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LIFTING OF ALMOST PERIODICITY OF A POINT THROUGH MORPHISMS OF FLOWS

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This article is dedicated to my mother, Naza Tanović-Miller, the best example of all

ABSTRACT. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of flows, y an almost periodic point of $\mathcal{Y}$, and $x \in f^{-1}(y)$. In general x is not necessarily almost periodic, but several conditions are known under which that happens. They fall into either "compact" or "noncompact" conditions, depending on whether $\mathcal{X}$ and $\mathcal{Y}$ are assumed to be compact or not. In "noncompact" conditions other assumptions are restrictive. We find a criterion for almost periodicity of x, which generalizes both "compact" and "noncompact" statements at the same time. We deduce theorems of Ellis, Markley, Kutaibi-Rhodes and Pestov as corollaries.

1. Introduction

The paper consists of eight sections, the first of which is this introduction.

Section 2 covers the notation, terminology, and some relevant basic facts.

Morphisms of flows with the same acting group were often investigated in Topological Dynamics (see the papers by R. Ellis and H. Gottschalk [7] and J. Auslander [1]). The case of not necessarily the same acting group was considered in only one paper so far, namely [9]. In Section 3 we call these morphisms "skew-morphisms" and give several natural situations where they appear. We use them in a systematic manner in the rest of the paper.

In Section 4 we introduce the notion of a continuous map good over a point, give examples and prove some statements with this notion.

The first important statement about lifting of almost periodicity was given by R. Ellis in [5] for compact flows. Later N. Markley and others obtained some statements for not necessarily compact flows. In [10] Markley said that his results "differ from other results of this genre in that we do not assume that either space is compact." Then S.H.A. Kutaibi, F. Rhodes and others ([9], [13], [14]) proved various "noncompact" statements under different assumptions. Some related results were also obtained by R. Sacker and G. Sell (see the Structure Theorem and the Equicontinuous Lifting Theorem in [15]) and by

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W. Shen and Y. Yi (see the Lifting Properties Theorem in [16]). Finally a result of V. Pestov in [12] can be considered as a lifting through a skew-morphism of flows.

Our goal is to find a theorem which unifies various known statements about lifting of almost periodicity of a point in both the compact and the not necessarily compact case. In Section 5 we give a first version of such a theorem.

In Section 6 we give a general version of such a theorem. In order to do that, we introduce the notion of a *skew-morphism good over a point with respect to orbit-closures*. In Section 7 we give several examples of this notion.

In Section 8 we show that various other statements about lifting of almost periodicity of a point ("compact" and "non-compact" as well) are corollaries of our criterion. We get, as corollaries, results of Ellis, Markley, Kutaibi-Rhodes, Pestov.

2. Notations and preliminaries

2.1. If X is a set, we denote its cardinality by $|X|$. All topological spaces are assumed to be Hausdorff. If T is a topological group, T_d denotes the group T equipped with the discrete topology.

2.2. Let X,Y be topological spaces, $f : X \to Y$ a continuous map. Then the map $g : X \to \mathrm{Gr}(f)$, defined by $g(x) = (x, f(x))$, is a homeomorphism. (Here $\mathrm{Gr}(f) = \{(x, f(x))|x \in X\}$ is considered as a subspace of $X \times Y$.)

2.3. $\mathbb{T}$ will denote the topological group of complex numbers of module 1. If T is an abelian group, the continuous homomorphisms $\chi : T \to \mathbb{T}$ are called *continuous characters* of T. The set of all continuous characters of T will be denoted by $\widehat{T}$.

2.4. Let T be a topological group. A subset A of T is *syndetic* if there exists a compact subset K of T such that $T = KA$. If S is a syndetic subgroup of T, the quotient space T/S is compact. A subset A of T is *discretely syndetic* if it is a syndetic subset of T_d.

2.5. Let $h : T \to T'$ be a surjective group homomorphism having the *compact-covering property* (i.e., for every compact K' in T' there is a compact K in T such that $h(K) = K'$). Then if S' is a syndetic subset of T', $h^{-1}(S')$ is a syndetic subset of T.

This statement is from [9]. The proof is similar to the proof of Lemma 5.2 below.

2.6 ([3]). Let X and Y be topological spaces, $f : X \to Y$ a continuous map. We say that (X, f) is a *covering* of Y if for each point $y \in Y$ there is an open neighborhood V of y such that $f^{-1}(V)$ is a nonempty disjoint union of open subsets U_i, $i \in I$, of X, on which the restrictions $f_i : U_i \to V$ of f are homeomorphisms.

An open neighborhood V of a point $y \in Y$ is called *elementary* if it satisfies the above condition. An open neighborhood U of a point $x \in X$ is called *elementary* if there is an elementary neighborhood V of the point $y = f(x)$ such that U is one of the disjoint open subsets U_i, $i \in I$, of X, whose union is equal to $f^{-1}(V)$.

A homeomorphism $g : X \to X$, $x \mapsto gx$, is called a *deck-transformation* of the covering (X, f) if $f(gx) = f(x)$ for all $x \in X$. The deck-transformations form a group $\triangle$ under composition (written as $(g, g') \mapsto gg'$). We say that $\triangle$ is *transitive* on the fiber $f^{-1}(y)$ of a point $y \in Y$ if for any two elements $x, x' \in f^{-1}(y)$ there is an element $g \in \triangle$ such that $x' = gx$.

If (X, f) is a covering of Y, the fibers of f are discrete. Also f is a surjective local homeomorphism. In particular, f is open.

2.7. A triple $\mathcal{X} = \langle T, X, \pi \rangle$ consisting of a topological group T, a topological space X and a continuous action $\pi : T \times X \to X$ of T on X is called a *flow* on X. We write $t \cdot x$ or tx for $\pi(t, x)$. We say that $\mathcal{X}$ is *compact* (resp. *abelian*), if X is compact (resp. if T is abelian). We say that $\mathcal{X}$ is *trivial* if $|X| = 1$. For $x \in X$ we denote by $\pi^x : T \to X$ the *orbital* map $t \mapsto t \cdot x$. For $t \in T$ we denote by π_t the *transition* homeomorphism $x \mapsto t \cdot x$.

