$ such that

$$P_{\mathcal{M}} = \sum_j M_{\phi_j} M^*_{\phi_j},$$

where the sum converges in the SOT-topology. Hence

$$P_{\mathcal{M}} k_z = \sum_j M_{\phi_j} M^*_{\phi_j} k_z = \sum_j M_{\phi_j} \overline{\phi_j(z)} k_z = \sum_j \overline{\phi_j(z)} \phi_j k_z.$$

But $\|P_{\mathcal{M}} k_z\|^2 = \langle P_{\mathcal{M}} k_z, P_{\mathcal{M}} k_z\rangle = \langle P_{\mathcal{M}} k_z, k_z\rangle + \langle P_{\mathcal{M}} k_z, P_{\mathcal{M}} k_z - k_z\rangle = \langle P_{\mathcal{M}} k_z, k_z\rangle$, so

$$\|P_{\mathcal{M}} k_z\|^2 = \langle P_{\mathcal{M}} k_z, k_z\rangle = \sum_j |\phi_j(z)|^2 \|k_z\|^2.$$

By Theorem 3.6,

$$E(z, A) = \log\frac{\|P_{\mathcal{M}} k_z\|}{\|k_z\|} = \frac{1}{2}\log\sum_j |\phi_j(z)|^2.$$

On the other hand, Green, Richter and Sundberg [8] have shown that the sequence $\{\phi_j\}$ can be chosen to be inner, i.e., that $\sum_j |\phi_j(z)|^2 \to 1$ as $z \to p$ for almost every $p \in \partial\mathbb{B}^m$. $\square$

Remark. It is natural to conjecture that $E(z, A) \to 0$ as $z \to p$ for all $p \in \partial\mathbb{B}^m \setminus \bar{A}$. Of course, if A is finite, then $E(z, A) \to 0$ everywhere on $\partial\mathbb{B}^m$.

4. Examples

Proposition 4.1. *If $w \in \mathbb{B}^m$, then*

$$E(z,w) = \frac{1}{2}\log\left(1 - \frac{(1-\|z\|^2)(1-\|w\|^2)}{|1-\langle z,w\rangle|^2}\right).$$

Proof. Clearly $E(z, w) = \log c_0$, where c_0 is the supremum over all $|c|$ such that

$$A = \begin{pmatrix} \dfrac{1-|c|^2}{1-\|z\|^2} & \dfrac{1}{1-\langle z,w\rangle} \\ \dfrac{1}{1-\langle w,z\rangle} & \dfrac{1}{1-\|w\|^2} \end{pmatrix} \geq 0.$$

Clearly $A \geq 0$ if and only if $|c| \leq 1$ and $\det A \geq 0$, i.e., iff

$$1 - |c|^2 \geq \frac{(1-\|z\|^2)(1-\|w\|^2)}{|1-\langle z,w\rangle|^2}.$$

Note that $E(z, w) = \delta^*(z, w) = g(z, w)$, where δ^* and g are the Carathéodory function and the pluricomplex Green function, respectively. $\square$

Proposition 4.2. *Let $r \in \mathbb{D}$ and let $w_1 = (r, 0)$, $w_2 = (-r, 0)$ and set $A = \{w_1, w_2\} \subset \mathbb{B}^2$. Then*

$$E(z,A) = \frac{1}{2}\log\left(1 - \frac{(1-|r|^4)(1+|z_1|^2)(1-\|z\|^2)}{|1+z_1\bar{r}|^2|1-z_1\bar{r}|^2}\right).$$

