SOME EXAMPLES FOR WEAK CATEGORY AND CONILPOTENCY

BY

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1. Introduction

We are concerned here with certain numerical invariants of homotopy type akin to the Lusternik-Schnirelmann category.

It is known that cat B, the Lusternik-Schnirelmann category of a space B (when renormalized) is an upper bound for conil B, the conilpotency class of the suspension of B [18; Theorem 2.10]. Furthermore if B is an $(n - 1)$-connected CW-complex of dimension $\leq (k + 2)n - 2$ and conil $B \leq k$ then cat B = conil B [2; Theorem 2].

Berstein and Hilton [3; (2.1)] gave a definition of category which is equivalent, for most classes of spaces, to the original one of Lusternik and Schnirelmann. This definition suggests two other invariants, wcat B, the weak category of a space B and wcat e, the weak category of the natural embedding map $e : B \rightarrow \Omega\Sigma B$ [3; (2.2)], [7; §5]. These two weak categories take values lying between those of cat B and conil B, but we will show by examples in Section 2 that all the invariants are different.

None of these definitions of category and weak category dualize easily in the sense of Eckmann-Hilton. So Ganea introduced yet another definition of category and weak category, in terms of a 'ladder' of fibrations, which does dualize. We will denote these invariants by G-cat and G-wcat respectively. (See Definition 6.1 of [5] for the cocategory of a space.) In Sections 3 and 4 we will show that G-cat B is the same invariant as cat B but that G-wcat B is different from wcat B.

We collect together the results on the relationships between the various invariants in the following theorem. All the numerical invariants in this paper will be normalized so as to take the value 0 on contractible spaces.

THEOREM 1.1. *Let B have the homotopy type of a simply connected countable* CW*-complex; then*

$$\text{cat } B = \text{G-cat } B \geq \text{G-wcat } B \geq \text{wcat } B \geq \text{wcat } e \geq \text{conil } B \geq \cup\text{-long } B$$

and furthermore all the inequalities can occur.

Here $\cup$-long B is the length of the longest nontrivial cup product of positive dimensional elements of $H^*(B; R)$, where R is any commutative ring.

Theorem 1.1 will follow from Theorems 3.4 and 3.5, [7; Theorems 4.4 and 5.2] and the remaining two inequalities follow directly from the definitions.

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Examples 4.7, 4.6, 2.4 and 2.3 will show that the first four inequalities can be strict. The example given at the end of [2] in which $B = S^2 \cup_\alpha e^5$, where $\alpha = \eta_2 \circ \eta_3$ is the generator of $\pi_4(S^2)$, shows that the last inequality can be strict.

All the examples will be spaces of the form $S^q \cup_\alpha e^n$, where $\alpha \in \pi_{n-1}(S^q)$. We will use Toda's notation [16] for the homotopy groups of spheres. All spaces in this paper have the homotopy type of countable CW-complexes and have a base point denoted by $*$ and all maps preserve base points. The constant map is denoted by 0. We will not usually distinguish between a map and its homotopy class.

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2. Weak category of the map e

In this section we recall the definitions of the various categories and find examples of spaces which distinguish wcat e from wcat B and conil B.

Let T_1^{k+1} be the subset of B^{k+1} consisting of points with at least one coordinate equal to $*$. Let $j : T_1^{k+1} \to B^{k+1}$ be the inclusion map and let $B^{(k+1)}$ be the quotient space B^{k+1}/T_1^{k+1} with identification map $q : B^{k+1} \to B^{(k+1)}$. Let $\Delta : B \to B^{k+1}$ be the diagonal map.

The category of a space B, cat B, is defined to be the least integer $k \geq 0$ for which there exists a map $\phi : B \to T_1^{k+1}$ with $j \circ \phi \simeq \Delta$. The weak category, wcat B, is the least integer $k \geq 0$ for which $q \circ \Delta \simeq 0$ and wcat e is the least integer $k \geq 0$ for which $q \circ \Delta \circ e \simeq 0 : B \to (\Omega\Sigma B)^{(k+1)}$. It is clear that wcat $B \geq$ wcat e but the two invariants are different as Example 2.4 will show.

It is proved in Theorem 3.20 of [3] that if B is a space of the form $S^q \cup_\alpha e^n$ then wcat $B \leq 1$ if and only if $\bar{H}(\alpha) = 0$ where $\bar{H} : \pi_{n-1}(S^q) \to \pi_n(S^q \wedge S^q)$ is the crude Hopf invariant [3; (2.11)]. The arguments used in the proof of this theorem may be adapted to prove the following proposition.