2.8. A flow $\mathcal{X}_S = \langle S, X, \pi\big|_{X \times S} \rangle$, where S is a subgroup of T, will be called a *restriction* of the flow $\mathcal{X} = \langle T, X, \pi \rangle$. Usually it is denoted simply by $\mathcal{X}_S = \langle S, X \rangle$. If a subset Y of X is invariant under the action of T, then the canonical flow $\langle T, Y \rangle$ is a *subflow* of $\mathcal{X}$.

2.9. Let $\mathcal{X} = \langle T, X \rangle$ and $\mathcal{Y} = \langle T, Y \rangle$ be flows. A map $f : X \to Y$ is a *morphism* of flows if it is continuous and $f(tx) = tf(x)$ for all $t \in T$ and $x \in X$. If f is surjective, $\mathcal{Y}$ is a *factor* of $\mathcal{X}$, and $\mathcal{X}$ is an *extension* of $\mathcal{Y}$.

2.10. Let $\mathcal{X} = \langle T, X \rangle$ be a flow. A continuous function $\eta : X \to \mathbb{T}$ is an *eigenfunction* of $\mathcal{X}$ if there is a continuous character $\chi \in \widehat{T}$ such that $\eta(t \cdot x) = \chi(t)\eta(x)$ for $(t, x) \in T \times X$. In that case χ is an *eigenvalue* of $\mathcal{X}$ (the eigenvalue which *corresponds* to η) and η is an eigenfunction which *corresponds* to χ.

2.11. A flow $\mathcal{X} = \langle T, X \rangle$ is *minimal* if the orbit $T \cdot x$ of every point $x \in X$ is dense in X. It is *totally minimal* if the flow $\mathcal{X}_S$ is minimal for every syndetic (equivalently, closed syndetic) subgroup of T. If $f : \mathcal{X} \to \mathcal{Y}$ is a surjective morphism of flows, then if $\mathcal{X}$ is minimal (resp. totally minimal), $\mathcal{Y}$ is minimal (resp. totally minimal).

2.12. Every compact flow contains a minimal set ([2], [6], [8], [17]).

2.13. For $x \in X$ and $U, V \subset X$, the *dwelling set* $D(U, V)$ (resp. $D(x, V)$) is the set of all $t \in T$ such that $t \cdot U \bigcap V \neq \emptyset$ (resp. $t \cdot x \in V$).

2.14. Let $\mathcal{X} = \langle T, X \rangle$ be a flow. A point $x \in X$ is *almost periodic* (in $\mathcal{X}$) if for every neighborhood U of x there is a syndetic subset A of T such that $Ax \subset U$, i.e., the dwelling set $D(x, U)$ is syndetic in T. A point $x \in X$ is *discretely almost periodic* if it is almost periodic in the flow $\mathcal{X}_d = \langle T_d, X \rangle$, where T_d is the group T equipped with the discrete topology. Every discretely almost periodic point is almost periodic. A flow $\mathcal{X}$ is *pointwise almost periodic* if every point $x \in X$ is almost periodic.

2.15. Let $\mathcal{X} = \langle T, X \rangle$ be a flow, $x \in X$. Let Y be an invariant subset of X which contains x and let $\mathcal{Y} = \langle T, Y \rangle$ be the subflow of $\mathcal{X}$ on Y. Then x is almost periodic in $\mathcal{X}$ if and only if x is almost periodic in $\mathcal{Y}$.

2.16. Let $\mathcal{X} = \langle T, X \rangle$ be a flow, $x \in X$. If x has a compact neighborhood, then x is almost periodic iff $\overline{Tx}$ is compact minimal. In particular, a point x in a compact flow $\mathcal{X}$ is almost periodic if and only if $\overline{Tx}$ is minimal ([2], [6], [8], [17]).

2.17. Let $\mathcal{X} = \langle T, X \rangle$ be a *compact* flow. Then ([2], [6], [8], [17]):

(i) A point $x \in X$ is almost periodic if and only if it is discretely almost periodic.
(ii) $\mathcal{X}$ is pointwise almost periodic if and only if every orbit closure in $\mathcal{X}$ is minimal.
(iii) If $\mathcal{X}$ is minimal, every point $x \in X$ is almost periodic.
(iv) There is at least one almost periodic point of $\mathcal{X}$.
(v) Let S be a syndetic normal subgroup of T, $\mathcal{X}_S = \langle S, X \rangle$ a restriction of $\mathcal{X}$, $x \in X$; then x is almost periodic in $\mathcal{X}$ if and only if x is almost periodic in $\mathcal{X}_S$.

3. The notion of a skew-morphism of flows

Definition 3.1. Let $\mathcal{X} = \langle T, X \rangle$, $\mathcal{Y} = \langle T', Y \rangle$ be two flows. A pair of maps (h, f), where $h : T \to T'$ is a continuous group homomorphism and $f : X \to Y$ is a continuous map, is called a *skew-morphism* of flows if

$$f(tx) = h(t)f(x)$$

for all $t \in T$ and all $x \in X$. We write $(h, f) : \mathcal{X} \to \mathcal{Y}$.

A skew-morphism (h, f) is called a *skew-isomorphism* if h is an isomorphism of topological groups and f is a homeomorphism.

Example 3.2. Let $\mathcal{X} = \langle T, X \rangle$, $\mathcal{Y} = \langle T, Y \rangle$ be two flows with the same acting group T and let $f : X \to Y$ be a morphism of flows. Then $(\mathrm{id}_T, f) : \mathcal{X} \to \mathcal{Y}$ is a skew-morphism. Also if $\mathcal{X}_d = \langle T_d, X \rangle$, then $(\mathrm{id}_T, \mathrm{id}_X) : \mathcal{X}_d \to \mathcal{X}$ is a skew-morphism (but not necessarily a skew-isomorphism).

