# SPECTRAL REPRESENTATIONS FOR A GENERAL CLASS OF OPERATORS ON A LOCALLY CONVEX SPACE

BY

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## 1. Introduction

The material in this paper is a generalization of the work of Maeda [15] which in turn generalized the work of Foias [8], C. Ionescu Tulcea [10], and Dunford [6].

Throughout the paper the symbol $N$ will denote the set $\{0, 1, 2, \cdots\}$, $Z$ the set of integers, $R$ the set of real numbers, $R^2$ or $C$ the Euclidean (complex) plane, and $T^1$ the unit circle in $R^2$ considered as a one-dimensional manifold. For any subset $S$ of $R^2$, $\mathfrak{K}(S)$ will be the set of all compact subsets of $R^2$ contained in $S$; $\mathfrak{K}$ will denote $\mathfrak{K}(R^2)$.

For a non-empty open subset $Q$ of $C$ the algebra $\mathcal{H}(Q)$ of complex-valued functions holomorphic on $Q$ will be endowed with the topology of uniform convergence on compact subsets of $Q$. A *holomorphic function over* $K \in \mathfrak{K}$ is a function holomorphic over some open neighborhood of $K$. Two holomorphic functions $f$ and $g$ over $K$ are equivalent if $f \mid Q = g \mid Q$ for some neighborhood $Q$ of $K$. The set $\mathcal{H}(K)$ of equivalence classes of functions holomorphic over $K$ is considered as an algebra in the natural way. When endowed with the "van Hove topology" (the inductive limit topology induced by the natural mappings of $\mathcal{H}(Q)$ into $\mathcal{H}(K)$), $\mathcal{H}(K)$ is a topological algebra with unit 1. The symbol $\lambda(\lambda \in C)$ will denote the element $\lambda 1$ of $\mathcal{H}(K)$; the symbol $z$, the identity function of $R^2$ onto itself considered as an element of $\mathcal{H}(K)$. Similarly, for $\lambda \in \mathbf{C}K(= C\backslash K)$, the function $\psi_\lambda$ defined by the equation $\psi_\lambda(z) = 1/(\lambda - z)$ is the inverse of $(\lambda - z)$ in $\mathcal{H}(K)$. The basic properties of $\mathcal{H}(Q)$ and $\mathcal{H}(K)$ are discussed in [21].

All vector spaces will be over the complex field $C$. If $E$ and $F$ are vector spaces, let $\mathcal{L}^*(E, F)$ be the set of linear mappings of $E$ into $F$; if $E$ and $F$ are topological vector spaces, let $\mathcal{L}(E, F)$ be the set of continuous linear mappings of $E$ into $F$. Whenever $E$ is a separated locally convex space, $\mathcal{L}(E)(=\mathcal{L}(E, E))$ will be assumed endowed with the topology of uniform convergence on sets of a family of bounded sets in $E$. $\mathcal{L}(E)$ is an algebra whose identity element will be denoted by $I$. Let $E' = \mathcal{L}(E, C)$, the (topological) dual of $E$.

If $T \in \mathcal{L}^*(D_T, E)$ where $D_T$ is a subspace of $E$, one says that $T$ is a transformation or mapping defined *in* $E$. A transformation $T$ in $E$ is said to have the *single-valued extension property* [6] if, for any open subset $Q$ of $C$ and any

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analytic function $h : Q \to D_T$ such that $(\lambda - T)h(\lambda) = 0$ for every $\lambda \in Q$, $h = 0$. If $T$ has the single-valued extension property, then, for each $x \in E$, there exists a maximally defined analytic function $f : \rho_T(x) \to D_T$ such that $(\lambda - T)f(\lambda) = x$ for every $\lambda \in \rho_T(x)$. The complement of $\rho_T(x)$ in $C$ is called the *spectrum of x with respect to T* and is denoted by $sp_T(x)$. For $F \subset C$, $\mathfrak{M}(T, F) = \{x \in E \mid sp_T(x) \subset F\}$.

Suppose now that $E$ is a separated locally convex space and that $T$ is a transformation defined in $E$. The *resolvent set* of $T$ [21], [14], denoted by $\rho(T)$, is the set of all $\lambda_0 \in C \cup \{\infty\}$ (the Riemann sphere) with the following property: there is a neighborhood $G$ of $\lambda_0$ in $C \cup \{\infty\}$ such that (1) for any $\lambda \in G\backslash\{\infty\}$, there exists $R(\lambda) \in \mathfrak{L}(E)$ such that $(\lambda - T)R(\lambda) = R(\lambda)(\lambda - T) = I$, and (2) $\{R(\lambda) \mid \lambda \in G\backslash\{\infty\}\}$ is bounded in $\mathfrak{L}(E)$. The set $sp(T) = (C \cup \{\infty\})\backslash\rho(T)$ is called the *spectrum* of $T$. It is obvious that $sp(T)$ is compact. $T$ is called *regular* if $sp(T) \subset C$. One sees easily that, if $T$ has the single-valued extension property, $sp_T(x) \subset sp(T)$ for every $x \in E$.