PROPOSITION 2.1. *Let* $B = S^q \cup_\alpha e^n$; *then* wcat $e \leq 1$ *if and only if*

$$(e \wedge e)_* \bar{H}(\alpha) = 0 \in \pi_n(\Omega\Sigma S^q \wedge \Omega\Sigma S^q).$$

The map $e : S^q \to \Omega\Sigma S^q$ is the natural embedding and $e \wedge e$ is the map from the smash product $S^q \wedge S^q$ which is e on each factor

LEMMA 2.2. *For q even, $\Omega\Sigma S^q \wedge \Omega\Sigma S^q$ has the same $(5q - 2)$-homotopy type as the cell complex*

$$T = S^{2q} \cup_\gamma e^{4q} \vee S^{3q} \vee S^{3q} \vee S^{4q} \vee S^{4q}$$

where $\gamma = 2[\iota_{2q}, \iota_{2q}] \in \pi_{4q-1}(S^{2q})$.

Proof. The space $\Omega\Sigma S^q$ is homotopic to S^q_∞, the reduced product complex of James [11], which has a cellular decomposition $S^q_\infty = S^q \cup_\xi e^{2q} \cup_\zeta e^{3q} \cup \cdots$. Milnor [12; Theorem 5] proves that

$$\Sigma\Omega\Sigma S^q \simeq S^{q+1} \vee S^{2q+1} \vee S^{3q+1} \vee \cdots.$$

Hence it follows that the suspensions of the attaching maps in S^q_∞ are trivial.

In the complex $S^q_\infty \wedge S^q_\infty$ the following are the cells of dimension less than $5q$. There is one 0-cell and one $2q$-cell. There are two $3q$-cells attached by the maps $\Sigma^q\xi = 0$ and two $4q$-cells attached by the maps $\Sigma^q\zeta = 0$. The remaining cell is a $4q$-cell with an attaching map which we shall call

$$\beta \in \pi_{4q-1}(S^{2q} \vee S^{3q} \vee S^{3q}).$$

By the direct sum decomposition in [9] we can consider β an element o $\pi_{4q-1}(S^{2q}) \oplus \pi_{4q-1}(S^{3q}) \oplus \pi_{4q-1}(S^{3q})$. Now both components of β in $\pi_{4q-1}(S^{3q})$ factor through $\Sigma^{2q}\xi = 0$. Let γ be the component of β in $\pi_{4q-1}(S^{2q})$.

From the cohomology ring of $\Omega\Sigma S^q$ [15], for q even, and the multiplication rule for the tensor product of two rings we see that the square of the cohomology generator of dimension $2q$ in $H^*(S^q_\infty \wedge S^q_\infty)$ is 4 times a generator of dimension $4q$. Hence by Steenrod's definition, the Hopf invariant of γ is 4. When $S^q_\infty \wedge S^q_\infty$ is suspended all the cells are attached trivially [12; Theorem 5], hence $\Sigma\gamma = 0 \in \pi_{4q}(S^{2q+1})$. Therefore by the delicate suspension theorem [17; (3.49)] γ is a multiple of $[\iota_{2q}, \iota_{2q}]$ and it follows from the Hopf invariant that $\gamma = 2[\iota_{2q}, \iota_{2q}]$.

Therefore T is the $(5q - 1)$-skeleton of $S^q_\infty \wedge S^q_\infty$ and it has the same $(5q - 2)$-homotopy type as $\Omega\Sigma S^q \wedge \Omega\Sigma S^q$. This completes the proof of the lemma.

Hence, for q even, there exists a map $k : T \to \Omega\Sigma S^q \wedge \Omega\Sigma S^q$ which induces isomorphisms in homotopy in dimensions $\leq 5q - 2$. Now it is clear that $(e \wedge e)_*$ factors into

$$\pi_n(S^{2q}) \xrightarrow{i_*} \pi_n(S^{2q} \cup_\gamma e^{4q}) \xrightarrow{j_*} \pi_n(T) \xrightarrow{k_*} \pi_n(\Omega\Sigma S^q \wedge \Omega\Sigma S^q)$$

where i and j are the inclusion maps and j_* maps monomorphically into a direct summand.

In Theorem 5.2 of [7] it is proved that wcat $e \geq$ conil B but it is mentioned that an example of strict inequality has not been produced. We will now use an example which occurs later in the above paper to show that the strict inequality can occur.

Example 2.3. Let $B = S^2 \cup_\alpha e^8$ where $\alpha = \eta_2 \circ \nu' \circ \eta_6$ is the generator of order 2 in $\pi_7(S^2)$. Then wcat $e = 2$ and conil $B = 1$.