EXAMPLE 3.3. Let $\mathcal{X} = \langle T, X\rangle$ be a flow, $f : X \to \mathbb{T}$ be an *eigenfunction* of $\mathcal{X}$ and $\chi \in \widehat{T}$ the corresponding *eigenvalue*. Let $\mathcal{T} = \langle \mathbb{T}, \mathbb{T}\rangle$ be the flow defined by the action of the unit circle $\mathbb{T}$ on itself by multiplication. Then $(f, \chi) : \mathcal{X} \to \mathcal{T}$ is a skew-morphism.

EXAMPLE 3.4. Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism, $y \in Y$, $x \in f^{-1}(y)$. Since $f(Tx) \subset T'y$, we have $f(\overline{Tx}) \subset \overline{T'y}$. Let $f_1 : \overline{Tx} \to \overline{T'y}$ be the restriction of f to these sets. Let $\mathcal{X}' = \langle T, \overline{Tx}\rangle$ and $\mathcal{Y}' = \langle T', \overline{T'y}\rangle$ be the canonical flows. Then $(h, f_1) : \mathcal{X}' \to \mathcal{Y}'$ is a skew-morphism of flows.

EXAMPLE 3.5. Let $\mathcal{X} = \langle T, X, \pi\rangle$ be a flow, S a normal subgroup of T, $x \in X$, $t \in T$. Consider the canonical flows $\mathcal{Y} = \langle S, \overline{Sx}\rangle$ and $\mathcal{Z} = \langle S, \overline{Stx}\rangle$. Notice that $\overline{Stx} = t\overline{Sx}$. Let $h = \mathrm{Int}_t : S \to S$, $h(s) = tst^{-1}$, and let $f = \pi_t : X \to X$, $\pi_t(x) = tx$. Then $(h, f) = (\mathrm{Int}_t, \pi_t) : \mathcal{Y} \to \mathcal{Z}$ is a skew-isomorphism of flows. If T is abelian, $\mathrm{Int}_t = \mathrm{id}_S$, so we have a skew-isomorphism $(\mathrm{id}_S, \pi_t) : \overline{Sx} \to \overline{Stx}$.

EXAMPLE 3.6. Let $\mathcal{X} = \langle T, X, \pi\rangle$ be a compact minimal abelian flow, S a syndetic subgroup of T. The orbit-closures under S form a partition of X. Let R be the equivalence relation on X defined by this partition, $\widetilde{X} = X/R$ and $p_X : X \to X/R$ the canonical map. For $x \in X$ let $S^x := \{t \in T \mid tx \in \overline{Sx}\}$. It is shown in [11] that $S^x = S^y$ for any $x, y \in X$. Let $S^* := S^x$, where x is an arbitrary element of X, and let $p_T : T \to T/S^*$ be the canonical homomorphism. For $x \in X$ denote by $\tilde{x}$ the element $p_X(x)$ of $\widetilde{X}$. The function $\tilde{\pi} : T/S^* \times X/R \to X/R$, given by $\tilde{\pi}(t + S^*, \tilde{x}) = \widetilde{tx}$, defines a flow $\widetilde{\mathcal{X}} = \langle T/S^*, X/R, \tilde{\pi}\rangle$ (see the proof of Theorem 4.3 in [11] for more details). Then $(p_T, p_X) : \mathcal{X} \to \widetilde{\mathcal{X}}$ is a skew-morphism of flows.

PROPOSITION 3.7. *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism.*

(i) *If h is surjective, then $f(X)$ is an invariant subset of $\mathcal{Y}$ (and hence $\langle T', f(X)\rangle$ is a subflow of $\mathcal{Y}$).*
(ii) *If $\mathcal{X}$ is minimal and f is surjective, then $\mathcal{Y}$ is minimal.*
(iii) *If $\mathcal{X}$ is totally minimal, h,f are both surjective and h has the compact-covering property, then $\mathcal{Y}$ is totally minimal.*

Proof. (i) and (ii) are easy.

(iii) Fix a syndetic subset S' of T' and an element $y \in Y$. By 2.5, $S = h^{-1}(S')$ is a syndetic subset of T. Let $x \in f^{-1}(y)$. Then $\overline{Sx} = X$. Hence $\overline{S'y} = \overline{h(S)y} = \overline{h(S)f(x)} = \overline{f(Sx)} \supset f(\overline{Sx}) = f(X) = Y$. So $\mathcal{Y}$ is totally minimal. $\square$

The proofs of the next two propositions are the same as the proofs in the case of morphisms.

Proposition 3.8 ([2], [6], [8], [17] for morphisms). *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective. Let $x \in X$, $y = f(x)$. Then if x is almost periodic in $\mathcal{X}$, y is almost periodic in $\mathcal{Y}$.*

Proposition 3.9 ([5], [17, II(7.10)] for morphisms). *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two compact flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective. Let $y \in Y$ be an almost periodic point of $\mathcal{Y}$. Then the set $f^{-1}(y)$ contains an almost periodic point of $\mathcal{X}$.*

Remark 3.10. Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-isomorphism, $x \in X$, $y = f(x)$. Then x is almost periodic in $\mathcal{X}$ if and only if y is almost periodic in $\mathcal{Y}$.

4. The notion of a continuous map good over a point

Definition 4.1. Let X and Y be topological spaces, $y \in Y$. A continuous map $f : X \to Y$, is said to be *good over* y if the fiber $f^{-1}(y) = \{x_i \mid i \in I\}$ is finite and if, given neighborhoods U_i of x_i, $i \in I$, there exists a neighborhood V of y, such that:

(G) $f^{-1}(V) \subset \bigcup_{i\in I} U_i$.