## 2. Basic definitions

DEFINITION. A commutative algebra $\mathfrak{A}$ together with a family $(\mathfrak{A}_K)_{K \in \mathfrak{K}}$ of ideals and, for each $K \in \mathfrak{K}$, a bilinear map $(\varphi, a) \to \varphi \times_K a$ of $\mathfrak{H}(K) \times \mathfrak{A}_K$ into $\mathfrak{A}_K$ is called a *distributional system* if

(1) $\mathfrak{A}_\phi = \{0\}$, $\mathfrak{A}_{K \cap L} = \mathfrak{A}_K \cap \mathfrak{A}_L$ for every $K \in \mathfrak{K}$, $L \in \mathfrak{K}$;

(2) if $K \in \mathfrak{K}, L \in \mathfrak{K}$, and $K \subset L$, then $\varphi \times_K a = \alpha \times_L \varphi$ for every $\varphi \in \mathfrak{H}(L)$, $a \in \mathfrak{A}_K$;

(3) for any $K \in \mathfrak{K}$, $a \in \mathfrak{A}_K$, $b \in \mathfrak{A}_K$, $\varphi \in \mathfrak{H}(K)$, $\psi \in \mathfrak{H}(K)$,

$$(\varphi\psi) \times_K a = \varphi \times_K (\psi \times_K a), \qquad \varphi \times_K (ab) = (\varphi \times_K a)b, \qquad 1 \times_K a = a.$$

The subscript $K$ on $\times_K$ will be omitted from here on since, by (2), no ambiguity can result.

Let $\mathfrak{A}$ be a distributional system. For $K \in \mathfrak{K}$ an element $u \in \mathfrak{A}$ will be called a *K-unit* of $\mathfrak{A}$ if $ua = a$ for every $a \in \mathfrak{A}_K$. The set of $K$-units of $\mathfrak{A}$ will be denoted by $\mathfrak{U}_K$. For $S \subset R^2$ define

$$\mathfrak{A}_S = \textstyle\bigcup_{K \in \mathfrak{K}(S)} \mathfrak{A}_K, \qquad \mathfrak{U}_S = \textstyle\bigcap_{K \in \mathfrak{K}(S)} \mathfrak{U}_K.$$

Write also $\mathfrak{A}_c = \mathfrak{A}_C$. An element $u \in \mathfrak{A}$ is *one over* $S \subset R^2$ if $u \in \mathfrak{U}_Q$ for some neighborhood $Q$ of $S$.

We collect here some definitions which will be used in different parts of this paper. A distributional system $\mathfrak{A}$ is called *separating* if, for any $K \in \mathfrak{K}$ and any neighborhood $Q$ of $K$, $\mathfrak{U}_K \cap \mathfrak{A}_Q \neq \emptyset$; $\mathfrak{A}$ is said to have *partitions of unity* if for any $K \in \mathfrak{K}$ and any finite open cover $\mathfrak{V}$ of $K$, there exists an element $(b_Q)_{Q \in \mathfrak{V}}$ of $\prod_{Q \in \mathfrak{V}} \mathfrak{A}_Q$ such that $\sum_{Q \in \mathfrak{V}} b_Q \in \mathfrak{U}_K$. One sees that a distributional system with partitions of unity is separating. $\mathfrak{A}$ is *modular* if, for any closed subset $F$ of $R^2$ and any $u \in \mathfrak{A}$ one over $F$, $a$-$ua \in \mathfrak{A}_{CF}$ for every $a \in \mathfrak{A}_c$.

Let $\mathfrak{A}$ be a distributional system, $E$ a separated locally convex space, and $U$ a representation of $\mathfrak{A}$ into $\mathfrak{L}(E)$. A net $(b_\alpha)$ of elements of $\mathfrak{A}_c$ is an *approxi-*

*mate identity* for $U$ if the net $(U(b_\alpha))$ converges simply (pointwise) to $I$ in $\mathcal{L}(E)$. $U$ is an *$\mathfrak{A}$-spectral representation* if (1) there exists an approximate identity for $U$, and (2) for any $K \in \mathfrak{K}$, $a \in \mathfrak{A}_K$, the mapping $\varphi \to U(\varphi \times a)$ of $\mathcal{K}(K)$ into $\mathcal{L}(E)$ is continuous. A closed subset $F$ of $R^2$ is said to *support* $U$ if $U(u) = I$ for every $u \in \mathfrak{A}$ which is one over $F$.

PROPOSITION 2.1. *Let $\mathfrak{A}$ be a modular distributional system, $U$ an $\mathfrak{A}$-spectral representation, and $F$ a closed subset of $R^2$. (1) If $U(a) = 0$ for every $a \in \mathfrak{A}_{CF}$, then $U$ is supported by $F$ (2). If $\mathfrak{A}$ is separating and $F \in \mathfrak{K}$, then $U$ is supported by $F$ if and only if $U(a) = 0$ for every $a \in \mathfrak{A}_{CF}$.*

*Proof.* (1) Let $(b_\alpha)$ be an approximate identity for $U$. If $u$ is one over $F$, then $b_\alpha - ub_\alpha \in \mathfrak{A}_{CF}$ for every $\alpha$ and therefore $I = \lim_\alpha U(b_\alpha) = \lim_\alpha U(ub_\alpha) = U(u)$ (the limit taken in the simple topology).