Proof. Here η_k generates $\pi_{k+1}(S^k)$ and ν' generates $\pi_6(S^3)$. This example occurs in [7; (6.1)] where it is proved that conil $B = 1$. (See [1; (1.8)] for the definition of the conilpotency class of a suspension.)

By Theorem II of [4] the sequence

$$\pi_8(S^7) \xrightarrow{\gamma *} \pi_8(S^4) \xrightarrow{i*} \pi_8(S^4 \cup_\gamma e^8)$$

is exact and hence Ker $i_* = \text{Im } \gamma_*$. Since $\pi_8(S^7)$ is the cyclic group of order 2 and $\gamma = 2[\iota_4, \iota_4]$ it follows that $\gamma_* = 0$ and Ker $i_* = 0$. Now Ker $(e \wedge e)_* = \text{Ker } (k_* \circ j_* \circ i_*) = 0$ since the kernels of each of the maps i_*, j_* and k_* are zero.

The crude Hopf invariant $\bar{H}(\alpha) = \Sigma\nu' \circ \eta_7 \neq 0 \in \pi_8(S^4)$. Hence $(e \wedge e)_* \bar{H}(\alpha) \neq 0$ and wcat $e > 1$ by Proposition 2.1.

In this example and in the later examples B is a complex containing three cells and so by the classical definition of the Lusternik-Schnirelmann category cat $B \leq 2$. Hence in this case wcat $e = 2$.

Example 2.4. Let $B = S^2 \cup_\alpha e^{10}$ where $\alpha = \eta_2 \circ \alpha_1(3) \circ \alpha_1(6)$ is the generator of order 3 in $\pi_9(S^2)$. Then wcat $B = 2$ and wcat $e = 1$.

Proof. The element $\alpha_1(k)$ is an element of order 3 in $\pi_{k+3}(S^k)$. Let $H_2 : \pi_{n-1}(S^q) \to \pi_{n-1}(S^{2q-1})$ be a Hopf invariant (see Definition 4.1). For $q = 2$ and $n \geq 4$ H_2 is an isomorphism and hence $H_2(\alpha) = \alpha_1(3) \circ \alpha_1(6) \in \pi_9(S^3)$

By Proposition 4.2 $\bar{H}(\alpha) = \Sigma H_2(\alpha) = \alpha_1(4) \circ \alpha_1(7) \neq 0$ and so wcat $B = 2$. by Theorem 3.20 of [3].

Hilton [8; p. 195] proves that $[[\iota_4, \iota_4], \iota_4] = \pm\alpha_1(4) \circ \alpha_1(7)$. Therefore $\bar{H}(\alpha) = \pm[[\iota_4, \iota_4], \iota_4] = \mp[\gamma, \iota_4]$ since $\gamma = 2[\iota_4, \iota_4]$ and $\bar{H}(\alpha)$ is of order 3. By the naturality of the Whitehead product

$$i_* \bar{H}(\alpha) = \mp i_* [\gamma, \iota_4] = \mp[i_* \gamma, i_* \iota_4] = 0 \in \pi_{10}(S^4 \cup_\gamma e^8)$$

since $i_* \gamma = 0$. Hence $(e \wedge e)_* \bar{H}(\alpha) = 0$ and wcat $e = 1$ by Proposition 2.1.

3. Ganea's definition of category

Let B be a simply connected space. Define the sequence of fibrations

$$\mathfrak{F}_k : F_k \xrightarrow{i_k} E_k \xrightarrow{p_k} B \qquad (k \geq 0)$$

as follows. $\mathfrak{F}_0$ is the standard fibration in which E_0 is the space of paths in B ending at $*$, F_0 is the space of loops in B and p_k maps a path onto its starting point. Suppose inductively that $\mathfrak{F}_k$ has been defined. Let $E'_{k+1} = E_k \cup CF_k$ be the cofibre of i_k and extend p_k to a map $p'_{k+1} : E'_{k+1} \to B$ by mapping CF_k to $*$. Convert p'_{k+1} into a homotopically equivalent fibre map $p_{k+1} : E_{k+1} \to B$; this then defines $\mathfrak{F}_{k+1}$.

Definition 3.1. *G-cat* B is the least integer $k \geq 0$ for which there exists a map $r : B \to E_k$ such that $p_k \circ r \simeq 1$; if no such integer exists G-cat $B = \infty$.

When p_k is converted into a cofibre map let C_k be its cofibre and $q_k : B \to C_k$ be the induced map.

DEFINITION 3.2. *G-wcat B is the least integer* $k \geq 0$ *for which* $q_k \simeq 0$; if no such integer exists G-wcat $B = \infty$.