Remark 4.2. Whenever the fiber $f^{-1}(y)$ is *empty*, f is good over y (the condition (G) being trivially satisfied).

Proposition 4.3. *Let X and Y be topological spaces, $y \in Y$. A continuous map $f : X \to Y$ is* good over y *if and only if the fiber $f^{-1}(y) = \{x_i \mid i \in I\}$ is nonempty finite and, given neighborhoods U_i of x_i, $i \in I$, there exist neighborhoods W_i of x_i, $i \in I$, and V of y, such that:*

(G1) *$W_i \subset U_i$, for all $i \in I$;*
(G2) *$f^{-1}(V) = \bigcup_{i\in I} W_i$.*

Proof. Clearly (G1) and (G2) imply (G). Conversely, if (G) holds, put $W_i := f^{-1}(V) \bigcap U_i$. □

Remark 4.4. Suppose that f is good over y. Then the W_i's and V can be chosen so that, in addition, they are *open* and that the condition

$$W_i \bigcap W_j = \emptyset \text{ for all } i, j \in I,\ i \neq j$$

is satisfied.

EXAMPLE 4.5. Any *homeomorphism* $f : X \to Y$ is good over every $y \in Y$. More generally, if X is a topological space and F a finite (discrete) space, then $\mathrm{pr}_1 : X \times F \to X$ is good over every $x \in X$.

REMARK 4.6. Let X, Y be topological spaces, $f : X \to Y$ a continuous map, $y \in Y$, and suppose that $f^{-1}(y) = \{x\}$. Then f is good over y if and only if for every neighborhood U of x there is a neighborhood V of y such that $f^{-1}(V) \subset U$. Then, in particular, the canonical surjection $f' : X \to f(X)$, deduced from f, is *open* at x.

PROPOSITION 4.7. *Let X be a compact space, $f : X \to Y$ a continuous map, $y \in Y$ a point with a finite fiber. Then f is good over y.*

Proof. The statement follows from the following standard fact:

Let X be a compact space, $f : X \to Y$ a continuous map and $y \in Y$. Then for every open neighborhood U of $f^{-1}(y)$ there is a neighborhood V of y with $f^{-1}(V) \subset U$.

(A continuous map f from a compact space X to a Hausdorff space Y is closed, so $f(X \setminus U)$ is closed in Y. Since the latter set cannot contain y, the open set $V = Y \setminus f(X \setminus U)$ is a neighborhood of y and has the desired property.) □

REMARK 4.8. Let X and Y be topological spaces, $f : X \to Y$ a continuous map and $y \in Y$ a point with a finite fiber.

(i) If X is compact, f is good over y but not necessarily a local homeomorphism. (Consider the subsets of $\mathbb{R}^2$: $X = \{(a,1) | -1 \le a \le 1\} \cup \{(0,2)\}$, $Y = \{(a,0) | -1 \le a \le 1\}$, the map $f : (x,y) \mapsto (x,0)$, and the point $y = (0,0)$.)

(ii) If (X, f) is a *covering* of Y, f is good over y and a local homeomorphism.

(iii) If f is a *local homeomorphism*, f is not necessarily good over y. (Consider the subsets of $\mathbb{R}^2$: $X = \{(a,b) | -1 \le a \le 1,\ b \in \{1,2\}\} \setminus \{(0,2)\}$, $Y = \{(a,0) | -1 \le a \le 1\}$, the map $f : (x,y) \mapsto (x,0)$, and the point $y = (0,0)$).

PROPOSITION 4.9. *Let X, Y be topological spaces, $f : X \to Y$ a continuous map, $y \in Y$.*

(i) *Let X_1 either be a* closed *subspace of X or contain $f^{-1}(y)$. Let $f_1 : X_1 \to Y$ be the restriction of f to X_1. Then if f is good over y, f_1 is good over y.*

(ii) *Let X_1 be a neighborhood of $f^{-1}(y)$ in X. Let $f_1 : X_1 \to Y$ be the restriction of f to X_1. Then f is good over y if and only if f_1 is good over y.*

(iii) *Let Y' be a subspace of Y which contains $f(X)$. Let $f' : X \to Y'$ be the canonical map deduced from f. Then f is good over y if and only if f' is good over y.*

Proof. (i) Suppose that f is good over y and let $f^{-1}(y) = \{x_i|\ i \in I\}$. For each $i \in I$ let U_i be an arbitrary neighborhood of x_i in X. Taking smaller neighborhoods if necessary we may assume that for each x_i which is not in X_1, $U_i \cap X_1 = \emptyset$. It is then clear that for each neighborhood V of y, $f^{-1}(V) \subset \bigcup_{i\in I} U_i$ implies $f_1^{-1}(V) \subset \bigcup_{i\in J}(U_i \cap X_1)$, where $J \subset I$ is such that $\{x_i|\ i \in J\} = f^{-1}(y) \cap X_1$.

(ii) Let $f^{-1}(y) = \{x_i|\ i \in I\}$ and for each $i \in I$ let U_i be an arbitrary neighborhood of x_i in X_1. The statement follows from the fact that then for each $i \in I$, U_i is a neighborhood of x_i in X as well.

(iii) Clear. □

Example 4.10. Let X, Y, f and y be as in Remark 4.8(i). Let $X_1 = X \setminus \{(0,1)\}$. Then f is good over y, but the restriction $f_1 : X_1 \to Y$ is not. This shows that 4.9(i) is no longer true if neither X_1 is closed in X nor X_1 contains $f^{-1}(y)$.

5. A criterion for lifting of almost periodicity of a point (first version)

The following lemma is a part of a proof from [12].