(2) If $a \in \mathfrak{A}_{CF}$, then $a \in \mathfrak{A}_M$ for some $M \in \mathfrak{K}(\mathbf{C}F)$. Let $L$ be a compact neighborhood of $F$ with $L \cap M = \emptyset$ and let $u \in \mathfrak{U}_L \cap \mathfrak{A}_{CM}$. Then $U(a) = U(u)U(a) = U(ua) = 0$ since $ua \in \mathfrak{A}_{CM} \cap \mathfrak{A}_M = \mathfrak{A}_{CM \cap M} = \mathfrak{A}_\emptyset = \{0\}$.

Suppose now that $\mathfrak{A}$ is a distributional system and that $U$ is an $\mathfrak{A}$-spectral representation. For an open subset $Q$ of $R^2$ define

$$\mathfrak{M}(U, Q) = \{U(a)x \mid a \in \mathfrak{A}_Q, x \in E\};$$

for $K \in \mathfrak{K}$ let

$$\mathfrak{M}(U, K) = \bigcap \{\mathfrak{M}(U, Q) \mid Q \text{ an open neighborhood of } K\}.$$

PROPOSITION 2.2. *Let $\mathfrak{A}$ be a separating distributional system; let $K \in \mathfrak{K}$ and $x \in E$. Then $x \in \mathfrak{M}(U, K)$ if and only if $U(u)x = x$ for every $u \in \mathfrak{A}$ which is one over $K$.*

*Proof.* If $x \in \mathfrak{M}(U, K)$, $L$ is a neighborhood of $K$, and $u \in \mathfrak{U}_L$, then $x = U(a)y$ for some $y \in E$, $a \in \mathfrak{A}_L$. Therefore $U(u)x = U(ua)y = U(a)y = x$. To show the converse result, let $Q$ be any open neighborhood of $K$, and let $u \in \mathfrak{U}_L \cap \mathfrak{A}_Q$. Then $x = U(u)x \in \mathfrak{M}(U, Q)$. By the arbitrariness of $Q$, $x \in \mathfrak{M}(U, K)$.

COROLLARY 1. *Each $\mathfrak{M}(U, K)$ is a closed subspace of $E$ invariant under each $U(a)(a \in \mathfrak{A})$.*

COROLLARY 2. *If $a \in \mathfrak{A}_K$, then $U(a)(E) \subset \mathfrak{M}(U, K)$.*

COROLLARY 3. *$\mathfrak{M}(U, \infty) = \bigcup_{K \in \mathfrak{K}} \mathfrak{M}(U, K)$ is dense in $E$.*

COROLLARY 4. *If $\mathfrak{A}$ is modular, then $x \in \mathfrak{M}(U, K)$ if and only if $U(a)x = 0$ for every $a \in \mathfrak{A}_{CK}$.*

*Proof.* If $x \in \mathfrak{M}(U, K)$ and $a \in \mathfrak{A}_{CK}$, then, since $\mathfrak{A}$ is assumed separating, $x = U(b)y$ for some $y \in E$, $b \in \mathfrak{A}$ with $ab = 0$. Thus $U(a)x = U(ab)y = 0$. Conversely, let $u \in \mathfrak{A}$ be one over $K$, and let $(b_\alpha)$ be an approximate identity for $U$. For each $\alpha$, $b_\alpha - ub_\alpha \in \mathfrak{A}_{CK}$ by modularity of $\mathfrak{A}$, so that $U(b_\alpha - ub_\alpha)x = 0$. Therefore $x = \lim_\alpha U(b_\alpha)x = \lim_\alpha U(ub_\alpha)x = U(u)x$.

## 3. $\mathfrak{A}$-spectral and $\mathfrak{A}$-scalar transformations

In this section we assume that $\mathfrak{A}$ is a separating modular distributional system, $E$ is a separated locally convex space, and $T$ is a transformation defined in $E$ with domain $D_T$.

DEFINITION. $T$ is an *$\mathfrak{A}$-spectral transformation* if there exists an $\mathfrak{A}$-spectral representation $U$ such that (1) $\mathfrak{M}(U, \infty) \subset D_T$, (2) $TU(a) \mid \mathfrak{M}(U, \infty) = U(a)T \mid \mathfrak{M}(U, \infty)$ for every $a \in \mathfrak{A}$, (3) $T^n U(a) \in \mathfrak{L}(E)$ for every $n \in N$, $a \in \mathfrak{A}_c$, and (4) $sp(T \mid \mathfrak{M}(U, K)) \subset K$ for every $K \in \mathfrak{K}$ such that $\mathfrak{M}(U, K) \neq \{0\}$. $U$ is then called an *$\mathfrak{A}$-spectral representation for $T$*.