Proof. Again, $E(z, w) = \log c_0$, where c_0 is the solution to

$$\det\begin{pmatrix} \dfrac{1-|c|^2}{1-\|z\|^2} & \dfrac{1}{1-\langle z,w_1\rangle} & \dfrac{1}{1-\langle z,w_2\rangle} \\ \dfrac{1}{1-\langle w_1,z\rangle} & \dfrac{1}{1-\langle w_1,w_1\rangle} & \dfrac{1}{1-\langle w_1,w_2\rangle} \\ \dfrac{1}{1-\langle w_2,z\rangle} & \dfrac{1}{1-\langle w_2,w_1\rangle} & \dfrac{1}{1-\langle w_2,w_2\rangle} \end{pmatrix}$$

$$= \det\begin{pmatrix} \dfrac{1-|c|^2}{1-\|z\|^2} & \dfrac{1}{1-z_1\bar{r}} & \dfrac{1}{1+z_1\bar{r}} \\ \dfrac{1}{1-r\bar{z}_1} & \dfrac{1}{1-|r|^2} & \dfrac{1}{1+|r|^2} \\ \dfrac{1}{1+r\bar{z}_1} & \dfrac{1}{1+|r|^2} & \dfrac{1}{1-|r|^2} \end{pmatrix} = 0.$$

An elementary but somewhat tedious computation leads to the formula given above. Alternatively, we could use Theorem 3.8 to prove this proposition. $\square$

Remark. In the two pole setting, Coman [5] has computed g and it follows from results in [7] that in this case $\delta^* = g$. However, E is strictly less than these functions unless $z_2 = 0$ or $z_1 = \pm r$.

Theorem 4.3. *Let $A = \{z_2 = 0\} \subset \mathbb{B}^2$. Then*

$$E(z, A) = \frac{1}{2} \log \frac{|z_2|^2}{1 - |z_1|^2}.$$

Proof. From Proposition 3.10, we see that $E(z, A) \leq \inf_{w \in A} E(z, w)$. Let $w = (z_1, 0)$. The mapping

$$T(\zeta_1, \zeta_2) = \left(\frac{z_1 - \zeta_1}{1 - \zeta_1 \bar{z}_1}, \frac{\sqrt{1 - |z_1|^2}\zeta_2}{1 - \zeta_1 \bar{z}_1} \right)$$

satisfies $T \in \mathrm{Aut}(\mathbb{B}^2)$, $T(w) = 0$ and $T(z) = (0, z_2/\sqrt{1 - |z_1|^2})$. Hence

$$E(z, w) = \log \frac{|z_2|}{\sqrt{1 - |z_1|^2}},$$

by Proposition 4.1. Furthermore $f(\zeta) = \zeta_2$ is the E-extremal function for $T(z)$ and 0 and hence $f_w = f \circ T^{-1}$ is the E-extremal function for z and w. On the other hand, the zero set of f is A and $T^{-1}(A) = A$, so $f_w^{-1}(0) = A$. By Proposition 3.10, $E(z, A) = E(z, w)$. $\square$

Remark. In this setting, we again have that $E = \delta^* = g$. See Lárusson and Sigurdsson [9] for the derivation of g in this case.

Remark. The fact that $E(z, A) = \inf_{w \in A} E(z, w)$ when $A = \{z_2 = 0\}$ is not an example of a general principle at work. In fact, the equality $E(z, A) = \inf_{w \in A} E(z, w)$ holds if and only if A is the intersection of $\mathbb{B}^m$ with a complex hyperplane. This can be seen using Proposition 3.10 and the fact that the zero set of a function that is E-extremal for a singleton is just a hyperplane.

Proposition 4.4. *Let $A = \{z_2 = 0\} \subset \mathbb{B}^2$ and let $a \in \mathbb{B}^2 \setminus A$. Define $A' = A \cup \{a\}$. Then*

$$E(z, A') = \frac{1}{2} \log \left(\frac{|z_2|^2}{1 - |z_1|^2} - \frac{|z_2|^2 (1 - |a_1|^2)(1 - \|a\|^2)(1 - \|z\|^2)}{|1 - z_1 \bar{a}_1|^2 |1 - \langle z, a \rangle|^2} \right).$$