Lemma 5.1. *Let $\mathcal{X} = \langle T, X, \pi\rangle$ be a flow, $x \in X$. Then for every neighborhood V of x there are a neighborhood W of x and a neighborhood O of the unit element $e \in T$ such that $OD(x,W) \subset D(x,V)$.*

Proof. Fix a neighborhood V of x. Since $\pi : T \times X \to X$ is continuous at (e,x), there is a neighborhood W of x and a neighborhood O of e such that $OW \subset V$. We claim that then $OD(x,W) \subset D(x,V)$. Indeed, let $o \in O$ and let $t \in D(x,W)$. Then $tx \in W$, hence $o(tx) \in OW$, and therefore $(ot)x \in V$, i.e., $ot \in D(x,V)$. □

Lemma 5.2. *Let $h : T \to T'$ be a surjective group homomorphism. Then for every discretely syndetic subset S' of T', $h^{-1}(S')$ is discretely syndetic in T.*

Proof. There is a finite subset $F' = \{b'_1, , \cdots , b'_n\}$ of T' such that $T' = F'S'$. For every $b'_i \in F'$ let $b_i \in T$ be such that $h(b_i) = b'_i$. Let $F = \{b_1, \cdots , b_n\}$. We claim that $T = Fh^{-1}(S')$. Indeed, for $t \in T$, let $h(t) = b's'$. Put $s = b^{-1}t$. Then $h(s) = h(b)^{-1}h(t) = b'^{-1}b's' = s'$, so $s \in h^{-1}(S')$. We have $t = b \cdot b^{-1}t \in F \cdot h^{-1}(S')$. □

The following lemma is a part of a proof from [10].

LEMMA 5.3. *Let T be a topological group, S a syndetic subset of T, $S_1, \cdots, S_n$ subsets of S such that $S = \bigcup_{i=1}^n S_i$, $t_1, \cdots, t_n$ elements of T. Then the set $\bigcup_{i=1}^n t_i S_i$ is syndetic.*

Proof. Let K be a compact subset of T such that $T = KS$. We have: $(\bigcup_{i=1}^n K t_i^{-1}) \cdot (\bigcup_{i=1}^n t_i S_i) \supset \bigcup_{i=1}^n K t_i^{-1} t_i S_i = \bigcup_{i=1}^n K S_i = K(\bigcup_{i=1}^n S_i) = KS = T$, and the set $\bigcup_{i=1}^n K t_i^{-1}$ is compact. So the set $\bigcup_{i=1}^n t_i S_i$ is syndetic. $\square$

The following theorem and its corollary are the first versions of the *criterion for lifting of almost periodicity of a point.*

THEOREM 5.4. *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective, $y \in Y$. Suppose that the following two conditions hold:*

(OC_1) *For any $x, x' \in f^{-1}(y)$, $\overline{Tx} = \overline{Tx'}$.*

(GR_1) *If x is an element of $f^{-1}(y)$, the restriction $f_1 : \overline{Tx} \to Y$ of f is good over y.*

Then for any $x \in f^{-1}(y)$, x is almost periodic in $\mathcal{X}$ if and only if y is almost periodic in $\mathcal{Y}$.

Proof. $\Longleftarrow$: Suppose y is almost periodic in $\mathcal{Y}$. Let $x \in f^{-1}(y)$. By the assumption (GR_1), the fiber $f^{-1}(y)$ is finite, say $f^{-1}(y) = \{x = x_1, x_2, \cdots, x_n\}$. Fix any open neighborhood U of x in $\overline{Tx}$. Put $U_1 = U$ and $t_1 = e$. It follows from the assumption (OC_1) that for each $i \in \{2, 3, \cdots, n\}$ there is an open neighborhood U_i of x_i in $\overline{Tx}$ and $t_i \in T$ such that $t_i U_i \subset U$. Choose disjoint open neighborhoods $W_i \subset U_i$, $i = 1, 2, \cdots, n$, of the points x_i in $\overline{Tx}$ and an open neighborhood V of y so that the conditions (G1), (G2) are satisfied. By Lemma 5.1, there is a neighborhood V' of y and a neighborhood O of the unit element $e_{T'}$ in T', such that $OD(y, V') \subset D(y, V)$. Also there is a compact set $K' \subset T'$ such that $T' = K'D(y, V')$. We have $K' \subset F'O$ for some finite subset F' of T'. Thus $T' \subset F'OD(y, V') \subset F'D(y, V) \subset T'$, so $T' = F'D(y, V)$. By Lemma 5.2, $S = h^{-1}(D(y, V))$ is discretely syndetic in T and hence syndetic in T. Because of (G2) we have $Sx \subset \bigcup_{i=1}^n W_i$ (since for every $s \in S$, $f(sx) = h(s)y \in V$). Let $S_i = \{s \in S | sx \in W_i\}$, $i = 1, 2, \cdots, n$. If for $s \in S$, $sx \in W_i$ for some $i = 1, 2, \cdots, n$, then $t_i sx \in t_i W_i \subset t_i U_i \subset U$ (here we used (G1)), hence for every $s \in S$, $s \in S_i$ implies $t_i s \in D(x, U)$. Consequently $D(x, U) \supset \bigcup_{i=1}^n t_i S_i$. Since $S = \bigcup_{i=1}^n S_i$, the set $\bigcup_{i=1}^n t_i S_i$ is syndetic in T by Lemma 5.3. Hence $D(x, U)$ is syndetic and so x is almost periodic in the subflow $\langle T, \overline{Tx}\rangle$ of $\mathcal{X}$. Consequently x is almost periodic in $\mathcal{X}$.

$\Longrightarrow$: Follows from Proposition 3.8. $\square$

COROLLARY 5.5. *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective, $y \in Y$. Suppose that f is good over y*

and that for any $x, x' \in f^{-1}(y)$, $\overline{Tx} = \overline{Tx'}$. *Then for any* $x \in f^{-1}(y)$, x *is almost periodic in* $\mathcal{X}$ *if and only if* y *is almost periodic in* $\mathcal{Y}$.