It is clear that G-cat $B \geq$ G-wcat B.

As in the last section let T_1^{k+1} be the subset of B^{k+1} with at least one coordinate equal to $*$ and let $j : T_1^{k+1} \to B^{k+1}$ be the inclusion map. Convert j into a fibre map

$$j' : E(B^{k+1}; B^{k+1}, T_1^{k+1}) \to B^{k+1}$$

whose domain is the space of paths in B^{k+1} starting in B^{k+1} and ending in T_1^{k+1}. Its fibre is $E(B^{k+1}; *, T_1^{k+1})$ which is homotopic to the join of $(k + 1)$ copies of ΩB [13; Theorem 2].

PROPOSITION 3.3. *The fibration* $\mathfrak{F}_k$ *is homotopic to the fibration induced by the diagonal map* $\Delta : B \to B^{k+1}$ *from the fibre map* j'.

Proof. The fibration induced by Δ from j' is

$$\mathfrak{R}_k : R_k \to Q_k \xrightarrow{h_k} B$$

where $Q_k = E(B^{k+1}; \Delta B, T_1^{k+1})$, $R_k = E(B^{k+1}; *, T_1^{k+1})$ and if $\xi \in Q_k$ then $h_k(\xi) = \pi_1 \xi(0)$, π_1 being the projection onto the first factor.

It is trivially true that $\mathfrak{F}_0$ is homotopic to $\mathfrak{R}_0$. Assume inductively that $\mathfrak{F}_{m-1}$ is homotopic to $\mathfrak{R}_{m-1}$.

$$\begin{tikzcd}
F_{m-1} \arrow[r] & E_{m-1} \arrow[r, "p_{m-1}"] \arrow[d] & B \arrow[d, equal] \\
& E_{m-1} \cup CF_{m-1} \arrow[r, "p_m'"] \arrow[d] & B \arrow[d, equal] \\
& Q_{m-1} \cup CR_{m-1} \arrow[r, "h_m'"] & B
\end{tikzcd}$$

The way E_m was constructed was to convert p_m' into a fibration. Now p_m' is homotopic to a map $h_m' : Q_{m-1} \cup CR_{m-1} \to B$ where $h_m' \mid Q_{m-1} = h_{m-1}$ and $h_m'(CR_{m-1}) = *$. Convert h_m' into the fibre map $v : U \to B$ where

$$U = \{(s\mu, \nu) \in Q_{m-1} \cup CR_{m-1} \times B^I \subset CQ_{m-1} \times B^I \mid h_m'(s\mu) = \nu(1)\}$$

and $v(s\mu, \nu) = \nu(0)$. Then v is homotopic to the map p_m and has fibre

$$V = \{(s\mu, \nu) \in Q_{m-1} \cup CR_{m-1} \times PB \mid h_m'(s\mu) = \nu(1)\}.$$

Let

$$\lambda : \{(\mu, \nu) \in Q_{m-1} \times B^I \mid h_{m-1}(\mu) = \nu(0)\} \to Q_{m-1}^I$$

be a path lifting map for the fibration h_{m-1}. Define the map

$$w : U \to Q_m = E(B^m \times B; \Delta B, T_1^m \times B \cup B^m \times *)$$

by $w(s\mu, \nu) = (\lambda(\mu, -\nu)(1)_s, \nu)$ where for any path $\xi \in Q_{m-1}$, ξ_s is the path defined by $\xi_s(t) = \xi(st)$. Then the right hand square in the following diagram commutes.

$$\begin{tikzcd}
V \arrow[rr] \arrow[d, "w'"] & & U \arrow[r, "v"] \arrow[dd, "w"] & B \arrow[dd, equal] \\
R_{m-1} \times PB \cup CR_{m-1} \times \Omega B \arrow[d, "w''"] & & & \\
R_m \arrow[rr] & & Q_m \arrow[r, "h_m"] & B
\end{tikzcd}$$

Hence w induces a map between the fibres V and R_m, which can be factored into two maps w' and w'' defined by $w'(s\mu, \nu) = (s\lambda(\mu, -\nu)(1), \nu)$ and

$$w''(s\xi, \nu) = (\xi_s, \nu) \in E(B^m \times B; *, T_1^m \times B \cup B^m \times *).$$

By the same arguments used in the proof of [5; Theorem 1.1] w' is a weak homotopy equivalence. By standard excision arguments it is clear that w'' induces homology isomorphisms and since B is simply connected w'' is also a weak homotopy equivalence. Hence by the homotopy exact sequence for fibrations, the 5-lemma and Whitehead's Theorem [19; Theorem 1] $\mathfrak{R}_m$ is homotopic to the fibration v and hence to $\mathfrak{F}_m$. The theorem follows by induction.