DEFINITION. $T$ is an *$\mathfrak{A}$-scalar transformation* if there exists an $\mathfrak{A}$-spectral representation $U$ such that (1) $\mathfrak{M}(U, \infty) \subset D_T$, and (2) $U(z \times a) = TU(a)$ for every $a \in \mathfrak{A}_c$. $U$ is then called an *$\mathfrak{A}$-scalar representation* for $T$.

*Remark.* An $\mathfrak{A}$-scalar representation for $T$ is an $\mathfrak{A}$-spectral representation for $T$. Therefore, an $\mathfrak{A}$-scalar transformation is an $\mathfrak{A}$-spectral transformation.

PROPOSITION 3.1. *If $T$ is an $\mathfrak{A}$-spectral transformation, and $U$ is an $\mathfrak{A}$-spectral representation for $T$, then $U$ is supported by $sp(T)$. If $T$ is a regular $\mathfrak{A}$-scalar transformation and $U$ is an $\mathfrak{A}$-scalar representation for $T$, then any $F \in \mathfrak{K}$ supporting $U$ contains $sp(T)$.*

*Proof.* Let $K \in \mathfrak{K}(\mathbf{C}sp(T))$, $a \in \mathfrak{A}_K$, and suppose that $U(a) \neq 0$. Then $\mathfrak{M}(U, K) \neq \{0\}$ (Corollary 2 of Proposition 2.2) from which one concludes that

$$sp(T \mid \mathfrak{M}(U, K)) \subset K \cap sp(T) = \phi,$$

a contradiction. By Proposition 2.1, $U$ is supported by $sp(T)$.

To prove the second assertion, fix $\lambda \in \mathbf{C}F$ and let $Q$ be a compact neighborhood of $\lambda$ such that $Q \cap F = \phi$. Since $\mathfrak{A}$ is separating, there exists a compact neighborhood $L$ of $\lambda$, a compact neighborhood $M$ of $sp(T) \cup Q$, $u \in \mathfrak{U}_L \cap \mathfrak{A}_Q$, and $v \in \mathfrak{U}_M \cap \mathfrak{A}_C$. Since $\mathfrak{A}$ is modular, $(v - uv) \in \mathfrak{A}_{\mathbf{C}\{\lambda\}}$ so that $\psi_\lambda \times (v - uv)$ is a well-defined element of $\mathfrak{A}$. A sample calculation shows that

$$(\lambda - T)U(\psi_\lambda \times (v - uv)) = U(v)(I - U(u)).$$

$U(v) = I$ since $sp(T)$ supports $U$, and $U(u) = 0$ by Proposition 2.1. Thus $\lambda \in \rho(T)$.

LEMMA 3.1 *Let $T$ be an $\mathfrak{A}$-spectral transformation, and let $U$ be an $\mathfrak{A}$-spectral representation for $T$ such that $TU(a)x = U(a)Tx$ for all $a \in \mathfrak{A}_c$, $x \in D_T$. If $y \in D_T$ and $\lambda \in C$ are such that $(\lambda - T)y = 0$, then $y \in \mathfrak{M}(U, \{\lambda\})$.*

*Proof.* Let $K \in \mathfrak{K}$ be such that $\lambda \notin K$, and let $a \in \mathfrak{A}_K$. Since

$$U(a)y \in \mathfrak{M}(U, K) \quad \text{and} \quad sp(T \mid \mathfrak{M}(U, K)) \subset K,$$

there exists $R \in \mathfrak{L}(\mathfrak{M}(U, K))$ such that $R(\lambda - T)U(a)y = U(a)y$ from which one concludes

$$U(a)y = R(\lambda - T)U(a)y = RU(a)(\lambda - T)y = 0.$$

Corollary 4 of Proposition 2.2 is now used to show that $y \in \mathfrak{M}(U, \{\lambda\})$.

THEOREM 3.1 *Let $T$ be an $\mathfrak{A}$-spectral transformation. Suppose there exists an $\mathfrak{A}$-spectral representation $U$ for $T$ such that $TU(a)x = U(a)Tx$ for all $a \in \mathfrak{A}_c$, $x \in D_T$. (This condition holds if $T \in \mathfrak{L}(E)$ or if $\mathfrak{M}(U, \infty) = D_T$.) Then $T$ has the single-valued extension property.*

*Proof.* Let $Q$ be an open subset of $C$ and $h$, an analytic function from $Q$ into $D_T$ such that $(\lambda - T)h(\lambda) = 0$ for all $\lambda \in Q$. Fix $\lambda_0 \in Q$. Let $L \in \mathfrak{K}(Q)$ be a neighborhood of $\lambda_0$, $M \in \mathfrak{K}(Q)$, a neighborhood of $L$, and $u \in \mathfrak{U}_L \cap \mathfrak{A}_M$. By Lemma 3.1, $h(\lambda) \in \mathfrak{M}(U, \{\lambda\})$ for each $\lambda \in Q$.