Proof. Since f is good over y, then (by Proposition 4.9) for any element $x \in f^{-1}(y)$ the restriction $f_1 : \overline{Tx} \to Y$ of f is good over y. So the statement follows from Theorem 5.4. $\square$

Example 5.6. Let $\mathcal{X} = \langle T, X\rangle$ be a compact flow. Consider the action of the group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ on $X \times X$ such that

$$(1 + 2\mathbb{Z}) \cdot (x, y) = (y, x),$$

for all $(x, y) \in X \times X$. It commutes with the canonical action of T on $X \times X$. Let $\mathcal{Y} := (\mathcal{X} \times \mathcal{X})/\mathbb{Z}_2$ be the canonical T-flow on the quotient space $Y := (X \times X)/\mathbb{Z}_2$ and let $q : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be the canonical map. Denote $[(x, y)] := q(x, y)$. Note that for all $x, y \in X$, $q^{-1}([(x, y)]) = \{(x, y), (y, x)\}$ if $x \neq y$, and $q^{-1}([(x, x)]) = \{(x, x)\}$. By Proposition 4.7, the map q is good over every point $[(x, y)]$ of Y. Also, for every $[(x, y)] \in Y$, if $(y, x) \in \overline{T(x, y)}$ then $(x, y) \in \overline{T(y, x)}$, so $\overline{T(x, y)}$ and $\overline{T(y, x)}$ are either equal to each other or disjoint. So, by Corollary 5.5, for every $(x, y) \in X \times X$, (x, y) is almost periodic in $\mathcal{X} \times \mathcal{X}$ if and only if $[(x, y)]$ is almost periodic in $(\mathcal{X} \times \mathcal{X})/\mathbb{Z}_2$.

6. A criterion for lifting of almost periodicity of a point (general version)

Definition 6.1. Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, y a point from Y. A skew-morphism $(h, f) : \mathcal{X} \to \mathcal{Y}$ is said to be *good over* y *with respect to orbit closures* if the following two conditions hold:

(OC) For any $x, x' \in f^{-1}(y)$, $x' \in \overline{Tx}$ implies $x \in \overline{Tx'}$.

(GR) For any $x \in f^{-1}(y)$, the restriction $f_1 : \overline{Tx} \to Y$ of f is good over y.

A morphism $f : \mathcal{X} \to \mathcal{Y}$ of flows $\mathcal{X} = \langle T, X\rangle$ and $\mathcal{Y} = \langle T, Y\rangle$ is said to be *good over a point* $y \in Y$ *with respect to orbit closures* if the skew-morphism $(\mathrm{id}_T, f) : \mathcal{X} \to \mathcal{Y}$ is good over y with respect to orbit closures.

Remark 6.2. The condition (OC) is weaker than the condition:

(OC′) For any $x, x' \in f^{-1}(y)$, $\overline{Tx}$ and $\overline{Tx'}$ are either equal to each other or disjoint.

If, for example, $\overline{Ty}$ is minimal, (OC) and (OC′) are equivalent. (This is the case in Proposition 7.3 below.)

The condition (GR) requires that all f_1's, but not necessarily f, are good over y. Hence the fiber $f^{-1}(y)$ can be infinite. (If f is good over y, the condition (GR) is automatically satisfied by Proposition 4.9.)

Example 6.3. Let $\mathcal{X} = \langle T, X\rangle$ and $\mathcal{Y} = \langle T', Y\rangle$ be two flows and $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h *surjective* and f a *homeomorphism.* Then for every $y \in Y$, (h, f) is good over y with respect to orbit-closures.

Theorem 6.4 (Criterion for lifting of almost periodicity of a point). *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective. Let $y \in Y$ be a point such that (h, f) is good over y with respect to orbit-closures. Then for any $x \in f^{-1}(y)$, x is almost periodic in $\mathcal{X}$ if and only if y is almost periodic in $\mathcal{Y}$.*

Proof. Let $x \in f^{-1}(y)$. Let $\mathcal{X}_1 = \langle T, \overline{Tx}\rangle$ be the subflow of $\mathcal{X}$ on $\overline{Tx}$. Let $f_1 : \overline{Tx} \to Y$ be the restriction of f to $\overline{Tx}$. Then the skew-morphism $(h, f_1) : \mathcal{X}_1 \to \mathcal{Y}$ satisfies conditions (GR_1), (OC_1) of Theorem 5.4. (The condition (GR_1) follows from the assumption (GR) and the condition (OC_1) follows from the assumption (OC).) Hence x is almost periodic in $\mathcal{X}_1$ iff y is almost periodic in $\mathcal{Y}$. By 2.15, x is almost periodic in $\mathcal{X}$ iff y is almost periodic in $\mathcal{Y}$. □

Corollary 6.5. *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective, $y \in Y$. Suppose that f is good over y and that for any $x, x' \in f^{-1}(y)$, $\overline{Tx}$ and $\overline{Tx'}$ are either equal to each other or disjoint. Then for any $x \in f^{-1}(y)$, x is almost periodic in $\mathcal{X}$ if and only if y is almost periodic in $\mathcal{Y}$.*

Proof. Follows from Proposition 4.9 and Theorem 6.4. □

7. Examples of skew-morphisms good over a point with respect to orbit-closures

Proposition 7.1. *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective. Suppose that (X, f) is a covering of Y. Let $y \in Y$ be a point with a finite fiber. Suppose that each deck-transformation of (X, f) is an automorphism of the flow $\mathcal{X}$ and that the group $\triangle$ of deck-transformations of (X, f) is transitive on $f^{-1}(y)$. Then:*

(i) *f is good over y.*
(ii) *(h, f) is good over y with respect to orbit closures.*

Proof. (i) Proved in Remark 4.8(ii).