The following theorem is also proved in [6; Proposition 2.2] directly from the classical Lusternik-Schnirelmann definition of category, instead of from G. W. Whitehead's definition used here.

THEOREM 3.4. G-cat B = cat B.

Proof. Suppose G-cat $B \leq k$ so that there exists a map $r : B \to E_k$ such that $p_k \circ r \simeq 1$. By Proposition 3.3 there exists a map

$$u : E_k \to T_1^{k+1}$$

such that $j \circ u \simeq \Delta \circ p_k : E_k \to B^{k+1}$. Let $\phi = u \circ r : B \to T_1^{k+1}$; then

$$j \circ \phi = j \circ u \circ r \simeq \Delta \circ p_k \circ r \simeq \Delta.$$

Hence cat $B \leq k$.

Conversely, suppose cat $B \leq k$ and that there exists a map $\phi : B \to T_1^{k+1}$ and a homotopy $\Psi_t : B \to B^{k+1}$ such that $\Psi_0 = \Delta$ and $\Psi_1 = j \circ \phi$. Define the map

$$r : B \to E(B^{k+1}; \Delta B, T_1^{k+1}) \quad \text{by} \quad r(b)(t) = \Psi_t(b).$$

This is a cross-section to $\mathfrak{R}_k$ and by Proposition 3.3 induces a cross-section to $\mathfrak{F}_k$; hence G-cat $B \leq k$.

THEOREM 3.5. G-wcat $B \geq$ wcat B.

Proof. The maps u and Δ induce a map Δ' between the cofibres of p_k and j such that the following diagram is homotopy commutative [14; (2.2)]. The cofibre of j is $B^{(k+1)}$, the $(k + 1)$-fold smash product of B.

$$\begin{tikzcd} E_k \arrow[r, "p_k"] \arrow[d, "u"] & B \arrow[r, "q_k"] \arrow[d, "\Delta"] & C_k \arrow[d, "\Delta'"] \\ T_1^{k+1} \arrow[r, "j"] & B^{k+1} \arrow[r, "q"] & B^{(k+1)} \end{tikzcd}$$

Now suppose G-wcat $B = k$; then $q_k \simeq 0$ and so

$$q \circ \Delta \simeq \Delta' \circ q_k \simeq 0 : B \to B^{(k+1)}.$$

Hence wcat $B \leq k$.

4. Weak category and the composite Hopf invariant

In this section we recall the properties of the various Hopf invariants that we will need. We will give a criterion for G-wcat $B \leq 1$ in terms of a composite Hopf invariant and then find examples which distinguish G-wcat B from wcat B and cat B.

Consider the following part of the ladder of fibrations used in defining G-wcat B.

$$\begin{tikzcd} \Omega B \arrow[r] & PB \arrow[r, "p_0"] \arrow[d] & B \arrow[r, "q_0"] \arrow[d, equal] & C_0 \\ \Omega B * \Omega B \arrow[r] & \Sigma\Omega B \arrow[r, "p_1"] & B \arrow[r, "q_1"] & C_1 \end{tikzcd}$$

Here p_0 is the standard fibre map $\mathfrak{F}_0$ with cofibre C_0. In the second fibration $\mathfrak{F}_1$ of the ladder p_1 is the evaluation map.

For the remainder of this section we will take B to be the cofibre of a map $\alpha : S^{n-1} \to Y$. In particular we will take B to be of the form $S^q \cup_\alpha e^n$. We will now define a composite higher Hopf invariant in order to use it to approximate $\Sigma\Omega B$ by a simpler space.

Let D be the infinite one point union $\bigvee_{i \geq 1} S^{i(q-1)+1}$ and let $\tau_k : D \to S^{k(q-1)+1}$ be the projection onto the k-th factor. Fix a homotopy equivalence

$$\psi : (\Sigma S_\infty^{q-1}, \Sigma S^{q-1}) \to (D, S^q)$$

which is the identity on S^q. This can be done by using the James' maps $S_\infty^{q-1} \to S_\infty^{k(q-1)}$ [10; p. 24]. Let $\theta : S_\infty^{q-1} \to \Omega S^q$ be the canonical weak homotopy equivalence of the reduced product complex [11]. Denote the suspension homomorphism by Σ and the Hurewicz isomorphism by

$$\rho : \pi_{n-1}(S^q) \to \pi_{n-2}(\Omega S^q).$$

DEFINITION 4.1 [10; p. 24]. The *composite higher Hopf invariant*

$$H : \pi_{n-1}(S^q) \to \pi_{n-1}(D)$$

is defined by $H = \psi_* \circ \Sigma \circ \theta_*^{-1} \circ \rho$.