For $\lambda \in Q \backslash M$, $U(u)h(\lambda) = 0$ (Corollary 4 of Proposition 2.2). Since $\lambda \to U(u)h(\lambda)$ is an analytic function from $Q$ into $E$ which is 0 on a non-empty open subset of $Q$, $U(u)h(\lambda) = 0$ for every $\lambda \in Q$. In particular $h(\lambda_0) = U(u)h(\lambda_0) = 0$. Since $\lambda_0$ is arbitrary, $h = 0$.

THEOREM 3.2. *Let $T$ be an $\mathfrak{A}$-spectral transformation and let $U$ be an $\mathfrak{A}$-spectral representation for $T$ such that $U(a)Tx = TU(a)x$ for every $a \in \mathfrak{A}_c$, $x \in D_T$. Then, for each $K \in \mathfrak{K}$, $\mathfrak{M}(U, K) = \mathfrak{M}(T, K)$.*

*Proof.* Suppose $x \in \mathfrak{M}(T, K)$. Let $a \in \mathfrak{A}_M$, $M \in \mathfrak{K}(\mathbf{C}K)$, and let $h : \mathbf{C}K \to D_T$ be the analytic function such that $(\lambda - T)h(\lambda) = x$ for all $\lambda \in \mathbf{C}K$. Define

$$f(\lambda) = U(a)h(\lambda) \quad \text{if} \quad \lambda \in \mathbf{C}K$$

and

$$f(\lambda) = ((\lambda - T) \mid \mathfrak{M}(U, M))^{-1}U(a)x \quad \text{if} \quad \lambda \in \mathbf{C}M.$$

To see that $f$ is well defined, write

$$U(a)x = U(a)(\lambda - T)h(\lambda) = (\lambda - T)U(a)h(\lambda) \tag{1}$$

for $\lambda \in \mathbf{C}K \cap \mathbf{C}M$. Since $U(a)x \in \mathfrak{M}(U, M)$, the desired result follows by multiplying both sides of (1) by $((\lambda - T) \mid \mathfrak{M}(U, M))^{-1}$.

$f$ is an entire function, and $(\lambda - T)f(\lambda) = U(a)x$ for all $\lambda \in C$. Since $sp(T \mid \mathfrak{M}(U, M))$ is compact, $\lim_{\lambda \to \infty} f(\lambda) = 0$ and therefore $f = 0$. One concludes that $U(a)x = 0$. Thus, by Corollary 4 of Proposition 2.1, $x \in \mathfrak{M}(U, K)$.

On the other hand, if $x \in \mathfrak{M}(U, K)$, the mapping

$$h : \lambda \to ((\lambda - T) \mid \mathfrak{M}(U, K))^{-1}x$$

is an analytic function of $\mathbf{C}K$ into $D_T$ such that $(\lambda - T)h(\lambda) = x$ for all $\lambda \in \mathbf{C}K$.

The following theorem is basic in the theory of spectral transformations. The proof is omitted here since it is analogous to the proofs of Theorem 4.1 and Theorem 4.2 of [15] (see Example 1 in Section 6).

THEOREM 3.3. *Let $\mathfrak{A}$ be a modular distributional system having partitions of unity. Suppose that $E$ is a separated locally convex space, that $\mathfrak{L}(E)$, endowed with the topology of uniform convergence on sets of a family of bounded subsets of $E$, is sequentially complete, and that $T$ is a transformation in $E$ with domain $D_T$.*

(1) *If $T$ is $\mathfrak{A}$-spectral, then there exists a closed $\mathfrak{A}$-scalar transformation $S$ in $E$ and $Q \in \mathfrak{L}^* (\mathfrak{M}(T, \infty))$ such that*

(i) $D_S \supset \mathfrak{M}(T, \infty)$,

(ii) $\lim_{n\to\infty} | \langle Q^n x, x' \rangle |^{1/n} = 0$ *for every* $x \in \mathfrak{M}(T, \infty)$, $x' \in E'$,

(iii) $T \mid \mathfrak{M}(T, \infty) = (S + Q) \mid \mathfrak{M}(T, \infty)$, *and*

(iv) $SQ \mid \mathfrak{M}(T, \infty) = QS \mid \mathfrak{M}(T, \infty)$.

(2) *Suppose $T$ is $\mathfrak{A}$-scalar, $U$ is an $\mathfrak{A}$-scalar representation for $T$, and $Q$ is a linear mapping defined in $E$ such that*

(i) $\mathfrak{M}(U, \infty) \subset D_Q$,

(ii) $QU(a) \mid \mathfrak{M}(U, \infty) = U(a)Q \mid \mathfrak{M}(U, \infty)$ *for every* $a \in \mathfrak{A}$,

(iii) $Q^n U(a) \in \mathfrak{L}(E)$ *for every* $a \in \mathfrak{A}_c$, $n \in N$, *and*

(iv) $\lim_{n\to\infty} | \langle Q^n x, x' \rangle |^{1/n} = 0$ *for every* $x \in \mathfrak{M}(U, \infty)$, $x' \in E'$.