(ii) The condition (GR) follows from (i). Let's check (OC). Observe that for every $g \in \triangle$ and $x', x'' \in X$, $x'' \in \overline{Tx'}$ implies $gx'' \in \overline{Tgx'}$ since g is an automorphism of $\mathcal{X}$. Fix any $x \in f^{-1}(y)$ and let $x' \in f^{-1}(y) \cap \overline{Tx}$. Let $g \in \triangle$ be such that $gx = x'$. From $gx \in \overline{Tx}$ we have (using the above observation) $g^2x \in \overline{Tgx} \subset \overline{Tx}$. Then $g^3x \in \overline{Tg^2x} \subset \overline{Tgx}$, etc. Since all elements $x, gx, g^2x, \cdots$ are in the finite set $f^{-1}(y)$, there is a smallest $n \geq 1$ such that $g^nx = x$. We have $\overline{Tx} = \overline{Tg^nx} \subset \overline{Tg^{n-1}x} \subset \cdots \subset \overline{Tgx} \subset \overline{Tx}$. Hence $\overline{Tx} = \overline{Tgx} = \overline{Tx'}$. Hence (OC) holds. □

Proposition 7.2. *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective. Let y be a point of Y which* *has a neighborhood V such that $f^{-1}(\overline{V})$ is compact and suppose that for any $x, x' \in f^{-1}(y)$, $\overline{Tx}$ and $\overline{Tx'}$ are either equal to each other or disjoint. Let $x \in f^{-1}(y)$ be such that $\overline{Tx} \cap f^{-1}(y)$ is finite. Let $f_1 : \overline{Tx} \to \overline{T'y}$ be the restriction of f to $\overline{Tx}$ and let $\mathcal{X}_1 = \langle T, \overline{Tx} \rangle$ be the subflow of $\mathcal{X}$ on $\overline{Tx}$. Then:*

(i) *f_1 is good over y.*
(ii) *$(h, f_1) : \mathcal{X}_1 \to \mathcal{Y}$ is good over y with respect to orbit closures.*

Proof. (i) Since $f_1^{-1}(\overline{V}) = f^{-1}(\overline{V}) \cap \overline{Tx}$, $K := f_1^{-1}(\overline{V})$ is compact. Also K is a neighborhood of $f_1^{-1}(y)$ in $\overline{Tx}$ and $f_1^{-1}(y) = f^{-1}(\overline{y}) \cap \overline{Tx}$ is finite by assumption. Since, by Proposition 4.7, the restriction $f_2 : K \to Y$ of f_1 is good over y, the map f_1 is also good over y (by Proposition 4.9).

(ii) Follows immediately from (i) and the assumption about orbit-closures of elements of $f^{-1}(y)$. □

Proposition 7.3. *Let $\mathcal{X} = \langle T, X \rangle$, $\mathcal{Y} = \langle T', Y \rangle$ be two flows, $\mathcal{X}$ compact, and let $(h, f) : \mathcal{X} \to \mathcal{Y}$ be a skew-morphism with h surjective and f locally injective. Let $y \in Y$ be an almost periodic point of $\mathcal{Y}$. Then:*

(i) *f is good over y.*
(ii) *(h, f) is good over y with respect to orbit closures.*

Proof. (i) For each point $z \in X$ we can choose an open neighborhood O_z such that f is injective on O_z. Since X is compact there are finitely many points $z_1, z_2, \cdots, z_n$ such that $O_{z_1} \cup \cdots \cup O_{z_n}$ covers X. Each of these sets can contain at most one element from $f^{-1}(y)$. Hence $f^{-1}(y)$ is finite. Now by Proposition 4.7, f is good over y.

(ii) The condition (GR) follows from (i). Let's check (OC). We may assume that f is surjective and hence Y compact. Fix any $x \in f^{-1}(y)$. Let x' be another point from $f^{-1}(y)$ and suppose $x' \in \overline{Tx}$. Suppose that $x \notin \overline{Tx'}$. Let $f^{-1}(y) = \{x = x_1, x_2, \cdots, x' = x_m, x_{m+1}, \cdots, x_n\}$. Without loss of generality we may assume that $\overline{Tx'} \cap f^{-1}(y) = \{x_m, x_{m+1}, \cdots, x_n\}$. Using the compactness of X and the fact that f is good over y, we can find (choosing conveniently neighborhoods U_i of x_i, $i = 1, \cdots, n$) open pairwise disjoint neighborhoods W_i of x_i, $i = 1, 2, \cdots, n$, and V of y, so that at the same time the conditions (G1), (G2) are satisfied, f is injective on each of W_i, $i = 1, 2, \cdots, n$, and $\overline{Tx'}$ is disjoint from every $\overline{W_i}$, $i = 1, 2, \cdots, m-1$. Let $S' = D(y, V)$. Then by 2.17(i), $T' = F'S'$, where F' is a finite subset of T'. Hence by Lemma 5.2, $T = Fh^{-1}(S')$, where F is a finite and $S = h^{-1}(S')$ a syndetic subset of T. There is a net $t_\alpha s_\alpha x_1 \to x_m$ with $t_\alpha \in F$ and $s_\alpha \in S$. The net (t_α) in F has a convergent subnet $t_\beta \to t$. Since $t_\beta s_\beta x_1 \to x_m$, we have $t s_\beta x_1 \to x_m$. Hence $s_\beta x_1 \to t^{-1} x_m$. Since $f(s_\beta x_1) = h(s_\beta) y \in V$, $s_\beta x_1 \in \bigcup_{i=1}^n W_i = f^{-1}(V)$. At the same time $t^{-1} x_m \in \overline{Tx'}$. Since $\overline{Tx'}$ is disjoint from each $\overline{W_i}$ for $i = 1, 2, \cdots, m-1$, we have that for $\beta \geq \beta_0$ (for some β_0) all $s_\beta x_1$ are in $\bigcup_{i=m}^n W_i$. Fix some $s_\beta x_1 \in W_j$, $j \in \{m, m+1, \cdots, n\}$. For each