For $k \geq 1$, the *higher Hopf invariants*

$$H_k : \pi_{n-1}(S^q) \to \pi_{n-1}(S^{k(q-1)+1})$$

are defined by $H_k = \tau_{k*} \circ H$.

Let $D' = \bigvee_{i \geq 2} S^{i(q-1)+1}$ and let $p' : D \to D'$ be the map which shrinks S^q to the base point. Define the *composite higher Hopf invariant* $H' : \pi_{n-1}(S^q) \to \pi_{n-1}(D')$ by $H' = p'_* \circ H$.

In the next proposition we recall from (3.10) and Theorem (3.19) of [10] the properties of the Hopf invariants we will need. We also state the connections between these Hopf invariants and the crude Hopf invariant

$$\bar{H} : \pi_{n-1}(S^q) \to \pi_n(S^q \wedge S^q)$$

and the delicate Hopf invariant

$$\mathcal{H} : \pi_{n-1}(S^q) \to \pi_n(S^q \times S^q, S^q \vee S^q)$$

as defined by Hilton in [3; (2.11)]. Part (iii) comes from (3.14) of [10]. It follows from Proposition 4.3 of [5] that the delicate Hopf invariant $\mathcal{H}$ is equal to a James type Hopf invariant which Ganea calls $\mathcal{H}''$. For $k \geq 2$, the higher Hopf invariants H_k can be obtained from $\mathcal{H}''$ by projecting from $\Omega S^q * \Omega S^q$ to the sphere $S^{k(q-1)+1}$ and hence $H_k(\alpha) = 0$ if $\mathcal{H}(\alpha) = 0$.

PROPOSITION 4.2.

(i) $H_1 = 1$, *the identity homomorphism*;
(ii) $H(\xi) \circ \Sigma\eta) = H(\xi) \circ \Sigma\eta$, *where* $\xi \in \pi_m(S^q)$ *and* $\eta \in \pi_{n-2}(S^{m-1})$;
(iii) $\mathcal{H} = \Sigma H_2$;
(iv) *if* $\mathcal{H}(\alpha) = 0$ *then* $H_k(\alpha) = 0$ *for* $k \geq 2$.

PROPOSITION 4.3. *Let*

$$\Sigma Z \xrightarrow{\alpha} Y \xrightarrow{\gamma} X$$

be a cofibration in which Y *is* $(q - 1)$*-connected and* Z *is* $(n - 3)$*-connected,* $(n - 1) \geq q \geq 3$. *Then there exists an* $(n + q - 2)$*-connected map* $m : \Sigma\Omega Y \cup_\beta C\Sigma Z \to \Sigma\Omega X$ *where* $\beta = \Sigma\bar{\alpha}$ *and* $\bar{\alpha} : Z \to \Omega Y$ *is the adjoint of* α.

Proof. Convert γ into a fibre map; let F be the fibre and $j : F \to Y$ be induced from the inclusion map of the fibre. Lift α to a map $d : \Sigma Z \to F$ such that $\alpha \simeq j \circ d$ and by Lemma 3.1 of [5] d is $(n + q - 3)$-connected. Let C be the cofibre of $\Omega j : \Omega F \to \Omega Y$ and extend $\Omega\gamma$ to a map $u : C \to \Omega X$. By Theorem 1.1 of [5] the fibre of u is homotopic to $\Omega F * \Omega^2 X$; hence u is $(n + q - 3)$-connected. Let $\beta = \Sigma\bar{\alpha}$; then $\beta \simeq \Sigma\Omega j \circ \Sigma\,\bar{d}$ and in the following diagram the horizontal sequences are cofibrations and v is induced from the maps between these cofibrations [14; (2.2)].

$$\begin{tikzcd}
\Sigma Z \arrow[r, "\beta"] \arrow[d, "\Sigma\bar{d}"'] \arrow[dr, "\Sigma\bar{\alpha}"'] & \Sigma\Omega Y \arrow[r] \arrow[d, equal] & \Sigma\Omega Y \cup_\beta C\Sigma Z \arrow[d, "v"] \\
\Sigma\Omega F \arrow[r, "\Sigma\Omega j"] & \Sigma\Omega Y \arrow[r] \arrow[dr, "\Sigma\Omega\gamma"'] & \Sigma C \arrow[d, "\Sigma u"] \\
 & & \Sigma\Omega X
\end{tikzcd}$$

Since $\Sigma\bar{d}$ is $(n+q-3)$-connected, by applying the 5-lemma to the homology exact sequence of the above cofibrations, we see that v is $(n+q-2)$-connected.