*Then $T + Q$ is an $\mathfrak{A}$-spectral transformation.*

## 4. Topologies on $\mathfrak{A}$

DEFINITION. Let $\mathfrak{A}$ be a distributional system. A topology on the underlying algebra of $\mathfrak{A}$ is *admissable* if, for every $K \in \mathfrak{K}$, $a \in \mathfrak{A}_K$, the mapping $\varphi \to \varphi \times a$ of $\mathfrak{K}(K)$ into $\mathfrak{A}$ is continuous. The *final topology*, $\tau_f$, on $\mathfrak{A}$ is the finest admissable topology on $\mathfrak{A}$ [1, pp 32–34]. The *final Michael topology*, $\tau_m$, is the supremum of all topologies, $\tau$, on $\mathfrak{A}$ coarser than the final topology and such that $(\mathfrak{A}, \tau)$ is a Michael algebra. (A *Michael algebra* [18] is a topological algebra such that there exists a fundamental system of neighborhoods of 0 consisting of convex, idempotent ($GG \subset G$) sets.)

Let $\mathfrak{A}$ be a distributional system, $\tau$ an admissable topology on $\mathfrak{A}$, and $E$ a separated locally convex space. A representation $U$ of $\mathfrak{A}$ into $\mathfrak{L}(E)$ is an $(\mathfrak{A}, \tau)$-*spectral representation* if (1) there exists an approximate identity for $U$, and (2) $U$ is continuous when $\mathfrak{A}$ is endowed with the topology $\tau$. One then defines $(\mathfrak{A}, \tau)$-*spectral transformation* and $(\mathfrak{A}, \tau)$-*scalar transformation* in the natural way.

PROPOSITION 4.1. *Let $\mathfrak{A}$ be a distributional system, $E$ a locally convex space, and $U$ a representation of $\mathfrak{A}$ into $\mathfrak{L}(E)$. Then the following assertions are equivalent*: (1) *$U$ is an $\mathfrak{A}$-spectral representation*; (2) *$U$ is an $(\mathfrak{A}, \tau)$-*

*spectral representation for some admissable topology* $\tau$ *on* $\mathfrak{A}$; (3) $U$ *is an* $(\mathfrak{A}, \tau_f)$-*spectral representation.*

*Remark.* Let $U$ be an $\mathfrak{A}$-spectral representation, and let $(b_\alpha)$ be an approximate identity for $U$. Then the initial toplogy $\tau$ induced on $\mathfrak{A}$ by $U$ [1, p 30] is admissable; in addition $(b_\alpha)$ is an approximate identity for $(\mathfrak{A}, \tau)$.

COROLLARY. *Let* $\mathfrak{A}$ *be a separating distributional system,* $E$ *a separated locally convex space, and* $T$ *a transformation in* $E$. *Then the following assertions are equivalent*: (1) $T$ *is an* $\mathfrak{A}$-*spectral* [$\mathfrak{A}$-*scalar*] *transformation*; (2) $T$ *is an* $(\mathfrak{A}, \tau)$-*spectral* [$(\mathfrak{A}, \tau)$-*scalar*] *transformation for some admissable topology* $\tau$ *on* $\mathfrak{A}$; $T$ *is an* $(\mathfrak{A}, \tau_f)$-*spectral* [$(\mathfrak{A}, \tau_f)$-*scalar*] *transformation.*

The remainder of this section is concerned with transformations defined in a Banach space. If $\mathfrak{A}$ is an algebra endowed with a topology $\tau_0$, let $m(\tau_0)$ be the finest of topologies $\tau$ on $\mathfrak{A}$ coarser than $\tau_0$ such that $(\mathfrak{A}, \tau)$ is a Michael algebra.

LEMMA 4.1. *Let* $\mathfrak{A}$ *be an algebra endowed with a topology* $\tau_0$, $\mathfrak{A}'$ *a normed algebra, and* $U$ *a representation of* $\mathfrak{A}$ *into* $\mathfrak{A}'$. *Then* $U$ *is* $\tau_0$-*continuous if and only if* $U$ *is* $m(\tau_0)$-*continuous.*

*Proof.* Since $m(\tau_0) \subset \tau_0$, every $m(\tau_0)$-continuous mapping is $\tau_0$-continuous. On the other hand, suppose $U$ $\tau_0$-continuous. Define a semi-norm $s$ on $\mathfrak{A}$ by the equation $s(a) = \| U(a) \| (a \in \mathfrak{A})$. One sees that, for any $\varepsilon > 0$, $\{a \in \mathfrak{A} \mid s(a) \leqq \varepsilon\}$ is an $m(\tau_0)$-neighborhood of 0 in $\mathfrak{A}$ which is mapped into $\{a \in \mathfrak{A}' \mid \| a \| \leqq \varepsilon\}$ by $U$.