Proof. Let $z \in \mathbb{B}^2$. From the proof of Theorem 4.3 we see that

$$EE_A^z(\zeta) = \frac{e^{i\theta} \zeta_2 \sqrt{1 - |z_1|^2}}{1 - \zeta_1 \bar{z}_1},$$

where $\theta \in \mathbb{R}$ is chosen so that $EE_A^z(z)$ is positive real. Hence, using Theorem 3.6, we obtain

$$EF_A^z(\zeta) = \frac{k_z(\zeta)}{\|k_z\|} EE_A^z(\zeta) = \frac{e^{i\theta}\zeta_2\sqrt{1-|z_1|^2}\sqrt{1-\|z\|^2}}{(1-\zeta_1\bar{z}_1)(1-\langle\zeta,z\rangle)},$$

and consequently, by Theorem 3.8,

$$\begin{aligned} E(z,A') &= \frac{1}{2}\log\left(\exp(E(z,A))^2 - (1-\|z\|^2)|EF_X^a(z)|^2\right) \\ &= \frac{1}{2}\log\left(\frac{|z_2|^2}{1-|z_1|^2} - \frac{|z_2|^2(1-|a_1|^2)(1-\|a\|^2)(1-\|z\|^2)}{|1-z_1\bar{a}_1|^2|1-\langle z,a\rangle|^2}\right). \qquad \square \end{aligned}$$

THEOREM 4.5. *Let $A = \{z_1z_2 = 0\} \subset \mathbb{B}^2$. Then*

$$E(z,A) = \frac{1}{2}\log\left(\frac{|z_1|^2|z_2|^2(2-\|z\|^2)}{(1-|z_1|^2)(1-|z_2|^2)}\right).$$

Proof. Let $z \in \mathbb{B}^2$ and let $B = \{z_2 = 0\} \cup \{(0,z_2)\}$. Then

$$\begin{aligned} E(z,A) \le E(z,B) &= \frac{1}{2}\log\left(\frac{|z_2|^2}{1-|z_1|^2} - \frac{|z_2|^2(1-|z_2|^2)(1-\|z\|^2)}{1-|z_2|^2}\right) \\ &= \frac{1}{2}\log\left(\frac{|z_1|^2|z_2|^2(2-\|z\|^2)}{(1-|z_1|^2)(1-|z_2|^2)}\right), \end{aligned}$$

by Propositions 3.10 and 4.4. On the other hand, if $X = \{z_2 = 0\}$, $z \in \mathbb{B}^2$ and $a = (0,z_2)$, then

$$P_Xk_z(\zeta) = \exp(E(z,X))EE_X^z(\zeta) = \frac{\bar{z}_2\zeta_2}{(1-\zeta_1\bar{z}_1)(1-\langle\zeta,z\rangle)}.$$

From the proof of Theorem 3.8 it follows that

$$\begin{aligned} P_Bk_z &= P_Xk_z - \frac{\langle k_z, P_Xk_a\rangle}{\|P_Xk_a\|^2}P_Xk_a \\ &= \frac{\bar{z}_2\zeta_2}{(1-\zeta_1\bar{z}_1)(1-\langle\zeta,z\rangle} - \frac{\bar{z}_2\zeta_2}{1-\zeta_2\bar{z}_2}. \end{aligned} \tag{4.1}$$

Now, $EF_B^z = P_Bk_z/\|P_Bk_z\|$, so the zero set of EF_B^z equals the zero set of P_Bk_z, and from Theorem 3.6 if also follows that the zero set of EE_B^z equals the zero set of P_Bk_z since the kernel function k_z is zero free. From Equation (4.1) we see that P_Bk_z and hence EE_B^z vanishes on A. Thus

$$E(z,A) = E(z,B) = \frac{1}{2}\log\left(\frac{|z_1|^2|z_2|^2(2-\|z\|^2)}{(1-|z_1|^2)(1-|z_2|^2)}\right),$$

by Proposition 3.10. $\square$

Remark. In this setting, E is smaller than g. The pluricomplex Green function with poles along $A = \{z_1z_2 = 0\}$ has been computed by Nguyen Quang Dieu [6], and even though δ^* is not completely known, it is clear that $\delta^*(z, A) \geq \log|2z_1z_2|$ (since $f(z) = 2z_1z_2$ is a H^∞ function bounded by one and f vanishes on A). In fact, on $D = \{z \in \mathbb{B}^2 : |z_1| \leq 1/\sqrt{2}, |z_2| \leq 1/\sqrt{2}\}$, $\delta^*(z, A) = g(z, A) = \log|2z_1z_2|$.

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