Let $m = \Sigma u \circ v : \Sigma\Omega Y \cup_\beta C\Sigma Z \to \Sigma\Omega X$ and then the proposition follows.

Remark 4.4. If $\Sigma Z = S^{n-1}$ and $Y = S^q$ in the above proposition then $X = S^q \cup_\alpha e^n$ and the map $\beta = \Sigma\bar{\alpha} \in \pi_{n-1}(\Sigma\Omega S^q)$. But

$$H(\alpha) = \psi_* \circ \Sigma \circ \theta_*^{-1} \circ \bar{\alpha} = \psi_* \circ (\Sigma\theta)_*^{-1} \circ \beta$$

and $\psi_* \circ (\Sigma\theta)_*^{-1}$ is an isomorphism. Hence in the above proposition we can consider β to be $H(\alpha)$ and m to be the map $m : D \cup_\beta e^n \to \Sigma\Omega B$.

THEOREM 4.5. *If* $B = S^q \cup_\alpha e^n$, $n-1 \geq q \geq 3$, *then* G-wcat $B \leq 1$ *if and only if* $\Sigma H'(\alpha) = 0 \in \pi_n(\Sigma D')$.

Proof. Let $m : D \cup_\beta e^n \to \Sigma\Omega B$ be the map defined in Proposition 4.3 and let C_1' be the cofibre of the map $p_1 \circ m$.

$$\begin{tikzcd}
D \cup_\beta e^n \arrow[r, "p_1\circ m"] \arrow[d, "m"] & B \arrow[r, "q_1'"] \arrow[d, equal] & C_1' \arrow[d, "m_1"] \\
\Sigma\Omega B \arrow[r, "p_1"] & B \arrow[r, "q_1"] & C_1
\end{tikzcd}$$

In the above diagram m induces a map of cofibres $m_1 : C_1' \to C_1$. By applying the 5-lemma to the homology exact sequence of the above cofibrations we see that m_1 is $(n+q-1)$-connected.

Now $C_1' = B \cup C(D \cup_\beta e^n) = (S^q \cup_\alpha e^n) \cup C((S^q \vee D') \cup_\beta e^n)$ and $p_1 \circ m$ maps S^q onto S^q with degree 1. Therefore since the embedding $S^q \subset C_1'$ can be pulled back to the cofibre $(S^q \vee D') \cup_\beta e^n$ it is nullhomotopic. Hence shrinking S^q to a point $C_1' \simeq C_\varepsilon = S^n \cup_\varepsilon C(D' \cup_\delta e^n)$ where $\delta = p' \circ \beta = H'(\alpha)$ and C_ε is the cofibre of $\varepsilon : D' \cup_\delta e^n \to S^n$ which is induced from $p_1 \circ m$.

We shall prove that C_1' is homotopic to $\Sigma D'$. Since $p_1 \circ m$ maps D into S^q, when we shrink S^q to a point ε maps D' to the base point.

Now $m_* : H_n(D \cup_\beta e^n) \to H_n(\Sigma\Omega B)$ is an isomorphism so that ε maps the n-cell onto S^n with degree ± 1. Hence, if the degree of ε on the n-cell is $+1$, ε is homotopic to the map ε' which occurs in the following cofibration sequence for δ:

$$S^{n-1} \xrightarrow{\delta} D' \to D' \cup_\delta e^n \xrightarrow{\varepsilon'} S^n \xrightarrow{\Sigma\delta} \Sigma D' \quad \cdots .$$

By [14; Satz 5], C_1' which is homotopic to C_ε, is also homotopic to $\Sigma D'$ and the inclusion map $S^n \to C_\varepsilon$ is homotopic to $\Sigma\delta$. If the degree of ε on the n-cell is -1 then the inclusion $S^n \to C_\varepsilon$ is homotopic to $-\Sigma\delta$.