PROPOSITION 4.2 [11]. *Let* $\mathfrak{A}$ *be a distributional system endowed with an admissable topology* $\tau_0$. *Suppose that* $E$ *is a Banach space and that* $\mathfrak{L}(E)$ *is endowed with the ordinary norm (uniform) topology. Then a representation of* $\mathfrak{A}$ *into* $\mathfrak{L}(E)$ *is* $(\mathfrak{A}, \tau_0)$-*spectral if and only if it is* $(\mathfrak{A}, m(\tau_0))$-*spectral.*

COROLLARY 1. *Let* $\mathfrak{A}$ *be a distributional system,* $E$ *a Banach space, and* $U$ *a representation of* $\mathfrak{A}$ *into* $\mathfrak{L}(E)$ *having an approximate identity. Then* $U$ *is an* $\mathfrak{A}$-*spectral representation if and only if* $U$ *is an* $(\mathfrak{A}, \tau_m)$-*spectral representation.*

COROLLARY 2. *Let* $\mathfrak{A}$ *be a separating distributional system,* $E$ *a Banach space, and* $T$ *a linear transformation in* $E$. *Then* $T$ *is* $\mathfrak{A}$-*spectral* [$\mathfrak{A}$-*scalar*] *if and only if* $T$ *is* $(\mathfrak{A}, \tau_m)$-*spectral* [$(\mathfrak{A}, \tau_m)$-*scalar*].

## 5. $\mathfrak{A}$-spectral operators and decomposable operators on Banach spaces

Let $E$ be a Banach space and $T \in \mathfrak{L}(E)$. Foias [9] has defined a *spectral maximal space* to be a closed subspace $D$ of $E$ invariant under $T$ and such that, if $Z$ is a closed subspace of $E$ invariant under $T$ such that $sp(T \mid Z) \subset sp(T \mid D)$, then $Z \subset D$. $T$ is *decomposable* if, for any finite open cover $\mathfrak{V}$ of

$sp(T)$, there exists a family $(D_Q)_{Q\epsilon\mathcal{V}}$ of spectral maximal spaces such that (i) $sp(T \mid D_Q) \subset Q$ for each $Q \epsilon \mathcal{V}$, and (ii) $E = \sum_{Q\epsilon\mathcal{V}} D_Q$.

THEOREM 5.1. *Let $\mathcal{A}$ be a modular distributional system having partitions of unity. Let $E$ be a Banach space, and let $T \epsilon \mathcal{L}(E)$ be $\mathcal{A}$-spectral. Then $T$ is decomposable.*

*Proof.* For every $K \epsilon \mathfrak{K}$, $\mathfrak{M}(T, K)$ is a spectral maximal sub-space and $sp(T \mid \mathfrak{M}(T, K)) \subset K$ [5, Lemma 5]. Suppose that $\mathcal{V}$ is a finite open cover of $sp(T)$. Let $K$ be a compact neighborhood of $sp(T)$ contained in $\bigcup_{Q\epsilon\mathcal{V}} Q$, and let $(b_Q)_{Q\epsilon\mathcal{V}}$ be an element of $\prod_{Q\epsilon\mathcal{V}} \mathcal{A}_Q$ such that $\sum_{Q\epsilon\mathcal{V}} b_Q \epsilon \mathcal{U}_K$. One sees (using Proposition 3.1 and Corollary 2 to Proposition 2.1) that the family $(\mathfrak{M}(T, K(Q)))_{Q\epsilon\mathcal{V}}$ $(b_Q \epsilon \mathcal{A}_{K(Q)})$ has properties (i) and (ii) in the definition of decomposable operator.

## 6. Examples of distributional systems

*Example* 1. The basic algebras of Maeda [15] are made into distributional systems by defining

$$\mathcal{A}_K = \{f \epsilon \mathcal{A} \mid \operatorname{supp}(f) \subset K\} \qquad (K \epsilon \mathfrak{K}) \tag{2}$$

and, for $\varphi \epsilon \mathcal{H}(K)$, $f \epsilon \mathcal{A}_K$

$$\begin{aligned} (\varphi \times f)(z) &= \varphi(z)f(z) \quad \text{if} \quad z \epsilon K \\ &= 0 \qquad\qquad \text{if} \quad z \notin K. \end{aligned} \tag{3}$$

These systems are separating (by assumption) and modular.

*Example* 2. Let $\mathcal{A}$ be a basic algebra of Maeda, and let $S \subset R^2$. Define

$$\mathcal{A}(S) = \{f \mid S \mid f \epsilon \mathcal{A}\}, \quad \mathcal{A}(S)_K = \{g \epsilon \mathcal{A}(S) \mid \operatorname{supp}(g) \subset K\},$$

and $\varphi \times (f \mid S) = (\varphi \times f) \mid S$ for $K \epsilon \mathfrak{K}$, $\varphi \epsilon \mathcal{H}(K)$, $f \mid S \epsilon \mathcal{A}(S)_K$. ($\varphi \times f$ is as in (3) above.) The importance of this example lies in the fact that, if $\mathcal{A}$ is a basic algebra, $U$ is an $\mathcal{A}$-spectral representation supported by some $K \epsilon \mathfrak{K}$, and $S$ is a bounded neighborhood of $K$, then there exists a unique $\mathcal{A}(S)$-spectral representation $U'$ such that $U'(f \mid S) = U(f)$ for every $f \epsilon \mathcal{A}$. $\mathcal{A}(S)$ has a unit element, and $z$ can be considered as an element of $\mathcal{A}(S)$.