$i = m, m+1, \cdots, n$, $s_\beta x_i \in \bigcup_{p=m}^{n} W_p$ (which must be in $\overline{Tx'}$ and in $\bigcup_{i=1}^{n} W_i$ at the same time). So there are two of the points $s_\beta x_1, s_\beta x_m, s_\beta x_{m+1}, \cdots, s_\beta x_n$ in one of the sets $W_m, \cdots, W_n$. The image under f_1 of each of them is $h(s_\beta)y$. Since f is injective on each of $W_m, \cdots, W_n$, these two points should be equal to each other, a contradiction. Hence $x \in \overline{Tx'}$, i.e., the condition (OC) is satisfied. □

Corollary 7.4. *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows and suppose that all orbit closures of $\mathcal{X}$ are compact. Let $(h, f) : \mathcal{X} \to \mathcal{Y}$ be a skew-morphism with h surjective, f locally injective and let $y \in Y$ be an almost periodic point of $\mathcal{Y}$. Then (h, f) is good over y with respect to orbit closures.*

Proof. For each point x from $f^{-1}(y)$, we can apply Proposition 7.3 to the skew-morphism $(h, f_1) : \mathcal{X}_1 \to \mathcal{Y}$, where $\mathcal{X}_1 = \langle T, \overline{Tx}\rangle$ is the subflow of $\mathcal{X}$ on $\overline{Tx}$ and $f_1 : \overline{Tx} \to Y$ is the restriction of f to $\overline{Tx}$. □

8. Applications of the criterion for lifting of almost periodicity of a point

Corollary 8.1. *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective and f a homeomorphism. Let $y \in Y$ and let $x \in f^{-1}(y)$. Then y is almost periodic in $\mathcal{Y}$ if and only if x is almost periodic in $\mathcal{X}$.*

Proof. Follows from Example 6.3 and Theorem 5.4. □

Corollary 8.2 ([12, Theorem]). *Let $\mathcal{X} = \langle T, X\rangle$ be a flow and x a point of X. Then x is almost periodic if and only if it is discretely almost periodic.*

Proof. Apply Corollary 8.1 to $(\mathrm{id}_T, \mathrm{id}_X) : \mathcal{X}_d \to \mathcal{X}$, where $\mathcal{X}_d = \langle T_d, X\rangle$. □

Corollary 8.3 ([10, Theorem 2.1] with $T = T'$ and $h = \mathrm{id}_T$). *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective. Suppose that (X, f) is a covering of Y all of whose fibers are finite. Let $y \in Y$ and let $x \in f^{-1}(y)$. Suppose that each deck-transformation of (X, f) is an automorphism of the flow $\mathcal{X}$ and that the group of deck-transformations of (X, f) is transitive on $f^{-1}(y)$. Then y is almost periodic in $\mathcal{Y}$ if and only if x is almost periodic in $\mathcal{X}$.*

Proof. Follows from Proposition 7.1 and Theorem 5.4. □

Corollary 8.4 ([9, Proposition 4.3] with $T = T'$ and $h = \mathrm{id}_T$). *Let $\mathcal{X} = \langle T, X\rangle$, $\mathcal{Y} = \langle T', Y\rangle$ be two flows, $(h, f) : \mathcal{X} \to \mathcal{Y}$ a skew-morphism with h surjective. Suppose that whenever $x_1, x_2 \in X$ are in the same fiber, their orbit-closures are either equal to each other or disjoint. Let y be a point of Y*

which has a neighborhood V such that $f^{-1}(\overline{V})$ is compact and let $x \in f^{-1}(y)$ be such that $\overline{Tx} \cap f^{-1}(y)$ is finite. Then y is almost periodic in $\mathcal{Y}$ if and only if x is almost periodic in $\mathcal{X}$.

Proof. Let $f' : \overline{Tx} \to \overline{T'y}$ be the restriction of f, $\mathcal{X}' = \langle T, \overline{Tx} \rangle$, $\mathcal{Y}' = \langle T', \overline{T'y} \rangle$ the canonical flows. Then, by Proposition 7.2, $(h, f') : \mathcal{X}' \to \mathcal{Y}'$ is good over y with respect to orbit-closures. Hence, by Theorem 5.4, y is almost periodic in $\mathcal{Y}'$ iff x is almost periodic in $\mathcal{X}'$. Also, by 2.15, y is almost periodic in $\mathcal{Y}$ iff y is almost periodic in $\mathcal{Y}'$ and x is almost periodic in $\mathcal{X}$ iff x is almost periodic in $\mathcal{X}'$. Thus y is almost periodic in $\mathcal{Y}$ iff x is almost periodic in $\mathcal{X}$. □

Corollary 8.5. *Let $\mathcal{X} = \langle T, X \rangle$ be a flow all of whose orbit-closures are compact and let $\mathcal{Y} = \langle T', Y \rangle$ be a compact flow. Let $(h, f) : \mathcal{X} \to \mathcal{Y}$ be a skew-morphism with h surjective and with f locally injective. Let $y \in Y$ be an almost periodic point in $\mathcal{Y}$ with a nonempty fiber. Then every $x \in f^{-1}(y)$ is an almost periodic point of $\mathcal{X}$.*

Proof. Follows from Proposition 7.3 and Theorem 5.4. □

Corollary 8.6 ([5, Proposition 3]). *Let $\mathcal{X} = \langle T, X \rangle$, $\mathcal{Y} = \langle T, Y \rangle$ be two compact flows and $f : \mathcal{X} \to \mathcal{Y}$ a surjective locally injective morphism. Let y be an almost periodic point of $\mathcal{Y}$. Then every $x \in f^{-1}(y)$ is an almost periodic point of $\mathcal{X}$.*

Proof. Follows from Corollary 8.5. □

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