Let $\tilde{\alpha}$ be the characteristic map of the n-cell in B. Factor q_1 through C_1' by means of q_1'.

$$\begin{tikzcd}
(CS^{n-1}, S^{n-1}) \arrow[r, "\tilde{\alpha}"] & (B, S^q) \arrow[r, "q_1"] \arrow[dr, "q_1'"'] & (C_1, *) \\
& & (C_1', *) \arrow[u, "m_1"']
\end{tikzcd}$$

Let $\phi' \in \pi_n(C_1')$ be the element represented by $q_1' \circ \tilde{\alpha}$. The inclusion map $S^n \to C_\varepsilon$ is in the homotopy class ϕ'. Hence

$$\phi' = \pm\Sigma\delta = \pm\Sigma H'(\alpha) \in \pi_n(\Sigma D').$$

Let $\phi = m_{1*}\phi' \in \pi_n(C_1)$ represent $q_1 \circ \tilde{\alpha}$; then we know that m_{1*} is an isomorphism in dimension n. If $n - 1 \geq q \geq 3$, C_1 has no cells in positive dimensions less than $q + 1$ and it follows that $q_1 \simeq 0$ if and only if $\phi = 0$. Hence the following five statements are equivalent:

(i) G-wcat $B \leq 1$.
(ii) $q_1 \simeq 0$.
(iii) $\phi = 0 \in \pi_n(C_1)$.
(iv) $\phi' = 0 \in \pi_n(C_1') \approx \pi_n(\Sigma D')$.
(v) $\Sigma H'(\alpha) = 0 \in \pi_n(\Sigma D')$.

Example 4.6. Let $B = S^3 \cup_\alpha e^{18}$ where $\alpha = \varepsilon_3 \circ \nu_{11} \circ \nu_{14} \in \pi_{17}(S^3)$ is an element of order 2, then cat $B = 2$ and G-wcat $B = 1$.

Proof. Recall from Chapter 6 of [16] that the element ε_3 of order 2 is the generator of $\pi_{11}(S^3)$ and is defined by the secondary composition $\{\eta_3, \Sigma\nu', \nu_7\}_1$. The element $\nu_n \in \pi_{n+3}(S^n)$ is the generator of order 8 in the stable 3-stem. Since ν_{11} and ν_{14} are both suspensions it follows from Proposition 4.2 (ii) that $H'(\alpha) = H'(\varepsilon_3) \circ \nu_{11} \circ \nu_{14}$.

Now $H'(\varepsilon_3) \in \pi_{11}(S^5 \vee S^7 \vee S^9 \vee S^{11})$ which by Theorem A of [9] is isomorphic to the direct sum decomposition

$$\pi_{11}(S^5) \oplus \pi_{11}(S^7) \oplus \pi_{11}(S^9) \oplus \pi_{11}(S^{11}) \oplus \pi_{11}(S^{11}).$$

By the definition of $H_k(\varepsilon_3)$ the projections of $H'(\varepsilon_3)$ on the first and third summands are $H_2(\varepsilon_3) \in \pi_{11}(S^5)$ and $H_4(\varepsilon_3) \in \pi_{11}(S^9)$. The projections on the other summands are zero since $\pi_{11}(S^7) = 0$ and $\pi_{11}(S^{11}) = Z$.

Now by (6.1) of [16] $H_2(\varepsilon_3) = \nu_5 \circ \nu_8$ and by a proof similar to that of (2.3) of [16] we see that

$$H_4(\varepsilon_3) \subset \{H_4(\eta_3), \Sigma\nu', \nu_7\}_1 = 0$$

since the coset consists of a single element. Thus the only non-zero component of $H'(\alpha)$ is

$$H_2(\alpha) = \nu_5 \circ \nu_8 \circ \nu_{11} \circ \nu_{14} \in \pi_{17}(S^5).$$

From the information on the 12-stem obtained in the proof of (7.6) of [16] we see that the suspension of $H_2(\alpha)$ is zero and hence by Proposition 4.2 we conclude that $\mathcal{H}(\alpha) \neq 0$ while $\Sigma H'(\alpha) = 0$.

Therefore $\operatorname{cat} B = 2$ by [3; (3.20)], while G-wcat $B = 1$ by Theorem 4.5 above.

Example 4.7. Let $B = S^3 \cup_\alpha e^{15}$ where $\alpha = \alpha_3(3)$ is an element of order 3 in $\pi_{14}(S^3)$; then G-wcat $B = 2$ and wcat $B = 1$.

Proof. The crude Hopf invariant $\bar{H}(\alpha)$ lies in $\pi_{15}(S^6)$ which contains no element of order 3. Hence $\bar{H}(\alpha) = 0$ and wcat $B = 1$ by [3; (3.20)].

By (13.10) of [16] $H_3(\alpha) = x.\alpha_2(7) \in \pi_{14}(S^7)$ for some $x \not\equiv 0 \pmod 3$. Therefore $\Sigma H_3(\alpha) = x.\alpha_2(8)$ which is non-zero. Hence $\Sigma H'(\alpha) \neq 0$ and by Theorem 4.5 G-wcat $B = 2$.

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