This example generalizes the algebras $C^n(\Delta)$ of [12].

*Example* 3 [13]. Let $\rho = (\rho_n)$ be a two-sided sequence of real numbers such that $1 \leqq \rho_{m+n} \leqq \rho_m\rho_n$ for all $m \epsilon Z$, $n \epsilon Z$, and $\rho_n = O(\mid n \mid^q)$ $(\mid n \mid \to \infty)$ for some $q \epsilon N$. Define

$$\mathcal{A} = \{f \epsilon C(T^1) \mid \sum \rho_n \mid c_n \mid < \infty\}$$

where $c_n$ is the $n^{\text{th}}$ Fourier coefficient of $f$. Define $\mathcal{A}_K$ by (2), and $\times$ by (3). One concludes, from the Riemann-Lebesgue Lemma, that $C^\infty(T^1) \subset \mathcal{A}$.

Thus $\mathfrak{A}$ is a modular distributional system having partitions of unity.

*Example* 4. Let $\Delta \in \mathfrak{K}$. Using the notation of [19], we consider $\mathfrak{A} = \prod_{|p| \leqq n} C(\Delta)$ with the usual vector space structure and with multiplication defined by the equations

$$(fg)_p = \sum_{q+r=p} C_p^q f_q g_r \qquad (|p| \leqq n)$$

for $f = (f_p)_{|p| \leqq n}$, $g = (g_p)_{|p| \leqq n}$ any two elements of $\mathfrak{A}$.

Define $\operatorname{supp}(f) = \bigcup_{|p| \leqq n} \operatorname{supp}(f_p)$. One sees that $\operatorname{supp}(f + g) \subset \operatorname{supp}(f) \cup \operatorname{supp}(g)$, $\operatorname{supp}(\lambda f) = \operatorname{supp}(f)$ if $\lambda \neq 0$, and $\operatorname{supp}(f) = \phi$ if and only if $f = 0$. One can therefore define ideals $\mathfrak{A}_K$ by (2) above. For $K \in \mathfrak{K}$, $\varphi \in \mathfrak{H}(K)$, $f \in \mathfrak{A}_K$, define

$$(\varphi \times f)_p(z) = \sum_{q+r=p} C_p^q (D^q\varphi(z)) f_r(z)$$

if $z \in K \cap \Delta$ and $(\varphi \times f)_p(z) = 0$ if $z \in \Delta \backslash K$. With these definitions $\mathfrak{A}$ becomes a modular distributional system having partitions of unity.

*Example* 5. Let $\Delta \in \mathfrak{K}$, $n \in N$. The mapping $f \to (D^p f \mid \Delta)_{|p| \leqq n}$ is a homomorphism of $C^n(R^2)$ (the algebra of $n$-times continuously differentiable functions on $R^2$) into the algebra $\mathfrak{A}$ of Example 4. Let $C^n(\Delta)$ be the image of $C^n(R^2)$ under this homomorphism, and define $C^n(\Delta)_K = C^n(\Delta) \cap \mathfrak{A}_K$ ($\mathfrak{A}_K$ as in Example 4) for $K \in \mathfrak{K}$. To define $\times$ simply note that, if $\varphi \in \mathfrak{H}(K)$, $a \in C^n(\Delta)_K$, then $\varphi \times a$, as defined in Example 4, is an element of $C^n(\Delta)_K$. $C^n(\Delta)$ is a modular distributional system having partitions of unity.

These algebras were characterized by Whitney [22]. See [20] for more details.

*Example* 6 [16], [17]. A mapping $\gamma : T^1 \to R^2$ is a $C^n$-*curve* ($n \in N \cup \{\infty\}$) if $\gamma$ can be extended to an injective mapping of a neighborhood of $T^1$ onto a neighborhood of $\gamma(T^1)$ such that both $\gamma$ and $\gamma^{-1}$ are $n$-times continuously differentiable. Given a $C^n$-curve $\gamma$, one makes $C^n(T^1)$ into a distributional system $C^n(\gamma)$ by defining

$$C^n(\gamma)_K = \{f \in C^n(T^1) \mid \operatorname{supp}(f) \subset \gamma^{-1}(K \cap \gamma(T^1))\}$$

for $K \in \mathfrak{K}$ and

$$\begin{aligned}(\varphi \times f)(z) &= \varphi(\gamma(z))f(z) && \text{if} \quad z \in \gamma^{-1}(K \cap \gamma(T^1)) \\ &= 0 && \text{if} \quad z \in T^1 \backslash \gamma^{-1}(K \cap \gamma(T^1)),\end{aligned}$$

$\varphi \in \mathfrak{H}(K)$, $f \in C^n(\gamma)_K$. $C^n(\gamma)$ is modular and has partitions of unity.

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